Soap Films: Statics and Dynamics

Frederik Brasz

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1 Introduction

Soap films and bubbles have long fascinated humans with their beauty. The study of soap films is believed to have started around the time of Leonardo da Vinci. Since then, research has proceeded in two distinct directions. On the one hand were the mathematicians, who were concerned with finding the shapes of these surfaces by minimizing their area, given some boundary. On the other hand, physical scientists were trying to understand the properties of soap films and bubbles, from the macroscopic behavior down to the molecular description.

A crude way to categorize these two approaches is to call the mathematical shape-finding approach "statics" and the physical approach "dynamics", although there are certainly problems involving static soap films that require a physical description. This is my implication in this paper when I divide the discussion into statics and dynamics.

In the statics section, the thickness of the film, concentration of surfactants, and gravity will all be quickly neglected, as the equilibrium shape of the surface becomes the only matter of importance. I will explain why for a soap film enclosing no volume, the mean curvature must everywhere be equal to zero, and why this is equivalent to the area of the film being minimized. For future reference, a **minimal surface** is defined as a two-dimensional surface with mean curvature equal to zero, so in general soap films take the shape of minimal surfaces. Carrying out the area minimization of a film using the Euler-Lagrange equation will result in a nonlinear partial differential equation known as **Lagrange's Equation**, which all minimal surfaces must satisfy. Finally I will show some examples of minimal surfaces known to mathematicians and how a computer program can generate them.

In the dynamics section, one has to take into account the previously neglected film thickness, surfactant concenctration, and sometimes gravity. I will start with the accepted molecular model for soap films. An important question in soap film dynamics is that of the stability of soap films and bubbles, compared to the completely unstable films of pure liquids (like water). This question will be discussed, and the reason is due to the elasticity of soap films. The thickness profile of a vertical soap film will be analyzed, and a comparison between soap film dynamics and the two dimensional Navier-Stokes equations will be made. At the end, some results from soap film experiments will be shared.

Before the statics and dynamics sections, we start with a motivation section, explaining some applications of the study of soap films to industry and other areas of science. Afterwards, we end by remarking on generalizations to the topics studied here.

2 Motivation

Understanding soap films is useful for a wide variety of applications. It is also relevant to fluid mechanics in that soap films provide one of the closest substitutes for a truly two-dimensional environment, which means experimenters use soap films to test features of the 2D Navier-Stokes equations. The regimes in which soap films actually do follow the 2D Navier-Stokes equations were found analytically by Chomaz [2].

One industrial application of soap films is curtain coating, a kind of coating process in which a film of the coating liquid (the “curtain”) is dropped onto the object to be coated. Another is foam, which one is often trying to minimize in a chemical process, or perhaps trying to generate. As (liquid) foams are just systems of connected films enclosing a gas, an understanding of the dynamics of soap films is crucial for the understanding of the behavior of the foam as a whole. Liquid films are also relevant to atomization and sprays in combustion, and to biology for membranes composed of lipid molecules, which have similarities to soap films.

The more mathematical topic of minimal surfaces also has applications. In materials science, exotic minimal surfaces show up in the molecular structure of the mesoporous material Germanate [4], for example. "Block copolymers," molecules made up of two repelling polymer strands, arrange themselves so the repelling strands interact as little as possible. Because of this, a complex minimal surface known as Sherk's first surface has been observed in a particular block copolymer [5].

3 Statics

3.1 Zero mean curvature

In our discussion of static soap films, we will soon treat the film as an infinitesimally thin two-dimensional surface, but first we allow it to have thickness to justify why a soap film at equilibrium should have zero mean curvature everywhere.

