A Corrigendum to “Buying Supermajorities in Finite Legislatures”

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Abstract

Banks (2000) investigates the optimal coalition size in the finite-voter version of the Groseclose and Snyder vote-buying model. This note points out that the proof of the main result (Proposition 1) needs some corrections. I show that Proposition 1 holds with suitable modification of the proof.

Jeffrey S. Banks (2000) analyzes the finite-voter version of the Groseclose and Snyder vote-buying model and characterize the optimal coalition size. This note points out that the proof of the main result (Proposition 1) needs some corrections. For the rest of the paper, we follow notation and assumptions of Banks (2000).

A crucial step in the original proof of Proposition 1 is to show that Party A bribes the first $k(a)$ voters.

Claim 1 *For $a \in \mathbb{R}_+^n$, we can assume $C(a) = \{1, \ldots, k(a)\}$ in the optimal coalition.*

The original proof of this argument goes as follows.

> For any $a \in \mathbb{R}_+^n$, let $k(a) = |C(a)|$, and suppose $a \in U^l(\mathbf{v}, W_B)$ is such that $v_i \geq v_j$ and $j \in C(a)$ but $i \notin C(a)$; that is, i is at least as favorable to x as is j, but j is bribed and i is not. Then, under A2, there exists $a' \in U^l(\mathbf{v}, W_B)$ with $S(a') \leq S(a)$, $k(a') = k(a)$, and $i \in C(a')$, but $j \notin C(a')$ by simply swapping the roles of i and j: $a'_i = t(a) - v_i$, $a'_j = 0$, and for all $m \notin \{i, j\}$, $a'_m = a_m$.

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However, there exists a counterexample for the proof. Firstly, we state the correct claims in Banks (2000) useful for this paper.

Claim 2 *Without loss of generality, we can assume* $a_i + v_i = t(a)$ *for all* $i \in C(a)$.

Claim 3 *Under assumption A2, A has to bribe no less than* $\frac{n+1}{2}$ *voters.*

Claim 4 *Under assumption A2,* $t(a) \geq v_1$.

Then, we construct the counterexample for **Claim 1**.

Example 5 *Assume* $n = 3$, $W_B = 2$, $v_1 = 1 - \varepsilon$, $v_2 = v_3 = -\varepsilon$ *with* $\varepsilon \in (0, 1)$.

Note that this satisfies A1 and A2. We consider the case with $k(a) = 3$. We concentrate on the case where $|C(a)| = 2$. Since v_2, v_3 are symmetric, it suffices to consider the following cases;

- $C(a) = \{1, 2\}$

  For B, since voter 3 votes for y without bribe, it is necessary and sufficient to pay $t(a)$ to one of $\{1, 2\}$ so that B wins. Therefore, to prevent B from participating, it is necessary and sufficient to have $t(a) \geq W_B$, which means $t(a) = 2$ and $S(a) = 2 \times 2 - (1 - \varepsilon - \varepsilon) = 3 + 2\varepsilon$.

- $C(a) = \{2, 3\}$

  For B, since no voter votes for y without bribe, it is necessary and sufficient to pay $t(a)$ to voter 1 and one of $\{2, 3\}$ so that B wins[1]. Therefore, to prevent B from participating, it is necessary and sufficient to have $1 - \varepsilon + t(a) \geq W_B$. Therefore, $t(a) \geq W_B - (1 - \varepsilon)$, which means $t(a) = 1 + \varepsilon$ and $S(a) = (1 + \varepsilon) \times 2 - (-\varepsilon - \varepsilon) = 2 + 4\varepsilon$.

Therefore, for sufficiently small $\varepsilon > 0$, for $a = \{0, 1 + 2\varepsilon, 1 + 2\varepsilon\}$, there is no a' with $C(a') = \{1, 2\}$, $a' \in U^l(\mathbf{v}, W_B)$, and $S(a') \leq S(a)$.

[1] Note that we verify below that voter 1 is "cheaper" than voters 2 and 3 for Party B since $1 - \varepsilon \leq t(a)$.

The intuition is as follows. Start from A's bribe schedule such that voter 1 is excluded. Since B has to bribe voter 1, if voter 1 is sufficiently costly to bribe, to prevent B from participating, A has to pay small amount of money to voters 2 and 3. Suppose A tries to change the bribe schedule such that voter 3 is excluded. Then, B does not need to bribe voter 3 since $v_3 \leq 0$. Therefore, to prevent B from participating, A has to pay a lot of money to voters 1 and 2. Under some parameter values, the former can be cheaper for A.

However, the above situation happens only if the initial bribe schedule is not cost minimizing, that is, we can still show that in the cost minimizing coalition, $v_j \in C(a)$ and $v_i > v_j$ imply $v_i \in C(a)$.

Lemma 6 *In the cost minimizing coalition, $v_j \in C(a)$ and $v_i > v_j$ imply $v_i \in C(a)$.*

Proof. Suppose $i \notin C(a)$ and $j \in C(a)$ with $v_i \geq v_j$ for the cost minimizing coalition.

