SUPPLEMENTAL MATERIAL 5:

PROOF OF THEOREM 1 for a General N-Player Game withOUT CHEAP TALK

In this Supplemental Material, we prove the dispensability of cheap talk and public randomization in the proof of Theorem 1 for a general N-player game with $N \geq 3$ (see the Supplemental Material 3 for the proof with cheap talk and public randomization). Remember that in the Supplemental Material 3, the coordination block uses the perfect cheap talk, the supplemental rounds use the noisy cheap talk, the report block uses the public randomization and perfect cheap talk, and the re-report block uses the perfect cheap talk.

First, in Section 46, we replace the perfect cheap talk in the coordination block with the noisy cheap talk. As seen in Section 4.7.2, with more than two players, we need to make sure that while the players exchange messages and infer the other players' messages from private signals in order to coordinate on x_i, there is no player who can induce a situation where some players infer x_i is G while the others infer x_i is B in order to increase her own equilibrium payoff. For this purpose, we need to use the communication through actions and to make new assumptions. In Section 45.1, we introduce these new assumptions and explain why they are necessary.

Second, in Section 48, we dispense with the noisy cheap talk in the coordination block (given the first step above) and supplemental rounds. See Section 45.2 for what assumption is necessary for this step.

Third, in Section 51, we dispense with the public randomization and the perfect cheap talk in the report and re-report blocks. See 45.3 for new assumptions for this step.

In this Supplemental Material, when we say player $i \notin \{1, ..., N\}$, without otherwise specified, it means player $i \pmod N$. In addition, without loss of generality, assume that

$$|A_1| |Y_1| \geq \cdots \geq |A_N| |Y_N| . \tag{152}$$

45 Notations and Assumptions

45.1 Assumptions for Dispensing with the Perfect Cheap Talk in Coordination Block

We explain how to replace the perfect cheap talk with the noisy cheap talk in the coordination block. As explained in Section 29, the noisy cheap talk is "private" in that when player j sends the message to player n via noisy cheap talk, the main signal $f\left[n\right]\left(m\right)$ is only observed by player n.

This creates the second problem in Section 4.6.3: If player i sent the message x_i to each of the other players $-i$ via noisy cheap talk separately, then player i could create a situation where some players infer x_i is G while the others infer x_i is B by telling a lie. Since the action that will be taken in the main blocks may not be included in $\left\{a(x)\right\}_x$ and we do not have any bound on player i's payoff in such a situation, it might be of player i's interest to tell a lie.

To prevent this situation, we consider the following message protocol: Let $N(i) = \{i, i+1, i+2\}$ be the set of players whose index is in $\{i, i+1, i+2\}$. In addition, let

$$n^*(i) \in \arg\min_{j \in \{i, i+2\}} |A_j| \, |Y_j| \tag{153}$$

be the player whose $|A_j| \, |Y_j|$ is smaller among $\{i, i+2\}$. Let

$$n^{**}(i) = \{i, i+2\} \setminus \{n^*(i)\} \tag{154}$$

be the other player. Note that $N(i) = \{n^*(i), i+1, n^{**}(i)\}$. The players communicate as follows:

1. First, player i sends the message about x_i to player $n^*(i)$.
2. Then, player $n^*(i)$ sends the message about x_i to players $N(i)$ via actions. This corresponds to "Phase 1" of Hörner and Olszewski (2006).

3. After that, each player j in $N(i)$ sends the message about x_i to each player $n \neq j$ via noisy cheap talk.

4. Finally, each player n infers x_i based on the messages from $N(i)$. This corresponds to "Phase 2" of Hörner and Olszewski (2006).

As Hörner and Olszewski (2006), to incentive each player $j \in N(i)$ to tell the truth in Step 3, for each $j \in N(i)$, if there exists player $n \in -j$ such that player n's inference of player j's message changes player n's inference of x_i in Step 4 (that is, if player j is "pivotal"), then player $j-1$ makes player j indifferent between any action profile sequence.

Given above, we will show that player $n^*(i)$ does not want to deviate in Step 2 in order to create a situation where player $n^*(i)$ herself will be pivotal with high probability in Step 3. Remember that we take $n^*(i)$ such that the set of player $n^*(i)$'s action-signal pairs is smaller than that of player $n^{**}(i)$ in (153). Heuristically, this guarantees that player $n^*(i)$ cannot infer player $n^{**}(i)$'s inference precisely, which prevents player $n^*(i)$ from creating the situation where player $n^*(i)$ is pivotal.

Given player $n^*(i)$'s truthtelling strategy in Step 2, the probability that player i is pivotal in Step 3 is almost independent of player i's strategy in Step 1. Since x_i controls player $(i+1)$'s payoff, players i and $n^*(i) \neq i+1$ do not have an incentive to manipulate the communication in Step 1.

Below, we explain which step requires exactly what assumption.

Let us consider Step 1 first. Suppose that player i wants to send the message $x_i \in \{G, B\}$ to player $n^*(i)$. If $n^*(i) = i$, then this is redundant. Otherwise, player i sends x_i by taking $a_i^{x_i}$ for $T^{\frac{1}{2}}$ periods. The other players are supposed to take a_{-i}^G. We want to make sure that player $n^*(i)$ can statistically infer player i's message regardless of deviations by the other players $-(i, n^*(i))$.

More generally, for each $i \in I$ and $n \in -i$, we want to construct a statistics $\psi_n^i(y_n)$ with which player n can infer player i's binary message regardless of the other players' deviation.

That is,

$$\mathbb{E}\left[\psi_n^i(y_n) \mid a_i, a_j, a_{-(i,j)}^G\right] = \begin{cases} q_2 & \text{if } a_i = a_i^G, \\ q_1 & \text{if } a_i = a_i^B \end{cases} \tag{155}$$

for all $j \in -(i,n)$ and $a_j \in A_j$.

A sufficient condition is as follows: Let $Q_n^i(a_i, a_j, a_{-(i,j)}^G) \equiv \left(q(y_n \mid a_i, a_j, a_{-(i,j)}^G)\right)_{y_n}$ be the vector expression of player n's signal distribution conditional on $a_i, a_j, a_{-(i,j)}^G$. It suffices to assume that all the vectors $Q_n^i(a_i, a_j, a_{-(i,j)}^G)$ with $j \in -(i,n)$, $a_i \in \{a_i^G, a_i^B\}$ and $a_j \in A_j$ are linearly independent.

Assumption 13 *For any $i \in I$ and $n \in -i$, there exist $\left\{a_i^G, a_i^B\right\} \subset A_i$ and $a_{-i}^G \in A_{-i}$ such that $Q_n^i(a_i, a_j, a_{-(i,j)}^G)$ with $j \in -(i,n)$, $a_i \in \left\{a_i^G, a_i^B\right\}$ and $a_j \in A_j$ are linearly independent.*

For notational convenience, we assume that a_i^G that is used for player i to send the message is the same as a_i^G that is player i's action in a_{-j}^G when player $j \in -i$ sends the message.

This assumption is generic since Assumption 2 implies that $|Y_n| \geq 2\sum_{j \neq i,n} |A_j|$. The following lemma shows that this assumption is sufficient for the existence of ψ_n^i.

Lemma 33 *If Assumption 13 holds, then for each $i \in I$ and $n \in -i$, there exist $q_2 > q_1$ and $\psi_n^i : Y_n \to (0,1)$ satisfying (155).*

Proof. The same as Lemma 3. ■

See (168) for how player $n^*(i)$ infers x_i using $\psi_n^i(y_n)$.

After player $n^*(i)$ infers x_i, player $n^*(i)$ sends the message about her inference of x_i to players $N(i) = \{n^*(i), i+1, n^{**}(i)\}$. To distinguish player $n^*(i)$'s inference of x_i from the true state x_i, let $w_i \in \{G, B\}$ denote player $n^*(i)$'s inference of x_i.

While player $n^*(i)$ sends w_i, player $n^*(i)$ takes $a_{n^*(i)}^{w_i}$, player $i+1$ takes $\alpha_{i+1}^* \in \Delta(A_{i+1})$, player $n^{**}(i)$ takes $\alpha_{n^{**}(i)}^* \in \Delta\left(A_{n^{**}(i)}\right)$, and each player $j \notin N(i)$ takes a_j^G for $T^{\frac{1}{2}}$ periods. That is, in equilibrium, the players take

$$\alpha(i, w_i) \equiv \left(a_{n^*(i)}^{w_i}, \alpha_{i+1}^*, \alpha_{n^{**}(i)}^*, \left\{a_j^G\right\}_{j \notin N(i)}\right)$$

for $T^{\frac{1}{2}}$ periods.

Take $n \in N(i) \setminus n^*(i)$. Suppose that player $j = N(i) \setminus \{n^*(i), n\}$ unilaterally deviates and takes $a_j \in A_j$. Then, the distribution of player n's action-signal pairs is

$$\mathbf{q}_n(a_j, \alpha_{-j}(i, w_i)) \equiv (q(a_n, y_n \mid a_j, \alpha_{-j}(i, w_i)))_{a_n \in A_n, y_n \in Y_n}.$$

Consider the following linear equations: For any $a_j \in A_j$,

$$\boldsymbol{i}_n(i)\mathbf{q}_n(a_j, \alpha_{-j}(i, w_i)) = \begin{cases} q_2 & \text{if } w_i = G, \\ q_1 & \text{if } w_i = B. \end{cases} \tag{156}$$

Here, $\boldsymbol{i}_n(i)$ is a $1 \times |A_n| |Y_n|$ vector. Intuitively, if player n uses $\boldsymbol{i}_n(i)\mathbf{1}_{a_{n,t}, y_{n,t}}$ after the history $(a_{n,t}, y_{n,t})$ to infer w_i, then player j cannot manipulate player n's inference.

Solve (156) for $\boldsymbol{i}_n(i)$. Suppose that there are $L_n(i)$ linearly independent solutions. Then, let

$$I_n(i) = \left(\boldsymbol{i}_n^l(i)\right)_{l=1}^{L_n(i)} \tag{157}$$

be the $L_n(i) \times |A_n| |Y_n|$ matrix collecting all the linearly independent $\boldsymbol{i}_n(i)$'s. Suppose that player n infers w_i is equal to $\hat{w}_i \in \{G, B\}$ if the realized frequency $\mathbf{x}$ of action-signal pairs satisfies

$$I_n(i)\mathbf{x} + \boldsymbol{\varepsilon} = q(\hat{w}_i)\mathbf{1}$$

for some $\|\boldsymbol{\varepsilon}\| \leq \varepsilon$ (imagine that ε is a small number). Here,

$$q(\hat{w}_i) = \begin{cases} q_2 & \text{if } \hat{w}_i = G, \\ q_1 & \text{if } \hat{w}_i = B. \end{cases}$$

We will take care of the case where there is no such $\hat{w}_i \in \{G, B\}$ later. Note that (156) implies that player $j = N(i) \setminus \{n^*(i), n\}$ cannot manipulate this inference.

In addition, consider the matrix projecting player $n^*(i)$'s history on the conditional expectation of player n's history given an action profile by players $-n^*(i)$ being equal to

$\alpha_{-n^*(i)}(i, w_i)$:

$$\underset{\left(|A_n||Y_n| \times \left|A_{n^*(i)}\right|\left|Y_{n^*(i)}\right|\right)}{Q_{n,n^*(i)}(i)},$$

where the element corresponding to (a_n, y_n) , $\left(a_{n^*(i)}, y_{n^*(i)}\right)$ is the conditional probability that player n observes (a_n, y_n) given $\left(a_{n^*(i)}, y_{n^*(i)}\right)$ and $\alpha_{-n^*(i)}(i, w_i)$:

$$q(a_n, y_n | \alpha_{-n^*(i)}(i, w_i), a_{n^*(i)}, y_{n^*(i)}).$$

Since $\alpha_{-n^*(i)}(i, w_i) = \left(\alpha^*_{i+1}, \alpha^*_{n^{**}(i)}, \left\{a_j^G\right\}_{j \notin N(i)}\right)$ is independent of w_i, $Q_{n,n^*(i)}$ is independent of w_i.