The reason mostly comes down to the Young-Laplace equation,

$$p_{in} - p_{out} = 2\gamma\kappa \tag{3.1}$$

where p_{in} is the pressure inside the liquid, p_{out} is the pressure outside, γ is the surface tension, and κ is the curvature. We assume that the film is not enclosing any volume, so the pressure on both sides of the film will equal the atmospheric pressure. Now suppose there is an area of nonzero mean curvature. If we assume that the film has constant thickness and treat the inside of the film as our inner fluid in the Young-Laplace equation, then one side of the film will have positive curvature, while the other will have negative curvature. But by the Young-Laplace equation, applying it to the positive curvature side will give a $p_{in} > p_{atm}$ and applying it to the negative curvature side will give a $p_{in} < p_{atm}$, a contradiction. Thus, for a constant thickness film, the mean curvature will equal zero everywhere.

Now allow the film to have a varying thickness, with bulges and depressions in the film. We know that the equilibrium shape of a soap film will minimize its energy, and the energy associated with the surface tension of a film is

$$E_S = 2\gamma A \tag{3.2}$$

where A is the area of the film, and the 2 is there because it has two sides. To minimize E_S, then, the film will minimize its area. Hence, a soap film at equilibrium will have a constant thickness, since variations would cost extra area, and it follows that soap films have mean curvature equal to zero at equilibrium.

3.2 Area minimization

It is also possible to start with the energy and area minimizations of soap films and see that the curvature has to equal zero. To do this, we let the surface of a soap film be defined by $z = f(x, y)$, where $f(x, y)$ is the height of the surface above the point (x, y) in the plane. The surface area of this film is given by the integral

$$A = \int_S \sqrt{1 + {f_x}^2 + {f_y}^2}\; dx\; dy \tag{3.3}$$

where S is the region of the plane on which the film is defined, and the value of f is specified on the boundary of S.

To choose the shape $f(x, y)$ that minimizes the area given the boundary conditions, we can apply the Euler-Lagrange equation (for two independent variables):

$$\frac{\partial \mathcal{L}}{\partial f} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial f_x} - \frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial f_y} = 0 \tag{3.4}$$

Here we have

$$\mathcal{L} = \sqrt{1 + {f_x}^2 + {f_y}^2} \tag{3.5}$$

and substituting in $\mathcal{L}$ to the Euler-Lagrange equation, we get

$$-\frac{\partial}{\partial x}\left(\frac{f_x}{\sqrt{1 + {f_x}^2 + {f_y}^2}}\right) - \frac{\partial}{\partial y}\left(\frac{f_y}{\sqrt{1 + {f_x}^2 + {f_y}^2}}\right) = 0 \tag{3.6}$$

It is here that we get the mean curvature equaling zero, since the left hand side of the above equation is the curvature of the function f. This can be shown by using that

$$2\kappa = \nabla \cdot \underline{n} \tag{3.7}$$

and

$$\underline{n} = \frac{\nabla(z - f(x,y))}{|\nabla(z - f(x,y))|} = \frac{\underline{e}_z - f_x \underline{e}_x - f_y \underline{e}_y}{\sqrt{1 + {f_x}^2 + {f_y}^2}} \tag{3.8}$$

Taking the divergence of this expression for $\underline{n}$ then gives the left hand side of 3.6. So from area minimization alone, we end up requiring that the surface has everywhere a curvature of zero.

We can also go further in simplifying 3.6.

$$\begin{aligned} 0 &= \frac{-f_{xx} - f_{yy}}{\sqrt{1 + {f_x}^2 + {f_y}^2}} + \frac{f_x(f_x f_{xx} + f_y f_{xy}) + f_y(f_y f_{yy} + f_x f_{xy})}{(1 + {f_x}^2 + {f_y}^2)^{3/2}} \\ &= -\frac{(1 + {f_x}^2) f_{yy} - 2\ f_x\ f_y\ f_{xy} + (1 + {f_y}^2) f_{xx}}{(1 + {f_x}^2 + {f_y}^2)^{3/2}} = 2\kappa \end{aligned}$$

Since this expression must equal zero, the numerator equals zero, and written in this form, we have what is known as **Lagrange's equation**,

$$(1 + {f_x}^2) f_{yy} - 2\ f_x\ f_y\ f_{xy} + (1 + {f_y}^2) f_{xx} = 0 \tag{3.9}$$