Suppose $0 \geq v_i > v_j$. Then, consider a' with switching the role of i and j (as in the original proof). B does not pay for i and has to pay $t(a)$ to buy j's vote under a. B does not pay for j and has to pay $t(a)$ to buy i's vote under a'. Since everything is symmetric, $a \in U^l(\mathbf{v}, W_B)$ implies $a' \in U^l(\mathbf{v}, W_B)$. Since $S(a') < S(a)$, a is not optimal.

Suppose $v_i > 0 \geq v_j$. Then, let $v^* = \max_{m \notin C(a)} v_m > 0$ and $i^* \in \arg\max_{m \notin C(a)} v_m$. Further, $k = \#\{m : m \notin C(a)\}$. From **Claim 3**,

$$k \leq n - \frac{n+1}{2} = \frac{n-1}{2}.$$

$\{m : m \notin C(a) \wedge v_m \leq 0\}$ vote for B without bribe. From **Claim 4**, $\{m : m \notin C(a) \wedge v_m > 0\}$ is cheaper than $m \in C(a)$ for B to buy. Therefore, B firstly bribes $\{m : m \notin C(a) \wedge v_m > 0\}$. Since $\#\{m : m \notin C(a) \wedge v_m \leq 0\} + \#\{m : m \notin C(a) \wedge v_m > 0\} = k$, what remains for B to pay is $t(a)\left(\frac{n+1}{2} - k\right)$. In total, the cost for B to buy supermajorities is

$$t(a)\left(\frac{n+1}{2} - k\right) + \sum_{\substack{m \notin C(a) \\ v_m > 0}} v_m.$$

It is necessary and sufficient to have

$$t(a)\left(\frac{n+1}{2}-k\right)+\sum_{\substack{m\notin C(a)\\ v_m>0}} v_m \geq W_B$$

to refrain B from participating, which means

$$t(a)=\frac{W_B-\sum_{\substack{m\notin C(a)\\ v_m>0}} v_m}{\frac{n+1}{2}-k}.$$

The total payment by A is

$$S(a)=\frac{W_B-\sum_{\substack{m\notin C(a)\\ v_m>0}} v_m}{\frac{n+1}{2}-k}(n-k)-\sum_{m\in C(a)} v_m$$

since $n-k=|C(a)|$.

Consider the following a': bribing $i^* \cup C(a)$. Then, what B has to pay is

$$t(a')\left(\frac{n+1}{2}-(k-1)\right)+\sum_{\substack{m\notin C(a)\\ v_m>0}} v_m - v^*$$

since B now has to bribe someone in $C(a)$ instead of v^*. It is necessary and sufficient to have

$$t(a')\left(\frac{n+1}{2}-(k-1)\right)+\sum_{\substack{m\notin C(a)\\ v_m>0}} v_m - v^* \geq W_B$$

to refrain B from participating, which means

$$t(a')=\frac{W_B-\sum_{\substack{m\notin C(a)\\ v_m>0}} v_m + v^*}{\frac{n+1}{2}-(k-1)}$$

and

$$S(a')=\frac{W_B-\sum_{\substack{m\notin C(a)\\ v_m>0}} v_m + v^*}{\frac{n+1}{2}-(k-1)}(n-(k-1))-\sum_{m\in C(a)} v_m - v^*.$$

What remains to show is $S(a) - S(a') > 0$.

$$S(a) - S(a') = \left(\frac{n-k}{\frac{n+1}{2}-k} - \frac{n-(k-1)}{\frac{n+1}{2}-(k-1)}\right)\left(W_B - \sum_{\substack{m \notin C(a) \\ v_m > 0}} v_m\right) + \left(1 - \frac{n-(k-1)}{\frac{n+1}{2}-(k-1)}\right)v^*.$$

Note that, by definition, $\sum_{\substack{m \notin C(a) \\ v_m > 0}} v_m \leq l v^*$,where $l = \{m : m \notin C(a) \wedge v_m > 0\}$. In addition, A2 implies $\frac{n+1}{2}v^* < W_B$. Since $\frac{n-k}{\frac{n+1}{2}-k} - \frac{n-(k-1)}{\frac{n+1}{2}-(k-1)} > 0$, $l \leq k$, and $2k \leq n-1$, we have

$$\begin{aligned} S(a) - S(a') &> \left(\frac{n-k}{\frac{n+1}{2}-k} - \frac{n-(k-1)}{\frac{n+1}{2}-(k-1)}\right)\left(\frac{n+1}{2}v^* - l v^*\right) + \left(1 - \frac{n-(k-1)}{\frac{n+1}{2}-(k-1)}\right)v^* \\ &= 2(n-1)\frac{k-l}{(2k-(n+3))(2k-(n+1))}v^* \geq 0. \end{aligned}$$

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