Given $Q_{n,n^*(i)}(i)$, the set of player $n^*(i)$'s histories such that player $n^*(i)$ believes that player n infers $\hat{w}_i \in \{G, B\}$ with a non-negligible probability is expressed by

$$\mathcal{I}_{n,n^*(i)}[\varepsilon](i, \hat{w}_i) \equiv \left\{ \begin{array}{c} \mathbf{x} \in \mathbb{R}_+^{\left|A_{n^*(i)}\right|\left|Y_{n^*(i)}\right|} : \exists \boldsymbol{\varepsilon} \in \mathbb{R}^{L_n(i)} \text{ such that} \\ \left\{ \begin{array}{c} \|\boldsymbol{\varepsilon}\| \leq \varepsilon, \\ I_n(i) Q_{n,n^*(i)}(i) \mathbf{x} = q(\hat{w}_i) \mathbf{1} + \boldsymbol{\varepsilon}. \end{array} \right. \end{array} \right\}$$

So that player $n^*(i)$ cannot induce the situation that players $n^{**}(i)$ and $i+1$ infer the different states, we want to make sure that, for sufficiently small ε,

$$\mathcal{I}_{n^{**}(i),n^*(i)}[\varepsilon](i, G) \cap \mathcal{I}_{i+1,n^*(i)}[\varepsilon](i, B) = \emptyset \tag{158}$$

and

$$\mathcal{I}_{n^{**}(i),n^*(i)}[\varepsilon](i, B) \cap \mathcal{I}_{i+1,n^*(i)}[\varepsilon](i, G) = \emptyset. \tag{159}$$

Therefore, we give a sufficient condition for (158) and (159).

In addition, we want to incentives each player $i' \in I$ to take a prescribed action by the reward function $\pi_{i'}^{x_{i'-1}}(n^*(i) \to N(i), a_{i'-1}, y_{i'-1})$ such that

- If player i' is player $n^*(i)$, then the ex ante payoff of player i' is constant for all $a_{i'} \in A_{i'}$:

$$\begin{aligned} & u_{i'}\left(a_{i'}, \alpha^*_{i+1}, \alpha^*_{n^{**}(i)}, a^G_{-N(i)}\right) + \mathbb{E}\left[\begin{array}{c} \pi_{i'}^{x_{i'-1}}(n^*(i) \to N(i), a_{i'-1}, y_{i'-1}) \\ \mid a_{i'}, \alpha^*_{i+1}, \alpha^*_{n^{**}(i)}, a^G_{-N(i)} \end{array}\right] \\ = & \text{ constant.} \end{aligned} \tag{160}$$

  A sufficient condition for this is that all the vectors of player $(i'-1)$'s signal distribution given $a_{i'}, \alpha^*_{i+1}, \alpha^*_{n^{**}(i)}, a^G_{-N(i)}$ are linearly independent with respect to $a_{i'}$. That is,

$$\left(q_{i'-1}(y_{i'-1} \mid a_{i'}, \alpha^*_{i+1}, \alpha^*_{n^{**}(i)}, a^G_{-N(i)})\right)_{y_{i'-1}}$$

  is linearly independent with respect to $a_{i'} \in A_{i'}$.

- If player i' is not player $n^*(i)$, then the ex ante payoff of player i' is constant for all $a_{i'} \in A_{i'}$ and player $n^*(i)$'s possible messages:

$$\begin{aligned} & u_{i'}\left(a_{i'}, \alpha_{-i'}(i,G)\right) + \mathbb{E}\left[\pi_{i'}^{x_{i'-1}}(n^*(i) \to N(i), a_{i'-1}, y_{i'-1}) \mid a_{i'}, \alpha_{-i'}(i,G)\right] \\ = & \; u_i\left(a_{i'}, \alpha_{-i'}(i,B)\right) + \mathbb{E}\left[\pi_{i'}^{x_{i'-1}}(n^*(i) \to N(i), a_{i'-1}, y_{i'-1}) \mid a_{i'}, \alpha_{-i'}(i,B)\right] \\ = & \text{ constant.} \end{aligned} \tag{161}$$

  A sufficient condition for this is that all the vectors of player $(i'-1)$'s signal distribution given $a_{i'}, \alpha_{-i'}(i, w_i)$ are linearly independent with respect to $a_{i'}$ and w_i. That is,

$$\left(q_{i'-1}(y_{i'-1} \mid a_{i'}, \alpha_{-i'}(i, w_i))\right)_{y_{i'-1}}$$

  is linearly independent with respect to $a_{i'} \in A_{i'}$ and $w_i \in \{G, B\}$.

In total, the following assumption is sufficient.

Assumption 14 *For any $i \in I$, there exist $\left\{a^G_{n^*(i)}, a^B_{n^*(i)}\right\}$, $\alpha^*_{i+1} \in \Delta(A_{i+1})$, $\alpha^*_{n^{**}(i)} \in \Delta\left(A_{n^{**}(i)}\right)$, $a^G_{-N(i)}$, q_2 and q_1 such that*

1. *$q_2, q_1 \in (0,1)$ and $q_2 > q_1$.*

2. *There exists $\mathbf{x} \in \mathbb{R}^{L_{i+1}(i)+L_{n^{**}(i)}(i)}$ such that*

$$\left[\begin{array}{c} I_{i+1}(i)Q_{i+1,n^*(i)}(i) \\ I_{n^{**}(i)}(i)Q_{n^{**}(i),n^*(i)}(i) \end{array}\right]' \mathbf{x} \leq \mathbf{0}, \left[\begin{array}{c} q_2\mathbf{1} \\ q_1\mathbf{1} \end{array}\right] \cdot \mathbf{x} > 0.$$

3. *There exists $\mathbf{x} \in \mathbb{R}^{L_{i+1}(i)+L_{n^{**}(i)}(i)}$ such that*

$$\left[\begin{array}{c} I_{i+1}(i)Q_{i+1,n^*(i)}(i) \\ I_{n^{**}(i)}(i)Q_{n^{**}(i),n^*(i)}(i) \end{array}\right]' \mathbf{x} \leq \mathbf{0}, \left[\begin{array}{c} q_1\mathbf{1} \\ q_2\mathbf{1} \end{array}\right] \cdot \mathbf{x} > 0.$$

4. *For $i' = n^*(i)$,*

$$\left(q_{i'-1}(y_{i'-1} \mid a_{i'}, \alpha^*_{i+1}, \alpha^*_{n^{**}(i)}, a^G_{-N(i)})\right)_{y_{i'-1}}$$

   is linearly independent with respect to $a_{i'} \in A_{i'}$.

5. *For $i' \in I \setminus \{n^*(i)\}$,*

$$\left(q_{i'-1}(y_{i'-1} \mid a_{i'}, \alpha_{-i'}(i, w_i))\right)_{y_{i'-1}}$$

   is linearly independent with respect to $a_{i'} \in A_{i'}$ and $w_i \in \{G, B\}$.

Since all the expressions are linear and $\mathbf{q}_n$ is a probability distribution, we can make sure that each element in $I_n(i)$ is in $(0,1)$. Further, for notational simplicity, we assume that $\left(a_j^G, a_j^B\right)_{j\in I}$ in Assumption 13 satisfies Assumption 14 for each i.[93]

This assumption is generic by the following reason: (156) puts $2(|A_j| - 1)$ constraints while we have $|A_n||Y_n| - 1$ degrees of freedom for $\boldsymbol{i}_n(i)$ if $\mathbf{q}_n(a_j, \alpha_{-j}(i, w_i))$ is linearly independent for each w_i and a_j except for the constraint that "if we add all the elements up, then it should be one." Hence, generically $L_n(i)$ is equal to $|A_n||Y_n| - 2|A_j| + 1$. Therefore, for each one of Conditions 2 and 3, we have $|A_{i+1}||Y_{i+1}| + \left|A_{n^{**}(i)}\right|\left|Y_{n^{**}(i)}\right| - 2|A_{i+1}| - 2\left|A_{n^{**}(i)}\right| + 1$

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[93] Remember that in Assumption 13, we assumed that a_i^G that is used for player i to send the message is the same as a_i^G that is player i's action in a_{-j}^G when player $j \in -i$ sends the message.

degrees of freedom for $\mathbf{x}$,[94] while we have $\left|A_{n^*(i)}\right| \left|Y_{n^*(i)}\right| + 1$ constraints. Hence, Assumption 2 together with (153) implies that we can generically find $\mathbf{x}$ for Conditions 2 and 3.

In addition, Condition 4 is generic if $|Y_{i'-1}| \geq |A_{i'}|$ and Condition 5 is generic if $|Y_{i'-1}| \geq 2\,|A_{i'}|$. Note that Assumption 2 implies that these inequalities are satisfied.

The following lemma shows that Assumption 14 is sufficient for (158) and (159).

Lemma 34 *If Assumption 14 is satisfied, then there exists* $\bar{\varepsilon} > 0$ *such that, for any* $\varepsilon < \bar{\varepsilon}$, *for any* $i \in I$, *(158) and (159) are satisfied.*

Proof. The same as Lemma 26. ■

In addition, the following lemma shows that Assumption 14 is sufficient for the construction of the reward stated above:

Lemma 35 *There exists* $\bar{u}$ *such that, for each* i *and* i', *there exist* $\pi_{i'}^{G}(n^*(i) \to N(i), \cdot, \cdot) : A_{i'-1} \times Y_{i'-1} \to [-\bar{u}, 0]$ *and* $\pi_{i'}^{B}(n^*(i) \to N(i), \cdot, \cdot) : A_{i'-1} \times Y_{i'-1} \to [0, \bar{u}]$ *such that*

1. *(160) is satisfied for* $i' = n^*(i)$ *and*
2. *(161) is satisfied for* $i' \in -n^*(i)$.

Proof. The same as Lemma 3. ■

45.2 Assumption for Dispensing with the Noisy Cheap Talk

We explain how player j sends a binary message $m \in \{G, B\}$ to player n via actions instead of the noisy cheap talk. Since we only use the noisy cheap talk with precision $p = \frac{1}{2}$, we concentrate on the case with $p = \frac{1}{2}$.

As in the two-player case, with η being a small number to be defined, the sender (player j) determines

$$z_j(m) = \begin{cases} m & \text{with probability } 1 - 2\eta, \\ \{G, B\} \setminus \{m\} & \text{with probability } \eta, \\ M & \text{with probability } \eta \end{cases}$$

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[94] Note that two rows are parallel to $\mathbf{1}$.

and player j takes

$$\alpha_j^{z_j(m)} = \begin{cases} a_j^G & \text{if } z_j(m) = G, \\ a_j^B & \text{if } z_j(m) = B, \\ \frac{1}{2}a_j^G + \frac{1}{2}a_j^B & \text{if } z_j(m) = M \end{cases}$$

for $T^{\frac{1}{2}}$ periods. The other players $-j$ take a_{-j}^G.

For each $i \in I$, let $\mathbf{y}_i$ be the realized frequency of player i's signal observation while player j sends m. In addition, let $\mathbf{q}_i(a) = (q_i(y_i \mid a))_{y_i}$ be player i's signal distribution with action profile a.

We want to construct $f[n](m) \in \{G, B\}$ from $\mathbf{y}_n$ and $g[n-1](m) \in \{m, E\}$ from $\mathbf{y}_{n-1}$ such that

- Player n infers the message correctly with high probability,
- Player $n-1$ has $g[n-1](m) = m$ with high probability,
- Given m, player n believes that $f[n](m) = m$ or $g[n-1](m) = E$ with high probability,
- Player n cannot manipulate $g[n-1](m)$, and
- The players other then the sender and receiver cannot manipulate $f[n](m)$ to increase their payoff.