All minimal surfaces must satisfy this nonlinear partial differential equation, although most minimal surfaces cannot be parametrized completely by $(x, y, f(x, y))$. For example, the catenoid, shown in Figure 1, is usually represented mathematically by the parametric equations

$$\begin{aligned} x &= c\cosh(\frac{v}{c})\cos u \\ y &= c\cosh(\frac{v}{c})\sin u \\ z &= v \end{aligned}$$

and any parametrization of the form $(x, y, f(x, y))$ will have one value of f for each x and y, so you can only get half of the catenoid in this form. But solving for such a z by inverting these equations does indeed give an $f(x, y)$ which satisfies Lagrange's equation.

3.3 Some examples of minimal surfaces

To conclude the section on statics, we will show some examples of minimal surfaces discovered by mathematicians. *Complete* minimal surfaces (minimal surfaces without

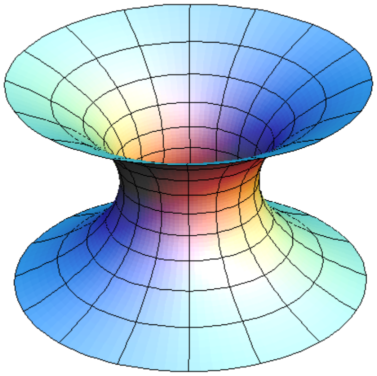

Figure 1: A catenoid, one of the first nontrivial minimal surfaces discovered, together with the helicoid.

boundaries) typically received the most attention. For complete minimal surfaces, the plane is of course the trivial example, and the catenoid (Figure 1) and helicoid (Figure 2) were the first nontrivial examples, discovered by Meusnier in 1776 [8]. The next complete minimal surface, the Costa surface (Figure 3), was not found until 1984 [8]. Since then, many more have been discovered.

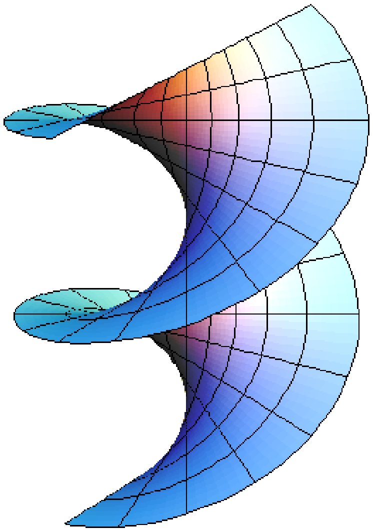

Figure 2: A helicoid, one of the first nontrivial minimal surfaces discovered, together with the catenoid.

As for minimal surfaces with specified boundaries, the minimal surface with a skew quadrilateral as its boundary was found by Schwarz in 1890 [8]. (A skew quadrilateral is a

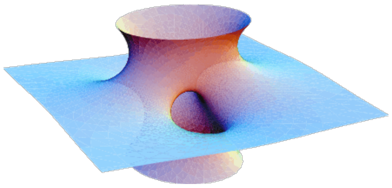

Figure 3: The Costa minimal surface.

quadrilateral not contained in a plane, but with the lines connecting midpoints of opposite sides intersecting each other.) They can also be solved for numerically. By starting with some guess that satisfies the boundary conditions, points with positive curvature can be pushed in and points with negative curvature can be pushed out. Repeating this process iteratively, the equilibrium shape can be found to a desired accuracy. This is basically what Brakke's Surface Evolver package [1] does (although it can accomodate many more complications like volume constraints), minimizing surface area by applying appropriate forces to vertices in the surface mesh iteratively. The surface shown in Figure 4 is an example of the kind of surface you can generate using this program.