As in the two player case, $g[n-1](m) = E$ if and only if $z_j(m) \neq m$ or $\mathbf{y}_{n-1}$ is not close to the affine hull of player $(n-1)$'s signal distribution with respect to player n's deviation, $\text{aff}(\{\mathbf{q}_{n-1}(a_j^m, a_n, a_{-(j,n)}^G)\}_{a_n})$. Using 2 of Notation 2 below,

1. $g[n-1](m) = m$ if $z_j(m) = m$ and $\mathbf{y}_{n-1} \in \mathcal{H}_{n-1}[\varepsilon](m)$.
2. $g[n-1](m) = E$ if $z_j(m) \neq m$ or $\mathbf{y}_{n-1} \notin \mathcal{H}_{n-1}[\varepsilon](m)$.

Here, we assume that player $n-1$ knew the true message m. As will be seen in Section 52, player j informs player $n-1$ of m in the re-report block. Since $g[n-1](m)$ only affects the reward function (does not affect $\sigma_{n-1}(x_{n-1})$), it suffices that player $n-1$ knows the information by the end of the review phase.

On the other hand, regardless of any player's deviation, with high probability, player n (receiver) receives $\mathbf{y}_n$ close to the affine hull of player n's signal distributions with respect to player i's action with $i \in -i$, that is,

$$\mathrm{aff}(\{\mathbf{q}_n(a_n^G, a_j, a_{-(j,n)}^G)\}_{a_j \in A_j}) \cup \mathrm{aff}(\{\mathbf{q}_n(a_n^m, a_j^G, a_i, a_{-(i,j,n)}^G)\}_{m \in \{G,B\}, i \neq j,n, a_i \in A_i}). \tag{162}$$

Using 4 of Notation 2, $\mathbf{y}_n \in \mathcal{G}_n[\varepsilon]$ with high probability.

If $\mathbf{y}_n \in \mathcal{G}_n[\varepsilon]$, then as in the two-player case, player n constructs $f[n](m)$ such that

- $f[n](m) = G$ if the conditional expectation of $\mathbf{y}_{n-1}$ given $m = G$ and $\mathbf{y}_n$ is close to $\mathcal{H}_{n-1}[\varepsilon]\,(G)$, and
- $f[n](m) = B$ if the conditional expectation of $\mathbf{y}_{n-1}$ given $m = B$ and $\mathbf{y}_n$ is close to $\mathcal{H}_{n-1}[\varepsilon]\,(B)$.

Using 6 of Notation 2,

- $f[n](m) = G$ if $\mathbf{y}_n \in \mathcal{H}_{n-1,n}[\varepsilon]\,(G)$, and
- $f[n](m) = B$ if $\mathbf{y}_n \in \mathcal{H}_{n-1,n}[\varepsilon]\,(B)$.

Further, so that players $-(j,n)$ cannot manipulate player n's inference (if $z_j(m) = m$), player n infers that m is $\hat{m} \in \{G, B\}$ if $\mathbf{y}_n$ is close to the affine full of player n's signal distributions under the message $\hat{m}$ with respect to a unilateral deviation of each player $i \in -(j,n)$, that is, if $\mathbf{y}_n$ is close to $\mathrm{aff}(\{\mathbf{q}_n(a_j^{\hat{m}}, a_i, a_{-(i,j)}^G)\}_{i \neq j,n, a_i \in A_i})$.

Using 8 of Notation 2,

- $f[n](m) = G$ if $\mathbf{y}_n \in \mathcal{J}_n[\varepsilon]\,(G)$, and
- $f[n](m) = B$ if $\mathbf{y}_n \in \mathcal{J}_n[\varepsilon]\,(B)$.

In total,

1. If $\mathbf{y}_n \in \mathcal{G}_n[\varepsilon]$, then

   (a) $f[n](m) = G$ if $\mathbf{y}_i \in \mathcal{H}_{n-1,n}[\varepsilon]\,(G) \cup \mathcal{J}_n[\varepsilon]\,(G)$,

   (b) $f[n](m) = B$ if $\mathbf{y}_i \in \mathcal{H}_{n-1,n}[\varepsilon]\,(B) \cup \mathcal{J}_n[\varepsilon]\,(B)$ or $\mathbf{y}_i \notin \mathcal{H}_{n-1,n}[\varepsilon]\,(G) \cup \mathcal{J}_n[\varepsilon]\,(G)$, and

2. If $\mathbf{y}_n \notin \mathcal{G}_n[\varepsilon]$, then player n infers $f[n](m)$ from the likelihood as in the two-player case.

Here, compared to the two-player case, $\mathcal{I}_i[\varepsilon](\hat{m})$ is not introduced since Lemma 15 does not have a counterpart of 3 of Lemma 2.

In addition, we want to incentives each player $i \in I$ to take a prescribed action by the reward function $\pi_i^{x_{i-1}}(j, a_{i-1}, y_{i-1})$ such that

- If player i is player j (sender), then the ex ante payoff of player i is constant for all $a_i \in A_i$:
  $$u_i\left(a_i, a_{-i}^G\right) + \mathbb{E}\left[\pi_i^{x_{i-1}}(j, a_{i-1}, y_{i-1}) \mid a_i, a_{-i}^G\right] = \text{constant}. \tag{163}$$
  A sufficient condition for this is that all the vectors of player $(i-1)$'s signal distribution given a_i, a_{-i}^G are linearly independent with respect to a_i. That is,
  $$\left(q_{i-1}(y_{i-1} \mid a_i, a_{-i}^G)\right)_{y_{i-1}}$$
  is linearly independent with respect to $a_i \in A_i$.

- If player i is not player j, then the ex ante payoff of player i is constant for all $a_i \in A_i$ regardless of player j's message:
  $$\begin{aligned} & u_i\left(a_i, a_j^G, a_{-(i,j)}^G\right) + \mathbb{E}\left[\pi_i^{x_{i-1}}(j, a_{i-1}, y_{i-1}) \mid a_i, a_j^G, a_{-(i,j)}^G\right] \\ = \; & u_i\left(a_i, a_j^B, a_{-(i,j)}^G\right) + \mathbb{E}\left[\pi_i^{x_{i-1}}(j, a_{i-1}, y_{i-1}) \mid a_i, a_j^B, a_{-(i,j)}^G\right]. \end{aligned} \tag{164}$$

A sufficient condition for this is that all the vectors of player $(i-1)$'s signal distribution given $a_i, a_j^m, a_{-(i,j)}^G$ are linearly independent with respect to a_i and m. That is,

$$\left(q_{i-1}(y_{i-1} \mid a_i, a_j^m, a_{-(i,j)}^G)\right)_{y_{i-1}}$$

is linearly independent with respect to $a_i \in A_i$ and $m \in \{G, B\}$.

We first give notations and then give a sufficient condition so that the above inference is well defined and that the reward function exists.

Notation 2 *For $a_j^G, a_j^B \in A_j$ and $a_{-j}^G \in A_{-j}$, for $m \in \{G, B\}$, we define the following:*

1. *A $(|Y_{n-1}| - |A_n| + 1) \times |Y_{n-1}|$ matrix $H_{n-1}(m)$ and a $(|Y_{n-1}| - |A_n| + 1) \times 1$ vector $\mathbf{p}_{n-1}(m)$ such that the affine hull of player $(n-1)$'s signal distributions with respect to player n's action when the other players take $a_j^m, a_{-(j,n)}^G$ is represented by*

$$\begin{aligned} & \operatorname{aff}(\{\mathbf{q}_{n-1}(a_j^m, a_n, a_{-(j,n)}^G)\}_{a_n \in A_n}) \cap \mathbb{R}_+^{|Y_{n-1}|} \\ = & \left\{\mathbf{y}_{n-1} \in \mathbb{R}_+^{|Y_{n-1}|} : H_{n-1}(m)\mathbf{y}_{n-1} = \mathbf{p}_{n-1}(m)\right\}. \end{aligned}$$

2. *The set of hyperplanes that are generated by perturbing RHS of the characterization of $\operatorname{aff}(\{\mathbf{q}_{n-1}(a_j^m, a_n, a_{-(j,n)}^G)\}_{a_n \in A_n}) \cap \mathbb{R}_+^{|Y_{n-1}|}$: For $\varepsilon \geq 0$,*

$$\mathcal{H}_{n-1}[\varepsilon](m) = \left\{ \begin{array}{c} \mathbf{y}_{n-1} \in \mathbb{R}_+^{|Y_{n-1}|} : \exists \boldsymbol{\varepsilon} \in \mathbb{R}^{|Y_{n-1}| - |A_n| + 1} \text{ such that} \\ \left\{ \begin{array}{c} \|\boldsymbol{\varepsilon}\| \leq \varepsilon, \\ H_{n-1}(m)\mathbf{y}_{n-1} = \mathbf{p}_{n-1}(m) + \boldsymbol{\varepsilon} \end{array} \right. \end{array} \right\}.$$

3. *Let G_i be a $(|Y_n| - |A_j| - 2\sum_{i \neq j,n} |A_i| + 1) \times |Y_n|$ matrix and $\mathbf{g}_n$ be a $(|Y_n| - |A_j| - 2\sum_{i \neq j,n} |A_i| + 1) \times 1$ vector such that (162) is represented by*

$$\left\{\mathbf{y}_n \in \mathbb{R}_+^{|Y_n|} : G_n\mathbf{y}_n = \mathbf{g}_n\right\}.$$

4. *The set of hyperplanes that are generated by perturbing RHS of the above characterization: For* $\varepsilon \geq 0$*,*

$$\mathcal{G}_i[\varepsilon] \equiv \left\{ \begin{array}{c} \mathbf{y}_n \in \mathbb{R}_+^{|Y_n|} : \exists \boldsymbol{\varepsilon} \in \mathbb{R}^{|Y_n|-|A_j|-2\sum_{i\neq j,n}|A_i|+1} \\ \text{such that} \left\{ \begin{array}{c} \|\boldsymbol{\varepsilon}\| \leq \varepsilon \\ G_n \mathbf{y}_n = \mathbf{g}_n + \boldsymbol{\varepsilon} \end{array} \right. \end{array} \right\}.$$

5. *The matrix projecting the distributions of player* n*'s signals on the conditional distribution of player* $(n-1)$*'s signals given an action profile* a*:*

$$Q_{n-1,n}(a) = \left[\begin{array}{ccc} q(y_{n-1,1} \mid a, y_{n,1}) & \cdots & q(y_{n-1,1} \mid a, y_{n,|Y_n|}) \\ \vdots & & \vdots \\ q(y_{n-1,|Y_{n-1}|} \mid a, y_{n,1}) & \cdots & q(y_{n-1,|Y_{n-1}|} \mid a, y_{n,|Y_n|}) \end{array} \right].$$

6. *For* $\hat{m} \in \{G, B\}$*, the set of player* n*'s signal frequencies such that player* n*'s conditional expectation of player* $(n-1)$*'s signal frequency is in* $\mathcal{H}_{n-1}[\varepsilon](\hat{m})$ *when the players take* $a_j^{\hat{m}}, a_{-j}^G$*:*

$$\mathcal{H}_{n-1,n}[\varepsilon](\hat{m}) = \left\{ \begin{array}{c} \mathbf{y}_n \in \mathbb{R}_+^{|Y_{n-1}|} \text{ such that} \\ \text{there exist } \boldsymbol{\varepsilon}_1 \in \mathbb{R}^{|Y_{n-1}|},\ \boldsymbol{\varepsilon}_2 \in \mathbb{R}^{|Y_{n-1}|-|A_n|+1} \text{ and } \mathbf{y}_{n-1} \in \mathbb{R}_+^{|Y_{n-1}|} \text{ satisfying} \\ \left\{ \begin{array}{c} \mathbf{y}_{n-1} = Q_{n-1,n}(a_j^{\hat{m}}, a_{-j}^G)\mathbf{y}_n + \boldsymbol{\varepsilon}_1, \\ H_{n-1}(\hat{m})\mathbf{y}_{n-1} = \mathbf{p}_{n-1}(\hat{m}) + \boldsymbol{\varepsilon}_2, \\ \|\boldsymbol{\varepsilon}_1\|, \|\boldsymbol{\varepsilon}_2\| \leq \varepsilon \end{array} \right. \end{array} \right\}.$$

7. *For* $\hat{m} \in \{G, B\}$*, a* $(|Y_n| - \sum_{i\neq j,n} |A_i| + 1) \times |Y_n|)$ *matrix* $J_n(\hat{m})$ *and a* $(|Y_n| - \sum_{i\neq j,n} |A_i| + 1) \times 1$ *vector* $\mathbf{r}_n(\hat{m})$ *such that the affine hull of player* n*'s signal distributions with respect to player* i*'s deviation with* $i \in -(j,n)$ *when the other players take* $a_j^{m}, a_{-(i,j)}^G$

is represented by

$$\begin{aligned} & \operatorname{aff}(\{\mathbf{q}_n(a_j^{\hat{m}}, a_i, a_{-(i,j)}^G)\}_{i\neq j,n,a_i\in A_i}) \cap \mathbb{R}_+^{|Y_n|} \\ = \; & \left\{\mathbf{y}_n \in \mathbb{R}_+^{|Y_n|} : J_n(\hat{m})\mathbf{y}_n = \mathbf{r}_n(\hat{m})\right\}. \end{aligned}$$

8. *The set of hyperplanes that are generated by perturbing RHS of the above characterization: For $\varepsilon \geq 0$,*

$$\mathcal{J}_n[\varepsilon](m) = \left\{ \begin{array}{c} \mathbf{y}_n \in \mathbb{R}_+^{|Y_n|} : \exists \boldsymbol{\varepsilon} \in \mathbb{R}^{|Y_n| - \sum_{i\neq j,n}|A_i|+1} \text{ such that} \\ \left\{ \begin{array}{c} \|\boldsymbol{\varepsilon}\| \leq \varepsilon, \\ J_n(\hat{m})\mathbf{y}_n = \mathbf{r}_n(\hat{m}) + \boldsymbol{\varepsilon} \end{array} \right. \end{array} \right\}.$$

Similar to Lemma 25, we can take $H_{n-1}(G)$, $H_{n-1}(B)$, $J_n(G)$ and $J_n(B)$ so that all the elements of all the matrices are in $(0, 1)$.

Assumption 15 *For each $j \in I$ and $n \in -j$, there exist $a_j^G, a_j^B \in A_j$ and a_{-j}^G such that the following seven conditions are satisfied:*

1. *There exists $\mathbf{x} \in \mathbb{R}^{|Y_n|-|A_j|-2\sum_{i\neq j,n}|A_i|+1+2(|Y_{n-1}|-|A_n|+1)}$ such that*

$$\begin{bmatrix} G_n \\ H_{n-1}(G)Q_{n-1,n}(a_j^G, a_{-j}^G) \\ H_{n-1}(B)Q_{n-1,n}(a_j^B, a_{-j}^G) \end{bmatrix}' \mathbf{x} \leq \mathbf{0}, \begin{bmatrix} \mathbf{g}_n \\ \mathbf{p}_{n-1}(G) \\ \mathbf{p}_{n-1}(B) \end{bmatrix} \cdot \mathbf{x} > 0.$$

2. *There exists $\mathbf{x} \in \mathbb{R}^{|Y_n|-|A_j|-2\sum_{i\neq j,n}|A_i|+1+2\left(|Y_n|-\sum_{i\neq j,n}|A_i|+1\right)}$ such that*

$$\begin{bmatrix} G_n \\ J_n(G) \\ J_n(B) \end{bmatrix}' \mathbf{x} \leq \mathbf{0}, \begin{bmatrix} \mathbf{g}_n \\ \mathbf{r}_n(G) \\ \mathbf{r}_n(B) \end{bmatrix} \cdot \mathbf{x} > 0.$$

3. *There exists* $\mathbf{x} \in \mathbb{R}^{|Y_n|-|A_j|-2\sum_{i\neq j,n}|A_i|+1+(|Y_{n-1}|-|A_n|+1)+|Y_n|-\sum_{i\neq j,n}|A_i|+1}$ *such that*

$$\begin{bmatrix} G_n \\ H_{n-1}(G)Q_{n-1,n}(a_j^G, a_{-j}^G) \\ J_n(B) \end{bmatrix}' \mathbf{x} \leq \mathbf{0}, \begin{bmatrix} \mathbf{g}_n \\ \mathbf{p}_{n-1}(G) \\ \mathbf{r}_n(B) \end{bmatrix} \cdot \mathbf{x} > 0.$$

4. *There exists* $\mathbf{x} \in \mathbb{R}^{|Y_n|-|A_j|-2\sum_{i\neq j,n}|A_i|+1+(|Y_{n-1}|-|A_n|+1)+|Y_n|-\sum_{i\neq j,n}|A_i|+1}$ *such that*

$$\begin{bmatrix} G_n \\ H_{n-1}(B)Q_{n-1,n}(a_j^B, a_{-j}^G) \\ J_n(G) \end{bmatrix}' \mathbf{x} \leq \mathbf{0}, \begin{bmatrix} \mathbf{g}_n \\ \mathbf{p}_{n-1}(B) \\ \mathbf{r}_n(G) \end{bmatrix} \cdot \mathbf{x} > 0.$$

5. *For each* $k \in \{1, \ldots, |Y_n|\}$, *we have*

$$q(y_{n,k}|a_j^G, \alpha_{-j}^G) \neq q(y_{n,k}|a_j^G, \alpha_{-j}^B).$$

6. *For* $i = j$,

$$\left(q_{i-1}(y_{i-1} \mid a_i, a_{-i}^G)\right)_{y_{i-1}}$$

is linearly independent with respect to $a_i \in A_i$.

7. *For* $i \in -j$,

$$\left(q_{i-1}(y_{i-1} \mid a_i, a_j^m, a_{-(i,j)}^G)\right)_{y_{i-1}}$$

is linearly independent with respect to $a_i \in A_i$ *and* $m \in \{G, B\}$.

For notational simplicity, we assume that $\left(a_j^G, a_j^B\right)_{j \in I}$ in Assumption 13 satisfies Assumption 15 for each j.[95]

As Assumption 13, we can show that Assumption 2 implies that we can generically find $\mathbf{x}$'s for each condition of Assumption 15 and that Conditions 6 and 7 are satisfied.

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[95] Remember that in Assumption 13, we assumed that a_i^G that is used for player i to send the message is the same as a_i^G that is player i's action in a_{-j}^G when player $j \in -i$ sends the message.

The next two lemmas show that Assumption 15 is actually sufficient so that the above inference $f[n](m)$ is well defined.

Lemma 36 *If Assumption 15 is satisfied, then there exists* $\bar{\varepsilon} > 0$ *such that for all* $\varepsilon < \bar{\varepsilon}$*, for each* $j \in I$ *and* $n \in -j$*, for any* $\mathbf{y}_n \in \Delta\left(\{\mathbf{1}_{y_n}\}_{y_n \in Y_n}\right)$*, at most one* $\hat{m} \in \{G, B\}$ *satisfies* $\mathbf{y}_n \in \mathcal{G}_n[\varepsilon] \cap (\mathcal{H}_{n-1,n}[\varepsilon](\hat{m}) \cup \mathcal{J}_n[\varepsilon](\hat{m}))$*.*

Proof. The same as in Lemma 26. ■

Lemma 37 *For each* $m \in \{G, B\}$*,* $j \in I$ *and* $n \in -j$*, if Assumption 15 is satisfied, then there exists a mapping from* $\mathbf{y}_n \in \Delta\left(\{\mathbf{1}_{y_n}\}_{y_n \in Y_n}\right)$ *to* $f[n](m) \in \{G, B\}$ *such that, for any* m *and* $\mathbf{y}_n$*, given* m*, player* n *puts a belief no less than* $1 - \exp(-O(T^{\frac{1}{2}}))$ *on the events that* $f[n](m) = m$ *or* $g[n-1](m) = E$*.*

Proof. The same as in Lemma 27. ■

We also provide the lemma to show that Assumption 15 is sufficient to construct the reward:

Lemma 38 *There exists* $\bar{u}$ *such that, for each* $j \in I$ *and* $i \in I$*, there exist* $\pi_i^G(j, \cdot, \cdot) : A_{i-1} \times Y_{i-1} \to [-\bar{u}, 0]$ *and* $\pi_i^B(j, \cdot, \cdot) : A_{i-1} \times Y_{i-1} \to [0, \bar{u}]$ *such that*

1. *(163) is satisfied for* $i = j$ *and*
2. *(164) is satisfied for* $i \in -j$*.*

If ε defined in (118) does not satisfy $\varepsilon < \bar{\varepsilon}$ in Lemmas 34 and 36, then re-take ε such that ε is smaller than $\bar{\varepsilon}$. This does not affect the consistency among the variables defined in Section 34.

45.3 Assumptions for Dispensing with the Public Randomization and Perfect Cheap Talk

First, to dispense with the public randomization, we need an assumption comparable to Assumption 11 in the two-player case. For each $i \in I$, with player j replaced with player

$i-1$ (the controller of player i's payoff), all the definitions about a^G, $Y^i_{i-1,1}$, $Y^i_{i-1,2}$, (132), (133), $Y^i_{i,1}$ and $Y^i_{i,2}$ in Section 38.2 are valid with more than two players.

Now, we formally state the more-than-two-player analogue of Assumption 11:

Assumption 16 *For each $i \in I$, there exists $a^G \in A$ such that there exist $Y^i_{i-1,1}$, $Y^i_{i-1,2}$, $\bar{p}_i$, $Y^i_{i,1}$ and $Y^i_{i,2}$ such that $Y^i_{i,1}$ and $Y^i_{i,2}$ satisfy*

1. *(132) and (133) with j replaced with $i-1$, and*
2. $$Y^i_{i,1} \neq \emptyset, Y^i_{i,2} \neq \emptyset, Y_i = Y^i_{i,1} \cup Y^i_{i,2}, Y_{i-1} = Y^i_{i-1,1} \cup Y^i_{i-1,2}.$$

For notational convenience, we assume that a^G is the same for each player and the same as in Assumption 13.[96]

Second, when player i with $i \geq 2$ sends the message, player $i-1$ wants to construct a statistics $\phi_{i-1}(a_{i-1}, y_{i-1})$ such that player $i-1$ can infer player i's message statistically and that the conditional independence property holds for player i, as $\phi_j(a_j, y_j)$ in Lemma 28: For some $a_i^G \in A_i$, $\bar{\alpha}_{i-1} \in \Delta(A_{i-1})$, $a^G_{-(i-1,i)} \in A_{-(i-1,i)}$, for all $y_i \in Y_i$,

$$\mathbb{E}\left[\phi_{i-1}(a_{i-1}, y_{i-1}) \mid \bar{\alpha}_{i-1}, a^G_{-(i-1,i)}, a_i, y_i\right] = \begin{cases} q_2 & \text{if } a_i = a_i^G, \\ q_1 & \text{if } a_i \neq a_i^G. \end{cases} \tag{165}$$

A sufficient condition for the existence of such $\phi_{i-1}(a_{i-1}, y_{i-1})$ is as follows: Let $\bar{Q}_{i-1}(\bar{\alpha}_{i-1}, a^G_{-(i-1,i)}, a_i, y_i) \equiv (q_{i-1}(a_{i-1}, y_{i-1} \mid \bar{\alpha}_{i-1}, a^G_{-(i-1,i)}, a_i, y_i))_{a_{i-1}, y_{i-1}}$ be the vector expression of the conditional probability of (a_{i-1}, y_{i-1}) after player i plays a_i and observes y_i, assuming that players $-i$ take $\bar{\alpha}_{i-1}, a^G_{-(i-1,i)}$. It is sufficient that $\bar{Q}_i(\bar{\alpha}_{i-1}, a^G_{-(i-1,i)}, a_i, y_i)$ is linearly independent with respect to a_i, y_i.