As an interesting side note, this Surface Evolver package was used to disprove Kelvin's Conjecture that a foam of bitruncated cubic honeycomb was the most efficient foam (in the sense of minimizing surface area). The Weaire-Phelan structure beats it by 0.3% [7], and this was calculated by the Surface Evolver. The Surface Evolver had to be used because even though the structures involved are approximately polyhedra, the faces and edges will have slight curvature in the equilbrium shape to conform to Plateau's rules on soap films. So the Surface Evolver starts with the polyhedron approximation and relaxes it to end up with the equilibrium and most efficient shape.

Of course, perhaps the most pleasing way to find minimal surfaces is to physically make them using soap solution. By dipping wire frames of various shapes into a soap solution, many different minimal surfaces can be physically realized as a soap film, as the helicoid is in Figure 5.

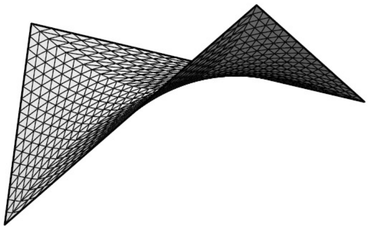

Figure 4: A minimal surface produced using Brakke's Surface Evolver. The boundary is a skew quadrilateral.

Figure 5: A helicoid made by dipping a wire frame into soap solution.

4 Dynamics

4.1 Soap film model

To understand the dynamics of soap films, we need a model that includes the properties we neglected earlier. Now the film will have some finite thickness depending on time and position, with an interface defined by $z = \frac{1}{2}H(x, y, t)$ (for a horizontal film; later we will have a vertical film with thickness $H(z)$). We will just define one side of the soap film and assume symmetry (so the thickness will be H).

Soap molecules are typically made of a hydrophilic polar head and a hydrophobic hydrocarbon tail. This encourages the molecules to move to the surface so the hydrophobic end can get away from the water, and makes them surfactants. Soaps often contain many types of surfactants in impure water, but we will consider a solution of just one type of molecule in pure water. We will define the initial concentration of soap in the liquid c_0.

Once a film is formed, the concentration of soap molecules in the bulk and on the surface of the film will be denoted as c and Γ, respectively. Soap films are often depicted as in Figure 6.

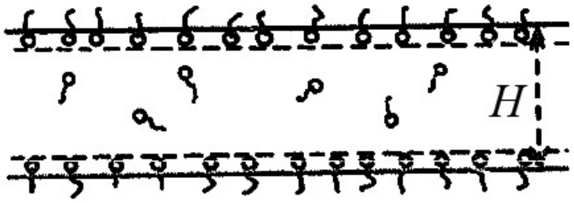

Figure 6: Cross section of a model soap film. (From [3].)

The relationship between Γ and c depends on how large the concentration is, being linear for small concentrations, and flattening out at a value of c called the critical micelles concentration, where Γ is essentially saturated. For ease of calculation, we will stick to the case of small concentrations, in which we have

$$\Gamma = kc \tag{4.1}$$

where k is typically of the order 4 μm. As the mean volume per molecule in the bulk is typically 3×10^4 nm^3, applying 4.1 gives a mean area per soap molecule of about 7 nm^2 at the surface, and we see that the concentration at the surface is much greater than in the bulk.

4.2 Surface tension

The importance of surfactant concentrations in soap film dynamics is their effect on surface tension. For the small concentrations that we will focus on, surface tension will decrease linearly with Γ. In general, we write

$$\gamma = \gamma_0 - f \tag{4.2}$$

where γ_0 is the surface tension of pure water and f is the spreading pressure of the surfactant. For small concentrations, f follows a two-dimensional perfect gas law,

$$f = RT\Gamma \tag{4.3}$$

4.3 Stability of soap films

The stability of soap films is due to their elasticity. If a particular area of a film is stretched, the surfactant concentration Γ decreases there, which causes the surface tension γ to increase. A larger surface tension stops the stretching from continuing, allowing the soap film to reach a new equilibrium. This changing surface tension in soap films is what allows them to last so much longer than pure liquid films, which have no such method of stabilization. The Marangoni effect is a name commonly given to this process, although in the literature [3] it is broken down into two types of elasticities depending on the time scale of the stretching.