At the same time, while player i sends a message by taking different a_i's, each player $n-1$ needs to incentivize player n to take the equilibrium strategy. To do so, we want to

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[96] Remember that in Assumption 13, we assumed that a_i^G that is used for player i to send the message is the same as a_i^G that is player i's action in a^G_{-j} when player $j \in -i$ sends the message.

construct the reward to cancel out the differences in the instantaneous utilities: If we pick a_i^B from $A_i \diagdown \{a_i^G\}$ properly, then for each $n \in I$, there exists a reward $\pi_n^{x_{n-1}}(\text{report}, i, a_{n-1}, y_{n-1})$ such that the ex ante payoff of player n is constant for all $a_n \in A_n$ and $a_i \in \{a_i^G, a_i^B\}$:

$$\begin{aligned} & u_i\left(a_n, \alpha_{-n}\right) + \mathbb{E}\left[\pi_n^{x_{n-1}}(\text{report}, i, a_{n-1}, y_{n-1}) \mid a_n, \alpha_{-n}\right] \\ = \; & \text{constant} \end{aligned} \tag{166}$$

for all $a_n \in A_n$ and

$$\begin{cases} \alpha_{-n} \in \left\{\left(\bar{\alpha}_{i-1}, a_{-(i-1,i,n)}^G, a_i^G\right), \left(\bar{\alpha}_{i-1}, a_{-(i-1,i,n)}^G, a_i^B\right)\right\} & \text{if player } n \text{ is not player } i \text{ (sender)}, \\ \alpha_{-n} \in \left\{\bar{\alpha}_{i-1}, a_{-(i-1,i)}^G\right\} & \text{if player } n \text{ is player } i. \end{cases} \tag{167}$$

A sufficient condition for the existence of such $\pi_n^{x_{n-1}}(\text{report}, i, a_{n-1}, y_{n-1})$ is as follows: Let

$\bar{Q}_{n-1}(i, a_n, \alpha_{-n}) = \left(q_{n-1}(y_{n-1} \mid a_n, \alpha_{-n})\right)_{y_{n-1}}$ be the vector expression of the conditional probability of y_{n-1} after the players play a_n, α_{-n}. It is sufficient that $\bar{Q}_n(i, a_n, \alpha_{-n})$ is linearly independent with respect to $a_n \in A_n$ and α_{-n} with (167).

Assumption 17 *For each $i \geq 2$, there exist $\bar{\alpha}_{i-1} \in A_{i-1}$ and $a_{-(i-1,i)}^G$ such that $\bar{Q}_i(\bar{\alpha}_{i-1}, a_{-(i-1,i)}^G, a_i, y_i)$ is linearly independent with respect to a_i, y_i. Further, there exist a_i^G, a_i^B such that for each $n \in I$, $\bar{Q}_n(i, a_n, \alpha_{-n})$ is linearly independent with respect to $a_n \in A_n$ and α_{-n} with (167).*

The former requirement is generic since we assume (152). In addition, the latter requirement is generic since $|Y_{n-1}| \geq 2\,|A_n|$.

Again, for notational convenience, for each i, a_i^G that player i uses to send a message and a_i^G that player i takes in $a_{-(j-1,j)}^G$ when another player j is a sender are the same. Moreover, assume that $\left(a_i^G, a_i^B\right)$ is the same as in Assumption 13.[97]

We can show that Assumption 17 is sufficient to have ϕ_{i-1} with conditionally independent property:

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[97] Remember that in Assumption 13, we assumed that a_i^G that is used for player i to send the message is the same as a_i^G that is player i's action in a_{-j}^G when player $j \in -i$ sends the message.

Lemma 39 *If Assumption 17 is satisfied, then there exist* $q_2 > q_1$ *such that for all* $i \in \{2, ..., N\}$*, there exist* $\phi_{i-1} : A_{i-1} \times Y_{i-1} \to (0,1)$ *such that (165) is satisfied.*

Proof. The same as Lemma 28. ■

In addition, Assumption 17 is sufficient to have $\pi_n^{x_{n-1}}(\text{report}, i, a_{n-1}, y_{n-1})$:

Lemma 40 *There exists* $\bar{u} > 0$ *such that, for each* $i \in I$ *and* $n \in I$*, there exist* $\pi_n^G(\text{report}, i, \cdot, \cdot) : A_{n-1} \times Y_{n-1} \to [-\bar{u}, 0]$ *and* $\pi_n^B(\text{report}, i, \cdot, \cdot) : A_{n-1} \times Y_{n-1} \to [0, \bar{u}]$ *such that (166) is satisfied.*

46 Coordination Block with the Noisy Cheap Talk

We consider the coordination block without the perfect cheap talk but with the noisy cheap talk with precision $p = \frac{1}{2}$.

As mentioned in Section 45.1, we define

$$\begin{aligned} N(i) &= \{i, i+1, i+2\}, \\ n^*(i) &\in \arg\min_{j \in \{i, i+2\}} |A_j| \, |Y_j| \, , \\ n^{**}(i) &= \{i, i+2\} \setminus \{n^*(i)\}. \end{aligned}$$

First, player i sends the message about $x_i \in \{G, B\}$ to player $n^*(i)$ via actions. Let $w_i \in \{G, B\}$ be player $n^*(i)$'s inference of this message. Second, player $n^*(i)$ sends the message about w_i to players $N(i)$ via actions. Each player $n \in N(i)$ constructs player n's inference of w_i, denoted $w_i(n) \in \{G, M, B\}$. Here, the inference M ("middle") is introduced so that it prevents player $n^*(i)$ from creating a situation where player $n^*(i)$ is pivotal. See 45.1 for the definition of "pivotal."

46.1 Structure of the Coordination Block

Formally, the coordination block proceeds as follows:

- The periods where the players coordinate on x_1:

  - The coordination round 1 for x_1. Player 1 sends the message about x_1 to player $n^*(1)$ via actions. If $n^*(1) = 1$, then this round does not exist.
  - The coordination round 2 for x_1. Player $n^*(1)$ sends the message about w_1 to players $N(1)$ via actions. Player $n \in N(1)$ creates the inference $w_1(n)$.
  - For each $j \in N(1) = \{1, 2, 3\}$ and $n \in -j$, we have the coordination rounds 3 for x_1 between j and n, where player j sends the message $w_1(j)$ via noisy cheap talk. The players take turns: First, player 1 sends $w_1(1)$ to player 2, second, player 1 sends $w_1(1)$ to player 3, and so on until player 1 sends $w_1(1)$ to player N. Then, player 2 sends $w_1(2)$ to player 1, and so on until player 2 sends $w_1(2)$ to player N. After player 2, player 3 sends $w_1(3)$ for each of the opponents -3 sequentially.

$\vdots$

- The periods where the players coordinate on x_i:
  - The coordination round 1 for x_i. Player i sends the message about x_i to player $n^*(i)$ via actions. If $n^*(i) = i$, then this round does not exist.
  - The coordination round 2 for x_i. Player $n^*(i)$ sends the message about w_i to players $N(i)$ via actions. Player $n \in N(i)$ creates the inference $w_i(n)$.
  - For each $j \in N(i)$ and $n \in -j$, we have the coordination rounds 3 for x_i between j and n, where player j sends the message $w_i(j)$ via noisy cheap talk. Again, the players take turns.

$\vdots$

- The periods where the players coordinate on x_N:
  - The coordination round 1 for x_N. Player N sends the message about x_N to player $n^*(N)$ via actions. If $n^*(N) = N$, then this round does not exist.
  - The coordination round 2 for x_N. Player $n^*(N)$ sends the message about w_N to players $N(N)$ via actions. Player $n \in N(N)$ creates the inference $w_N(n)$.

- For each $j \in N(N)$ and $n \in -j$, we have the coordination rounds 3 for x_N between j and n, where player j sends the message $w_N(j)$ via noisy cheap talk. Again, the players take turns.

For notational convenience, let $T(i \rightarrow_{x_i} n^*(i))$ be the set of periods in the coordination round 1 for x_i, where player i sends the message x_i to player $n^*(i)$ via actions. Similarly, let $T(n^*(i) \rightarrow_{w_i} N(i))$ be the set of periods in the coordination round 2 for x_i, where player $n^*(i)$ sends the message w_i to players $N(i)$ via actions

We explain each round in the sequel.

46.2 Coordination Round 1 for x_i

If player i is the same person as player $n^*(i)$, then this round does not exist. Let $w_i = x_i$ be player $n^*(i)$'s inference (player i's inference in other words).

Otherwise, player i takes $a_i^{x_i}$ and the other players $-i$ take a_{-i}^G for $T^{\frac{1}{2}}$ periods. Remember that $T(i \rightarrow_{x_i} n^*(i))$ be the set of periods in this round.

Player $n^*(i)$ creates her inference of x_i denoted by w_i as follows: First, player $n^*(i)$ creates $\Psi^i_{n^*(i),t} \in \{0,1\}$ from $\psi^i_{n^*(i)}(y_{n^*(i),t})$ as player i creates $\Psi^{a(x)}_{i,t}$ from $\psi^{a(x)}_i(y_{i,t})$. See Lemma 33 for the definition of $\psi^i_{n^*(i)}(y_{n^*(i),t})$.

Second, player $n^*(i)$ randomly picks $t_{n^*(i)}(i \rightarrow_{x_i} n^*(i))$ from $T(i \rightarrow_{x_i} n^*(i))$.

Finally, player $n^*(i)$ infers x_i from $\left\{\Psi^i_{n^*(i),t}\right\}_{T(i \rightarrow_{x_i} n^*(i))}$ but excludes period $t_{n^*(i)}(i \rightarrow_{x_i} n^*(i))$. That is, with $T_{n^*(i)}(i \rightarrow_{x_i} n^*(i)) \equiv T(i \rightarrow_{x_i} n^*(i)) \setminus \left\{t_{n^*(i)}(i \rightarrow_{x_i} n^*(i))\right\}$, player $n^*(i)$ infers $w_i = G$ if

$$\frac{1}{T^{\frac{1}{2}} - 1} \sum_{t \in T_{n^*(i)}(i \rightarrow_{x_i} n^*(i))} \Psi^i_{n^*(i),t} \geq \frac{q_1 + q_2}{2} \tag{168}$$

and $w_i = B$ otherwise.

Lemma 33 directly implies the following:

Lemma 41 *For any $i \in I$ and $x_i \in \{G, B\}$, if players i and $n^*(i)$ follow the equilibrium* *strategy, then*

$$\Pr\left(\{w_i = x_i\} \mid x_i\right) \geq 1 - \exp(-O(T^{\frac{1}{2}}))$$

and the conditional distribution of w_i *given* x_i *is independent of another player* $j \in -(i, n^*(i))$*'s unilateral deviation.*

46.3 Coordination Round 2 for x_i

This is the round where player $n^*(i)$ sends w_i to players $N(i)$. Player $n^*(i)$ takes $a_{n^*(i)}^{w_i}$, player $i+1$ takes α_{i+1}^*, player $n^{**}(i)$ takes $\alpha_{n^{**}(i)}^*$, and each player $j \notin N(i)$ takes a_j^G for $T^{\frac{1}{2}}$ periods. See Assumption 14 for the definition of α_{i+1}^* and $\alpha_{n^{**}(i)}^*$. Remember that $T(n^*(i) \rightarrow_{w_i} N(i))$ be the set of periods in this round.