For short time scales, in which soap molecules do not have time to diffuse from the bulk to the surface, a stretching will simply reduce Γ without affecting c. This is because the surface is made larger, but the bulk is just made thinner. The resulting increase in surface tension that opposes the stretching is known as the Marangoni elasticity, and it allows films to remain stable after a quick stretching.

For time scales long enough that diffusion maintains the equilibrium between Γ and c, a stretching of the surface will reduce Γ, but c will also decrease as some of the molecules in the bulk travel to the surface. This increase in surface tension is called the Gibbs elasticity, and the decrease in Γ will not be as large as for the Marangoni elasticity.

Figure 7 shows the two different responses of a soap film, and the paths AB and AC in Figure 8 identify the behavior of Γ and c for the two responses.

The transition from Marangoni to Gibbs elasticity should occur at $\tau_D = H^2/D$, where D is the diffusion constant. $D = 4 \times 10^{-6}$ cm^2/s, so for a film 1 μm thick, $\tau_D \sim 0.01$ sec, but according to [6], impurities make equilibrium tougher to reach so that Marangoni elasticity is observed up to times of order a second.

4.4 Quantitative description of elasticity

The Gibbs elasticity is given by the elasticity modulus of a film, which is

$$E = 2\frac{d\gamma}{d\ln A} \tag{4.4}$$

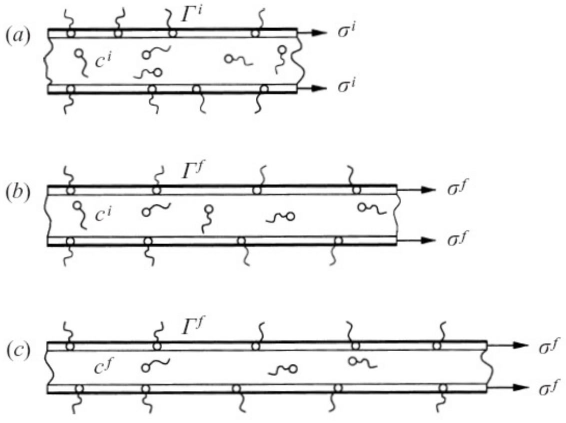

Figure 7: Evolution of a membrane under an increase of the applied tension: (a) initial state, (b) instantaneous response (Marangoni elasticity), (c) long-time response (Gibbs elasticity). (From [2].)

Due to the incompressibility of the fluid, the Gibbs elasticity can be written as

$$E_G = 2\frac{d\gamma}{d\ln A} = 2A\frac{d\gamma}{dA} = -2H\frac{d\gamma}{dH} \tag{4.5}$$

For small concentrations, the Marangoni elasticity is defined as [3]

$$E_M = 2f = 2RT\Gamma \tag{4.6}$$

We can get an expression for Γ in terms of H if we assume that the global concentration of soap is everywhere equal to c_0, so that

$$c_0 = c + 2\Gamma/H \tag{4.7}$$

Combining this with 4.1, we arrive at

$$\Gamma = c_0\frac{Hk}{H+2k} \tag{4.8}$$

This yields a Gibbs elasticity

$$E_G = 4RTc_0\frac{Hk^2}{(H+2k)^2} = \frac{2E_M k}{H+2k} \tag{4.9}$$

or in dimensionless form, the ratio of Gibbs elasticity to Marangoni elasticity is

$$\frac{E_G}{E_M} = \frac{1}{1+H/2k} \tag{4.10}$$

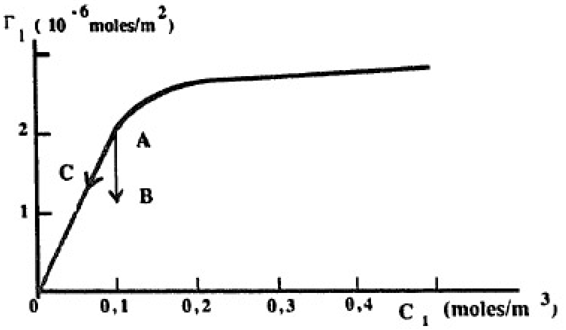

Figure 8: The surface concentration Γ as a function of bulk concentration c. AB and AC represent Marangoni and Gibbs elasticities responding to a local stretching, respectively. (From [6, 3].)

Thus, if the thickness gets much larger than k, the Gibbs elasticity will become much smaller than the Marangoni elasticity, but in the limit of thin films ($H \ll 2k$), they approximately equal each other, because the bulk fluid has so few soap molecules to move to the surface. It is important to remember that this is still only for small concentrations. For a sense of order of magnitude, a 2 μm thick film of a dilute solution of Sodium Dodecyl Sulfate (S.D.S.) has a Gibbs elasticity $E_G \sim 30 \times 10^{-3}$ N/m [6]. Using our typical k of 4 μm, this gives a corresponding Marangoni elasticity of $E_M \sim 38 \times 10^{-3}$ N/m.

4.5 Vertical soap films

A rectangular soap film placed vertically initially exhibits turbulent motion, but after some time it stops, showing horizontal interference fringes that imply a wedge-like thickness profile. This thickness profile can be solved for, although the film is still slowly thinning.

We ignore the effects of the frame, which are actually significant to the overall dynamics due to the negative pressure of the meniscus by the frame. Then we can write the balance between the surface tension gradient and shear stress at the surface as

$$\frac{d\gamma}{dz} = \mu \left(\frac{\partial w}{\partial x} \right)_{x=H/2} \tag{4.11}$$

where $x = 0$ corresponds to the center of the film, $x = H/2$ is the surface of the film, and w is the component of the velocity in the z direction.

Since the film surfaces are stationary, we treat the flow in the film as a viscous flow between fixed parallel interfaces, for which the velocity profile is

$$w(\hat{x}) = \frac{H^2}{2\mu}\frac{\partial p}{\partial z}\left[\left(\frac{\hat{x}}{H}\right)^2 - \left(\frac{\hat{x}}{H}\right)\right] = \frac{H^2}{2\mu}\rho g\left[\left(\frac{\hat{x}}{H}\right)^2 - \left(\frac{\hat{x}}{H}\right)\right] \tag{4.12}$$

where $\hat{x} = x - H/2$ to make the equation cleaner.

Then

$$\left(\frac{\partial w}{\partial x}\right)_{x=H/2} = \frac{H\rho g}{2\mu} \tag{4.13}$$

so that 4.11 becomes

$$\frac{d\gamma}{dz} = \frac{\rho g H(z)}{2} \tag{4.14}$$

Writing this in the form

$$2\frac{df}{dz} = -\left(\rho H(z)\right)g \tag{4.15}$$

shows the connection with static equilibrium in a classic fluid, where $(\rho H(z))$ is the effective density (two-dimensional) and $2\frac{df}{dz}$ is the gradient of an effective hydrostatic pressure (also two-dimensional). Since $H(z)$ will vary under the effect of gravity, we have a varying effective density, and the film behaves like a compressible fluid.

Solving for the thickness profile, we will consider a film with thickness $\bar{H}$ and surface tension $\bar{\gamma}$ at a reference height $z = 0$. For simplicity, we suppose our surfactant is insoluble, so that $\Gamma \neq 0$ but $c = 0$.