See (156) and (157) for the definition of the $L_n(i) \times |A_n| |Y_n|$ matrix $I_n(i)$. Based on $I_n(i)$, each player $n \in N(i) \setminus \{n^*(i)\}$ constructs a random variable $\boldsymbol{I}_{n,t}(i)$ as follows: After taking a_n and observing y_n, player n calculates $I_n(i)\, \mathbf{1}_{a_n,y_n}$. Here, $\mathbf{1}_{a_n,y_n}$ is a $|A_n| |Y_n| \times 1$ vector such that the element corresponding to a_n, y_n is equal to 1 and the other elements are 0. Hence, $I_n(i)\, \mathbf{1}_{a_n,y_n}$ is a $L_n(i) \times 1$ vector. Then, player n draws $L_n(i)$ random variables independently from the uniform distribution on $[0,1]$. If the lth realization of these random variables is less than the lth element of $I_n(i)\, \mathbf{1}_{a_n,y_n}$, then the lth element of $\boldsymbol{I}_n(i)$ is equal to 1. Otherwise, the lth element of $\boldsymbol{I}_n(i)$ is equal to 0. We have

$$\Pr(\{(\boldsymbol{I}_n(i))_l = 1\} \mid a, y) = \boldsymbol{i}_n^l(i) \mathbf{1}_{a_n,y_n}. \tag{169}$$

Given $\{\boldsymbol{I}_{n,t}(i)\}_{t \in T(n^*(i) \rightarrow_{w_i} N(i))}$, player $n \in N(i)$ infers w_i as follows: Player n randomly picks $t_n(n^*(i) \rightarrow_{w_i} N(i))$ from $T(n^*(i) \rightarrow_{w_i} N(i))$. Player n infers w_i from $\{\boldsymbol{I}_{n,t}(i)\}_{T(n^*(i) \rightarrow_{w_i} N(i))}$ but excludes period $t_n(n^*(i) \rightarrow_{w_i} N(i))$. Specifically, with $T_n(n^*(i) \rightarrow_{w_i} N(i)) \equiv T(n^*(i) \rightarrow_{w_i} N(i)) \setminus \{t_n(n^*(i) \rightarrow_{w_i} N(i))\}$,

1. Player $n^*(i)$ infers her own message straightforwardly: $w_i(n^*(i)) = w_i$.

2. Player $n \in N(i) \setminus \{n^*(i)\}$ infers as follows:

(a) If

$$\left\| \frac{1}{T^{\frac{1}{2}}-1} \sum_{t \in T_n(n^*(i) \to_{w_i} N(i))} \boldsymbol{I}_{n,t}(i) - q_2 \mathbf{1} \right\| \leq \varepsilon,$$

then $w_i(n) = G$.

(b) If

$$\left\| \frac{1}{T^{\frac{1}{2}}-1} \sum_{t \in T_n(n^*(i) \to_{w_i} N(i))} \boldsymbol{I}_{n,t}(i) - q_1 \mathbf{1} \right\| \leq \varepsilon,$$

then $w_i(n) = B$.

(c) Otherwise, $w_i(n) = M$ (the posterior is not skewed enough for $w_i = G$ or $w_i = B$ and so player $n^*(i)$ infers that the message is “middle”).

Assumption 14 implies the following Lemma:

Lemma 42 *For any $\varepsilon < \bar{\varepsilon}$, for any $i \in I$ and $w_i \in \{G, B\}$,*

1. *For any $n \in N(i^*)$,*
   - *(a) If players $n^*(i)$ and n follow the equilibrium strategy, then*

$$\Pr(\{w_i(n) = w_i\} \mid w_i) \geq 1 - \exp(-O(T^{\frac{1}{2}})).$$

   - *(b) The distribution of $w_i(n)$ given w_i is independent of player $j = N(i) \setminus \{n^*(i), n\}$'s unilateral deviation.*

2. *For any history of player $n^*(i)$ at the end of the coordination round 2 for x_i, player $n^*(i)$ puts a belief no more than $\exp(-O(T^{\frac{1}{2}}))$ on the event*

$$\{G, B\} \ni w_i(n^{**}(i)) \neq w_i(i+1) \in \{G, B\}.$$

Proof.

1. Follows from (156) and (169).
2. Follows from Lemma 34 and Hoeffding's inequality. By Assumption 3, excluding period $t_n(n^*(i) \rightarrow_{w_i} N(i))$ does not affect the probability so much.

■

As we will see, as long as the noisy cheap talk by the other players transmits correctly in the coordination round 3 for x_i (this is true ex ante at the end of the coordination round 2 for x_i), player $n^*(i)$ is pivotal for some player's inference of x_i if and only if $\{G, B\} \ni w_i(n^{**}(i)) \neq w_i(i+1) \in \{G, B\}$. 2 of Lemma 42 guarantees that, after any history (including those after player $n^*(i)$'s deviation), the probability that player $n^*(i)$ is pivotal is negligible for the almost optimality.

For each $n \in N(i) \setminus \{n^*(i)\}$, consider player $j = N(i) \setminus \{n^*(i), n\}$. As we will see, player j is not pivotal if players $n^*(i)$ and n infer the same state w_i. Therefore, 1 of Lemma 42 guarantees that player j cannot manipulate player n's inference to create a situation where player j is pivotal.

46.4 Coordination Round 3 for x_i Between Players j and n

This is the round where player $j \in N(i)$ sends $w_i(j)$ to player $n \in I$. Let $w_i(j)(n) \in \{G, B, M\}$ be player n's inference of player j's message. Here, we assume that the noisy cheap talk is available. See Section 48 for how to dispense with the noisy cheap talk.

If player j is the same player as player n, then $w_i(j)(n) = w_i(j)$, that is, player j infers her own message straightforwardly.

Otherwise, player j sends messages as follows. From $w_i(j) \in \{G, M, B\}$, player j constructs a sequence of two binary messages $w_i(j)\{1\}, w_i(j)\{2\} \in \{G, B\}^2$: If $w_i(j) = G$, then $w_i(j)\{1\} = w_i(j)\{2\} = G$; If $w_i(j) = B$, then $w_i(j)\{1\} = w_i(j)\{2\} = B$; If $w_i(j) = M$, then $w_i(j)\{1\} = G$ and $w_i(j)\{2\} = B$ with probability $\frac{1}{2}$ and $w_i(j)\{1\} = B$ and $w_i(j)\{2\} = G$ with probability $\frac{1}{2}$.

Player j sends the two messages $w_i(j)\{1\}$ and $w_i(j)\{2\}$ sequentially via noisy cheap talk.

With abuse of notation, we define $g[n-1](w_i(j)) \in \{w_i(j), E\}$ and $f[n](w_i(j)) \in \{G, M, B\}$ as follows: For $g[n-1](w_i(j))$,

1. $g[n-1](w_i(j)) = w_i(j)$ if and only if player $n-1$ thinks that there is no error for $f[n](w_i(j)\{1\})$ and $f[n](w_i(j)\{2\})$, that is, $g[n-1](w_i(j)\{1\}) = w_i(j)\{1\}$ and $g[n-1](w_i(j)\{2\}) = w_i(j)\{2\}$.
2. $g[n-1](w_i(j)) = E$ otherwise.

For $f[n](w_i(j))$, player i infers $f[n](w_i(j))$ from $f[n](w_i(j)\{1\})$ and $f[n](w_i(j)\{2\})$, using the mapping between $w_i(j)$ and $w_i(j)\{1\}, w_i(j)\{2\}$.

1. $f[n](w_i(j)) = G$ if and only if $f[n](w_i(j)\{1\}) = f[n](w_i(j)\{2\}) = G$.
2. $f[n](w_i(j)) = B$ if and only if $f[n](w_i(j)\{1\}) = f[n](w_i(j)\{2\}) = B$.
3. $f[n](w_i(j)) = M$ if and only if "$f[n](w_i(j)\{1\}) = G$ and $f[n](w_i(j)\{2\}) = B$" or "$f[n](w_i(j)\{1\}) = B$ and $f[n](w_i(j)\{2\}) = G$."

$g_2[n-1](w_i(j))$ and $f_2[j-1](w_i(j))$ are analogously defined.

Finally, player n infers $w_i(j)$ as $w_i(j)(n) = f[n](w_i(j))$.

46.5 Player n's Inference of x_i

Based on these rounds, player n infers x_i as follows. Let $x_i(n) \in \{G, B\}$ be player n's inference of x_i. From $\{w_i(j)(n)\}_{j \in N(i)}$, player n constructs $x_i(n)$ such that

$$x_i(n) = \begin{cases} G \quad \text{if} \begin{cases} w_i(n^{**}(i))(n) = w_i(i+1)(n) = G, \\ w_i(n^{**}(i))(n) = M, w_i(i+1)(n) = G, \\ w_i(n^{**}(i))(n) = G, w_i(i+1)(n) = M, \\ w_i(n^{**}(i))(n) = B, w_i(i+1)(n) = G, w_i(n^{*}(i))(n) = G, \\ w_i(n^{**}(i))(n) = G, w_i(i+1)(n) = B, w_i(n^{*}(i))(n) = G, \end{cases} \\ B \quad \text{otherwise.} \end{cases} \tag{170}$$

Finally, let

$$x\left(n\right)=\left\{x_i(n)\right\}_{i\in I}$$

be the profile of the inferences.

46.6 Definition of $\theta_{i-1}(\mathrm{c}) \in \{G, B\}$

Based on the realization of the coordination block, if some events happen, then player $i-1$ makes player i indifferent between any action profile sequence in the main blocks. $\theta_{i-1}(\mathrm{c}) = B$ implies that such an event happens while $\theta_{i-1}(\mathrm{c}) = G$ implies that such an event does not happen.

We will define the events to induce $\theta_{i-1}(\mathrm{c}) = B$: For each $j \in I$, while the players coordinate on x_j,

1. There exists player $j' \in -i$ with $j' \in N(j)$ such that when player j' sends the message $w_j(j')$ to player i, player $i-1$ has $g[i-1](w_j(j')) = E$.
2. There exist players $j' \in -i$ and $n \in -i \cap N(j)$ such that when player j' sends the message $w_j(j')$ to player n, player n has a wrong signal $f[n](w_j(j')) \neq w_j(j')$.
3. Player i is in $N(j)$ and consider the following inference:

$$x_j\left(n\right)=\begin{cases} G \quad \text{if} \begin{cases} w_j\left(n^{**}(j)\right)=w_j\left(j+1\right)=G, \\ w_j\left(n^{**}(j)\right)=M, w_j\left(j+1\right)=G, \\ w_j\left(n^{**}(j)\right)=G, w_j\left(j+1\right)=M, \\ w_j\left(n^{**}(j)\right)=B, w_j\left(j+1\right)=G, w_j\left(n^{*}(j)\right)=G, \\ w_j\left(n^{**}(j)\right)=G, w_j\left(j+1\right)=B, w_j\left(n^{*}(j)\right)=G, \end{cases} \\ B \qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{171}$$

   Note that this is what we replace player n's inference of the messages in the coordination round 3 in (170) with the true messages. We have $\theta_{i-1}(\mathrm{c}) = B$ if there exist $n \in I$ and $j \in I$ such that player i's message $w_j(i)$ matters for $x_j(n)$ in (171). That is,

(a) If player i is $n^*(j)$, then

$$\{G, B\} \ni w_j\left(n^{**}(j)\right) \neq w_j\left(j+1\right) \in \{G, B\}. \tag{172}$$

(b) If player i is in $N(j) \setminus \{n^*(j)\}$, then

$$w_j \equiv w_j\left(n^*(j)\right) \neq w_j\left(i'\right). \tag{173}$$

with $i' = N(j) \setminus \{i, n^*(j)\}$.

Note that, although player n can be player i herself, whether or not $w_j(i)$ matters in (171) is determined by the other players' messages $\{w_j(i')\}_{i' \neq i}$.

In the definition of $\theta_{i-1}(\mathrm{c})$, player $i-1$ uses the information owned by players $-(i-1,i)$. Section 52 explains how players $-(i-1,i)$ inform player $i-1$ of their history necessary to create $\theta_{i-1}(\mathrm{c})$ in the re-report block. Since $\theta_{i-1}(\mathrm{c})$ only affects the reward function (that is, does not affect $\sigma_{i-1}(x_{i-1})$), it suffices that player $i-1$ knows the information by the end of the review phase.