By 4.3, we can rewrite 4.15 as

$$2RT\frac{d\Gamma}{dz} = -\rho g H(z) \tag{4.16}$$

Since $c = 0$, by 4.7,

$$\Gamma = \frac{c_0 H}{2} = \frac{\bar{H}}{H}\bar{\Gamma} \tag{4.17}$$

Then we have

$$\frac{dH}{dz} = -\frac{\rho g\bar{H}}{2RT\bar{\Gamma}}H = -\frac{\rho g\bar{H}}{2(\gamma_0 - \bar{\gamma})}H \tag{4.18}$$

We nondimensionalize H and z as

$$\hat{H} = \frac{H}{\bar{H}} \tag{4.19}$$

$$\hat{z} = \frac{z}{L} \tag{4.20}$$

where $L = \frac{2(\gamma_0 - \bar{\gamma})}{\rho g \bar{H}}$.

We then get the simple relation

$$\hat{H}(\hat{z}) = \frac{1}{2} e^{-\hat{z}} \tag{4.21}$$

This is plotted in Figure 9. To get a sense of how quickly the thickness changes, taking $\bar{H} \sim 2\mu$m and $\gamma_0 - \gamma \sim 30 \times 10^{-3}$ N/m, I get $L \sim 3$ m, which means the top of the film in Figure 9 would have to be 9 m high. (This does not agree with [3], so I probably made a mistake somewhere, because it should vary more quickly.)

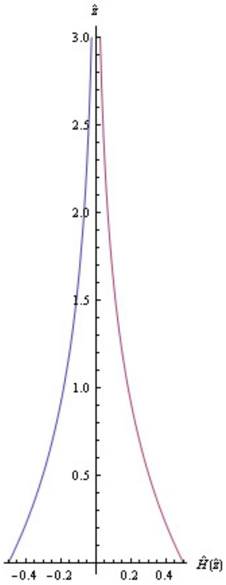

Figure 9: The thickness profile of a vertical film with insoluble soap, in dimensionless form.

It is important to note here that the thickness profile is a good approximation instantaneously, but $\bar{H}$ and $\bar{\gamma}$ will change over time as liquid drains out of the film onto the boundary, and this will change the shape over time.

4.6 Two-dimensional hydrodynamics

An interesting question in soap film dynamics is, how well do soap films comply with the two-dimensional Navier-Stokes equations? As soap films provide a natural venue for visualizing two-dimensional flows, with variations in thickness giving rise to interference fringes, it is important to know whether experiments are uncovering behavior of general two-dimensional flows or behavior specific to soap films alone. This is the focus of [2], and the mathematics is rather complicated, so I will not go into it here, but I will quote the very interesting results.

Before doing so, I just need to define the elastic Mach number $M_e = u/v_e$, where v_e is the Marangoni elastic wave velocity ($v_e \sim 10$ m/s for a film 1 μm thick) and u is the velocity in the film at a point.

The results of [2] are then that "the equations describing leading-order soap films are two-dimensional but do not correspond to any classical two-dimensional dynamics except in two limit cases:"

(1) When the elastic Mach number is small and initial variations in film thickness and total soap concentration are small, the soap film does obey the two-dimensional incompressible Navier-Stokes equations.

(2) When the elastic Mach number is of order one, the Reynolds number is large, and the solubility is neglected, the soap film dynamics at leading order obey the compressible Euler equation for a two-dimensional gas. A strange feature of this is that the effective ratio of specific heats, γ, equals 1 for these dynamics.

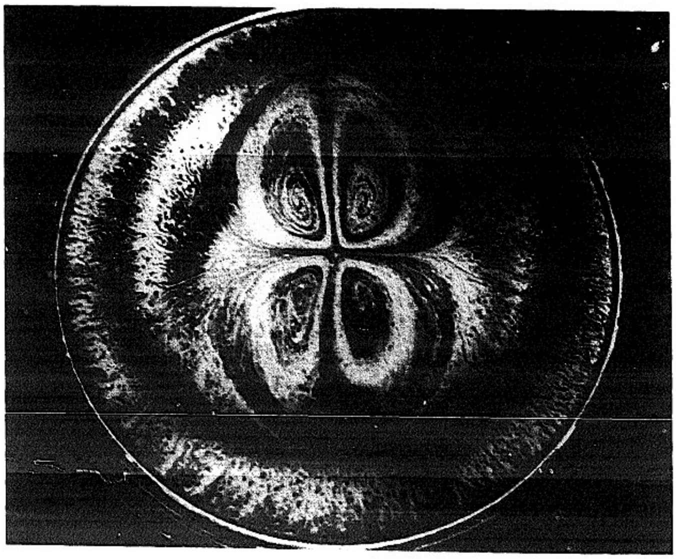

Figure 10: The circulations observed in a horizontal film when an air jet is aimed perpendicularly to its surface. (From [3]).