We verify that the distribution of $\theta_{i-1}(\mathrm{c})$ is almost independent of player i's strategy: For Cases 1 and 2, we need to verify that player i cannot manipulate $\theta_{i-1}(\mathrm{c})$ by affecting some player's message m. The definition of the noisy cheap talk implies that the probability of $g[i-1](m) = E$ when player i is a receiver and that of $f[n](m) \neq m$ when player $j \in -i$ is a sender and player $n \in -i$ is a receiver are almost independent of m.[98]

For Case 3-(a), 2 of Lemma 42 implies that player i puts a belief no more than $\exp(-O(T^{\frac{1}{2}}))$ on (172) after *any history* (including those after player i's deviation) at the end of the coordination round 2 for x_j. Since $w_j\left(n^{**}(j)\right)$ and $w_j\left(j+1\right)$ are fixed at the end of the coordination round 2 for x_j, whether (172) happens or not is almost independent of player i's strategy.

For Case 3-(b), note that if $w_j(n^{**}(j)) = w_j$, then (173) is not the case. In addition, regardless of w_j, this event happens with probability no more than $\exp(-O(T^{\frac{1}{2}}))$ from the

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[98] Note that m can be affected by player i's strategy before the round where player j sends m to player n.

perspective at the end of the coordination round 1 for x_j by 1 of Lemma 42.[99] Therefore, no player can change the distribution of $\theta_{i-1}(\mathrm{c})$ by more than $\exp(-O(T^{\frac{1}{2}}))$.

In summary, we have shown the following lemma:

Lemma 43 *If*

1. *the probability of $g[i-1](m) = E$ when player i is a receiver of a message m is almost independent of m and*
2. *the probability of $f[n](m) \neq m$ when player $j \in -i$ is a sender of a message m and player $n \in -i$ is a receiver is almost independent of m,*

then, the distribution of $\theta_{i-1}(\mathrm{c}) \in \{G, B\}$ is almost independent of player i's strategy.

The premise of lemma is stated to clarify what assumption about the noisy cheap talk is used, expecting that we will dispense with it later.

46.7 Incentives in the Coordination Block

First, Lemma 43 implies that player i does not have an incentive to manipulate $\theta_{i-1}(\mathrm{c})$.

Second, we consider player i's incentive to tell the truth about $w_n(i)$ with $i \in N(n)$ for the coordination round 3 for x_n between i and $i' \in -i$. If player i' with $i' \in -i$ received a wrong signal $f[i'](w_n(j))$ for some $n \in I$ and $j \in -i$, then Case 2 of $\theta_{i-1}(\mathrm{c})$ implies $\theta_{i-1}(\mathrm{c}) = B$. Hence, together with Case 3 of $\theta_{i-1}(\mathrm{c})$, whenever player i's message matters for $x_n(i')$ for some $i' \in -i$, then $\theta_{i-1}(\mathrm{c}) = B$ and player i is indifferent between any action profile sequence. Therefore, it is optimal for player i to tell the truth.

Third, we consider the incentive of player i in the coordination rounds 1 and 2 for x_n. If player i is player $n^*(n)$, then since x_n controls the value of player $n+1 \neq n^*(n)$, player $n^*(n)$ is indifferent between coordinating on $x_n(j) = G$ for all $j \in I$ or $x_n(j) = B$ for all $j \in I$. (170) and 2 of of Lemma 42 imply that, if the messages in the coordination round

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[99] Note that w_j can be affected by some player's strategy before the end of the coordination round 1 for x_j.

3 transmit correctly if a sender is not player $n^*(n)$ (this is true with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$), then player $n^*(n)$ at the end of the coordination round 2 puts a conditional belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the event that $x_n(j) = G$ for all $j \in I$ or $x_n(j) = B$ for all $j \in I$ regardless of player $n^*(n)$'s history. Therefore, player $n^*(n)$ (player i) is almost indifferent between any strategy in the coordination rounds 1 and 2 for x_n.

If player i is player n (the initial holder of state x_n) but not player $n^*(n)$, then again, since x_n controls the value of player $n+1 \neq n$, player n is indifferent between coordinating on $x_n(j) = G$ for all $j \in I$ or $x_n(j) = B$ for all $j \in I$. 1 of of Lemma 42 implies that regardless of player n's strategy in the coordination rounds 1 and 2 for x_n, players $n^*(n)$ and $n+1$ have $w_n(n^*(n)) = w_n(n+1) \in \{G, B\}$ with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$. Then, (170) implies that, if the messages in the coordination round 3 transmit correctly if a sender is not player n (again, this is true with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$), then either $x_n(j) = G$ for all $j \in I$ or $x_n(j) = B$ for all $j \in I$. Therefore, player n (player i) is almost indifferent between any strategy in the coordination rounds 1 and 2.

If player i is not player n or player $n^*(n)$, then Lemma 41 and 1 of of Lemma 42 imply that, regardless of player i's strategy in the coordination rounds 1 and 2 for x_n, players $n^*(n)$ and at least one player $i' \in N(n) \setminus \{i\}$ have $w_n(n^*(n)) = w_n(i') = x_n$ with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$. Then, (170) implies that, if the messages in the coordination round 3 transmit correctly if a sender is not player i (this is true with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$), then either $x_n(j) = G$ for all $j \in I$ or $x_n(j) = B$ for all $j \in I$. Therefore, player i is almost indifferent between any strategy in the coordination rounds 1 and 2.

Finally, we show that the definition of $\theta_{i-1}(\mathrm{c}) = B$ implies that, for any i, for any t in the main blocks, for any h_i^t, player i puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the event that $x(j) = x(i)$ for all $j \in -i$ or $\theta_{i-1}(\mathrm{c}) = B$ by the following reasons: (i) If player i's signal $f[i](w_n(j))$ was wrong for some $n \in I$ and $j \in -i$, then, given $w_n(j)$, $g[i-1](w_n(j)) = E$ with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$. Since $g[i-1](w_n(j))$ is not revealed by players $(-i)$'s continuation strategy in the main blocks, player i believes that $\theta_{i-1}(\mathrm{c}) = B$

because of Case 1. (ii) If player i' with $i' \in -i$ received a wrong signal $f[i'](w_n(j))$ for some $n \in I$ and $j \in -i$, then Case 2 of $\theta_{i-1}(\mathrm{c})$ implies $\theta_{i-1}(\mathrm{c}) = B$. From (i) and (ii), player i who considers almost optimality can condition that $f[i'](w_n(j)) = w_n(j)$ for all $i' \in I$, $n \in I$ and $j \in -i$. (iii) If player i is pivotal for player i''s inference of x_n with $i' \in I$ and $n \in I$, then $\theta_{i-1}(\mathrm{c}) = B$. Hence, $w_n(i)$ does not matter for player i's value. Therefore, in total, $x(j) = x(i)$ for all $j \in -i$ or $\theta_{i-1}(\mathrm{c}) = B$.

The following lemma summarizes the above discussion:

Lemma 44 *The following two statements are true:*

1. *If, for each player $i \in I$,*

    (a) *the probability of $g[i-1](m) = E$ when player i is a receiver of a message m is almost independent of m,*

    (b) *the probability of $f[n](m) \neq m$ when player $j \in -i$ is a sender of a message m and player $n \in -i$ is a receiver is almost independent of m, and*

    (c) *for all n with $i \neq n+1$, player i's value is almost the same between $x_n(j) = G$ for all $j \in I$ and $x_n(j) = B$ for all $j \in I$ regardless of $\{x_{n'}(j)\}_{j \in I, n' \leq n-1}$ ($n' \leq n-1$ implies that the coordination rounds for $x_{n'}$ comes before those for x_n),*

    then it is almost optimal for player i to follow the equilibrium strategy in the coordination block.

2. *For any $i \in I$, for any t in the main blocks, for any h_i^t, player i puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the event that $x(j) = x(i)$ for all $j \in -i$ or $\theta_{i-1}(\mathrm{c}) = B$.*

Note that, for the second statement, 1-(a), 1-(b) and 1-(c) are not necessary.

47 Structure of the Review Phase

Replacing the perfect cheap talk in the coordination block with the noisy cheap talk, the structure of the coordination block is as explained in Section 46.1. Now, the coordination block has at most $N(1+1+3(N-1))$ rounds.[100] After the coordination block, the structure is the same as in Section 31 of the Supplemental Material 3. As in the Supplemental Material 3, let r be a generic serial number for a round.

If we replace the noisy cheap talk with messages via actions, then as we will see in Section 48, we treat rounds where a player sends one message and rounds where a player send two messages separately. Each round where the sender sends one message has $T^{\frac{1}{2}}$ periods. Section 48 explains how the sender sends the message. On the other hand, each round where the sender would send two messages via noisy cheap talk is now divided into two rounds each of which has $T^{\frac{1}{2}}$ without the noisy cheap talk. Using the first $T^{\frac{1}{2}}$-period round, the sender sends the first message as we will explain in Section 48. After that, using the second $T^{\frac{1}{2}}$-period round, the sender sends the second message. With abuse of notation, let r again be a generic serial number for a round and $T(r)$ be the set of periods in round r.

48 Dispensing with the Noisy Cheap Talk

We consider how player j sends a binary message $m \in \{G, B\}$ to player i in some round. Let r be the serial number of this round and $T(r)$ be the set of periods in round r.

As mentioned in Section 45.2, with η being a small number to be defined, the sender (player j) determines

$$z_j(m) = \begin{cases} m & \text{with probability } 1-2\eta, \\ \{G,B\} \setminus \{m\} & \text{with probability } \eta, \\ M & \text{with probability } \eta \end{cases}$$

and player j takes

$$\alpha_j^{z_j(m)} = \begin{cases} a_j^G & \text{if } z_j(m) = G, \\ a_j^B & \text{if } z_j(m) = B, \\ \frac{1}{2}a_j^G + \frac{1}{2}a_j^B & \text{if } z_j(m) = M \end{cases}$$

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[100] The precise number depends on whether $n^*(i) = i$ or not for each i.

for $T^{\frac{1}{2}}$ periods. The other players $-j$ take a_{-j}^{G}.

Then, intuitively, as in Section 45.2, $g[n-1](m)$ is determined as follows:

1. $g[n-1](m) = m$ if $z_j(m) = m$ and $\mathbf{y}_{n-1} \in \mathcal{H}_{n-1}[\varepsilon]\,(m)$.
2. $g[n-1](m) = E$ if $z_j(m) \neq m$ or $\mathbf{y}_{n-1} \notin \mathcal{H}_{n-1}[\varepsilon]\,(m)$.

Instead of using $\mathbf{y}_{n-1}$ directly, as in the two-player case, we consider the following construction of $g[n-1](m)$.

48.1 Formal: $g[n-1](m) \in \{m, E\}$

In the definition of $g[n-1](m)$, player $n-1$ uses m, which is the information owned by player j. Section 52 explains how player j informs player $n-1$ of m. Since $g[n-1](m)$ only affects the reward function (does not affect $\sigma_{n-1}(x_{n-1})$), it suffices that player $n-1$ knows the information by the end of the review phase.

If $z_j(m) \neq m$, then $g[n-1](m) = E$ as in Section 45.2. Let us concentrate on the case with $z_j(m) = m$. Let $\mathbf{y}_{n-1}(r)$ be the frequency of player $(n-1)$'s signals in round r.