4.7 Experimental results

A large number of experiments have been performed on soap films, and here I just summarize two from [3] to give a flavor for them.

In the first, they examined soap films set into motion by air flow. The soap film was circular and oriented horizontally, with a narrow jet of air aimed at the center. The center of the film is stretched slightly by the jet, and four regular steady circulations result, as shown by Figure 10.

The stretching at the center increases the surface tension there drawing fluid inward, but the deflected air from the jet drives fluid radially outward. The instability in these two opposing forces is believed to be the cause for the circulations observed.

In the second experiment, the two dimensional evolution of a turbulent wake was observed by towing a disk of aluminum paper suspended in a soap film. As Figure 11 shows, the two modes of a Von Karman wake were observed, but there were also deviations from strict two-dimensional behavior. A general damping was observed and assumed to be due to air friction.

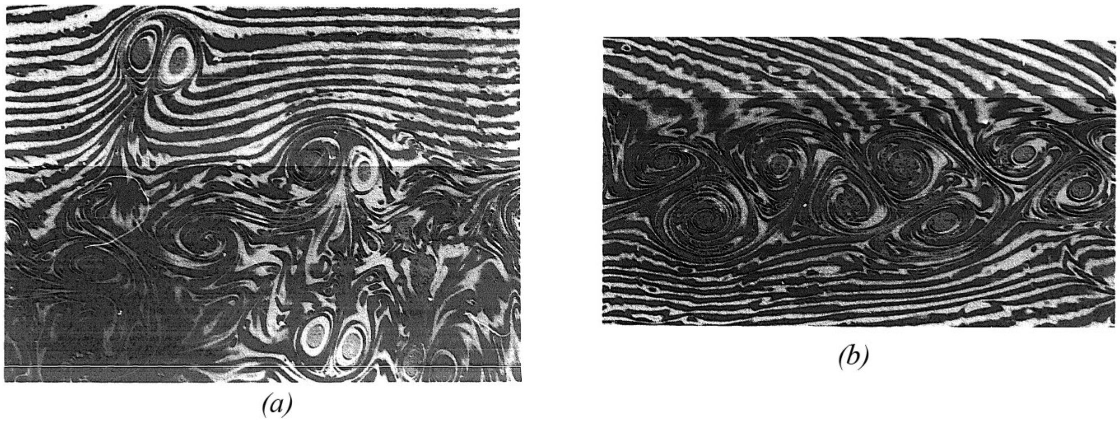

Figure 11: The two modes of destabilization of a Von Karman wake. (From [3]).

5 Generalizations

With a topic like soap films: statics and dynamics, it's hard to get more general. The actual features of each that I examined, however, are small and particular parts of the subjects.

For the minimal surfaces, I only considered surfaces that can be parametrized as $(x, y, f(x, y))$, and most minimal surfaces are really described by $(x(u, v), y(u, v), z(u, v))$ for parameters u and v. Learning how to manipulate surfaces in this kind of framework would be the next step for studying minimal surfaces further, and this would take me into the world of differential geometry.

For the dynamics sections, a quantitative description of the soap film for high concentrations would be the next step here, along with the vertical film thickness profile for soluble surfactants (which is given in [3]). Also, I wanted to come up with a model problem to be able to understand how the Marangoni and Gibbs elasticities really function, but I wasn't able to. Trying to work out the response of a soap bubble to some initial perturbations in thickness would be interesting, but very challenging.

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