First, player $n-1$ randomly picks $t_{n-1}(r)$ from $T(r)$, the set of periods in round r. With $T_{n-1}(r) \equiv T(r) \setminus \{t_{n-1}(r)\}$, player $n-1$ constructs random variables $\left\{\mathbf{\Omega}_{n-1,t}^{H}\right\}_{t\in T_{n-1}(r)}$ as follows. After taking a_{n-1} ($a_{n-1} = a_{n-1}^{m}$ if player $n-1$ is the sender ($n-1=j$) since we concentrate on $z_j(m) = m$ and $a_{n-1} = a_{n-1}^{G}$ if player $n-1$ is not the sender) and observing $y_{n-1,t}$, player $n-1$ calculates $H_{n-1}(m)\mathbf{1}_{y_{n-1,t}}$. Then, player $n-1$ draws $(|Y_{n-1}| - |A_n| + 1)$ random variables independently from the uniform distribution on $[0,1]$. If the lth realization of these random variables is less than the lth element of $H_{n-1}(m)\mathbf{1}_{y_{n-1,t}}$, then the lth element of $\mathbf{\Omega}_{n-1,t}^{H}$ is equal to 1. Otherwise, the lth element of $\mathbf{\Omega}_{n-1,t}^{H}$ is equal to 0. Since all the elements of $H_{n-1}(m)$ are in $(0,1)$, we have

$$\Pr\left(\left\{\left(\mathbf{\Omega}_{n-1,t}^{H}\right)_l = 1\right\} \mid a, y\right) = \left(H_{n-1}(m)\mathbf{1}_{y_{n-1,t}}\right)_l \in (0,1) \tag{174}$$

for all a and y.

We define $g[n-1](m) = m$ if and only if

$$\left\| \frac{1}{T^{\frac{1}{2}}-1} \sum_{t \in T_{n-1}(r)} \mathbf{\Omega}^H_{n-1,t} - \frac{1}{T^{\frac{1}{2}}-1} \sum_{t \in T_{n-1}(r)} H_{n-1}(m) \mathbf{1}_{y_{n-1,t}} \right\| \leq \frac{\varepsilon}{4} \tag{175}$$

and

$$\left\| \frac{1}{T^{\frac{1}{2}}-1} \sum_{t \in T_{n-1}(r)} \mathbf{\Omega}^H_{n-1,t} - \mathbf{p}_{n-1}(m) \right\| \leq \frac{\varepsilon}{2}. \tag{176}$$

In summary, there are following cases:

1. $g[n-1](m) = E$ if $z_{n-1}(m) \neq m$, (175) is not satisfied, or (176) is not satisfied.
2. $g[n-1](m) = m$ if $z_{n-1}(m) = m$, (175) is satisfied, and (176) is satisfied.

48.2 Formal: $f[n](m) \in \{G, B\}$

On the other hand, let us consider how the receiver (player n) infers the message. Let $\mathbf{y}_n(r)$ be the frequency of player n's signal observations in round r. Instead of using $\mathbf{y}_n(r)$ directly as in Section 45.2, we consider the following procedure to construct $f[n](m)$.

First, player n randomly picks $t_n(r)$ from $T(r)$, the set of periods in round r. With $T_n(r) \equiv T(r) \setminus \{t_n(r)\}$, player n constructs $f[n](m)$ only depending on $\{y_{n,t}\}_{t \in T_n(r)}$. For notational convenience, let $\mathbf{y}_n(r, T_n(r))$ be the frequency of player n's signal observations in $T_n(r)$.

$f[n](m)$ is determined as in Section 45.2 with $\mathbf{y}_n(r)$ replaced with $\mathbf{y}_n(r, T_n(r))$:

1. If $\mathbf{y}_n(r, T_n(r)) \in \mathcal{G}_n[\varepsilon]$, then
   (a) $f[n](m) = G$ if $\mathbf{y}_i(r, T_n(r)) \in \mathcal{H}_{n-1,n}[\varepsilon](G) \cup \mathcal{J}_n[\varepsilon](G)$.
   (b) $f[n](m) = B$ if $\mathbf{y}_i(r, T_n(r)) \in \mathcal{H}_{n-1,n}[\varepsilon](B) \cup \mathcal{J}_n[\varepsilon](B)$ or $\mathbf{y}_i(r, T_n(r)) \notin \mathcal{H}_{n-1,n}[\varepsilon](G) \cup \mathcal{J}_n[\varepsilon](G)$, and
2. If $\mathbf{y}_n(r, T_n(r)) \notin \mathcal{G}_n[\varepsilon]$, then player n infers $f[n](m)$ from the likelihood as in the two-player case.

By Assumption 3 (full support), neglecting $\left(a_{n,t_n(r)}, y_{n,t_n(r)}\right)$ does not affect the posterior so much.

48.3 Definition of $\theta_{i-1}(j \to_m n) \in \{G, B\}$

While player $j \in I$ sends a message m to player $n \in -j$, for each $i \in I$, player $i-1$ creates $\theta_{i-1}(j \to_m n) \in \{G, B\}$. As for $\theta_{i-1}(\mathrm{c})$, $\theta_{i-1}(j \to_m n) = B$ implies that player $i-1$ makes player i indifferent between any action profile sequence in the subsequent rounds.

Again, player $i-1$ uses the information owned by players $-(i-1,i)$. Section 52 explains how players $-(i-1,i)$ inform player $i-1$ in the re-report block. Since $\theta_{i-1}(j \to_m n)$ only affects the reward function (does not affect $\sigma_{i-1}(x_{i-1})$), it suffices that player $i-1$ knows the information by the end of the review phase.

To create $\theta_{i-1}(j \to_m n)$, player $i-1$ calculates the following variables:

Construction of $\mathbf{\Omega}^G_{n,\tau}$ If $i-1 \neq n$, then player n informs player $i-1$ of how many times player n observes y_n for each $y_n \in Y_n$ in $T(r)$ (while receiving the message). Let $T(r, y_n)$ be this number.

For each $y_n \in Y_n$, player $i-1$ calculates $G_n \mathbf{1}_{y_n}$. Then, repeat the following process $T(r, y_n)$ times: Player $i-1$ draws $\left(|Y_n| - |A_j| - 2\sum_{i' \neq j,n} |A_{i'}| + 1\right)$ random variables independently from the uniform distribution on $[0,1]$. If the lth realization of these random variables is less than the lth element of $G_n \mathbf{1}_{y_n}$, then the lth element of $\mathbf{\Omega}^G_n$ is equal to 1. Otherwise, the lth element of $\mathbf{\Omega}^G_n$ is equal to 0. Since player $i-1$ repeats this process $T(r, y_n)$ times, it generates $T(r, y_n)$ *i.i.d.* random variables $\left\{\mathbf{\Omega}^G_{n,\tau}\right\}_{\tau=1}^{T(r,y_n)}$. Since all the elements of G_n are in $(0,1)$,

$$\Pr\left(\left\{\left(\mathbf{\Omega}^G_{n,\tau}\right)_l = 1\right\}\right) = \left(G_n \mathbf{1}_{y_n}\right)_l.$$

In total, $\left\{\left\{\mathbf{\Omega}^G_{n,\tau}\right\}_{\tau=1}^{T(r,y_n)}\right\}_{y_n \in Y_n}$ is constructed.

Construction of $\mathbf{\Omega}^J_{n,\tau}(m)$ In addition to player n informing player $i-1$ of $T(r, y_n)$, player j informs player $i-1$ of m in the re-report block.

For each $y_n \in Y_n$, player $i-1$ calculates $J_n(m)\mathbf{1}_{y_n}$. Then, repeat the following process $T(r, y_n)$ times: Player $i-1$ draws $\left(|Y_n| - \sum_{i' \neq j,n} |A_{i'}| + 1\right)$ random variables independently from the uniform distribution on $[0,1]$. If the lth realization of these random variables is less than the lth element of $J_n(m)\mathbf{1}_{y_n}$, then the lth element of $\mathbf{\Omega}_n^J(m)$ is equal to 1. Otherwise, the lth element of $\mathbf{\Omega}_n^J(m)$ is equal to 0. Since player $i-1$ repeats this process $T(r, y_n)$ times, it generates $T(r, y_n)$ *i.i.d.* random variables $\left\{\mathbf{\Omega}_{n,\tau}^J(m)\right\}_{\tau=1}^{T(r,y_n)}$. Since all the elements of $J_n(m)$ are in $(0,1)$,

$$\Pr\left(\left\{\left(\mathbf{\Omega}_{n,\tau}^J(m)\right)_l = 1\right\}\right) = \left(J_n(m)\mathbf{1}_{y_n}\right)_l.$$

In total, $\left\{\left\{\mathbf{\Omega}_{n,\tau}^J(m)\right\}_{\tau=1}^{T(r,y_n)}\right\}_{y_n \in Y_n}$ is constructed.

Definition of $\theta_{i-1}(j \to_m n) \in \{G, B\}$ For player $i \in \{j, n\}$ (sender or receiver), $\theta_{i-1}(j \to_m n) = G$ for any history.

If player i is in players $-(j,n)$, then player $i-1$ has $\theta_{i-1}(j \to_m n) = G$ if

1. The frequency of $\left\{\left\{\mathbf{\Omega}_{n,\tau}^G\right\}_{\tau=1}^{T(r,y_n)}\right\}_{y_n \in Y_n}$ is close to $\mathbf{g}_n$:

$$\left\| \frac{1}{T^{\frac{1}{2}}} \sum_{y_n \in Y_n} \sum_{\tau=1}^{T(r,y_n)} \mathbf{\Omega}_{n,\tau}^G - \mathbf{g}_n \right\| \leq \frac{\varepsilon}{2}.$$

   Regardless of player i's deviation, this is the case with probability $1 - \exp(-O(T^{\frac{1}{2}}))$ by Notation 2 and the law of large numbers.[101]

2. The frequency of $\left\{\left\{\mathbf{\Omega}_{n,\tau}^G\right\}_{\tau=1}^{T(r,y_n)}\right\}_{y_n \in Y_n}$ is close to $\left\{\frac{T(r,y_n)}{T} G_n \mathbf{1}_{y_n}\right\}_{y_n}$ (the frequency of

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[101] While player $n^*(i)$ excludes one period $t_{n^*(i)}(i \to_{x_i} n^*(i))$ from (168), player $i-1$ does not exclude a period from $\{T(r, y_n)\}_{y_n}$.

The reason why player $n^*(i)$ excludes one period $t_{n^*(i)}(i \to_{x_i} n^*(i))$ from (168) is to prevent the continuation play of player $n^*(i)$ from revealing player $n^*(i)$'s signal observation too much. This is important to incentivize player $n^*(i)+1$ to tell the truth in the report block.

On the other hand, since $\theta_{i-1}(j \to_m n)$ is not revealed by player $(i-1)$'s continuation play in the main blocks, player $i-1$ does not need to exclude one period here.

The same causion is applicable for the other three inequalities to determine $\theta_{i-1}(j \to_m n)$.

$G_n \mathbf{1}_{y_n}$ using player n's true signal observation):

$$\left\| \frac{1}{T^{\frac{1}{2}}} \sum_{y_n \in Y_n} \sum_{\tau=1}^{T(r,y_n)} \mathbf{\Omega}_{n,\tau}^{G} - \frac{1}{T^{\frac{1}{2}}} \sum_{y_n \in Y_n} \sum_{\tau=1}^{T(r,y_n)} G_n \mathbf{1}_{y_n} \right\| \leq \frac{\varepsilon}{4}.$$

Ex post (after conditioning $\{a_t, y_t\}_{t \in T(r)}$), this is true with probability $1 - \exp(-O(T^{\frac{1}{2}}))$ by the law of large numbers.

3. The frequency of $\left\{ \{\mathbf{\Omega}_{n,\tau}^{J}(m)\}_{\tau=1}^{T(r,y_n)} \right\}_{y_n \in Y_n}$ is close to $\mathbf{r}_n(m)$:

$$\left\| \frac{1}{T^{\frac{1}{2}}} \sum_{y_n \in Y_n} \sum_{\tau=1}^{T(r,y_n)} \mathbf{\Omega}_{n,\tau}^{J}(m) - \mathbf{r}_n(m) \right\| \leq \frac{\varepsilon}{2}.$$

Regardless of player i's deviation, this is the case with probability $1 - \exp(-O(T^{\frac{1}{2}}))$ by Notation 2 and the law of large numbers.

4. The frequency of $\left\{ \{\mathbf{\Omega}_{n,\tau}^{J}(m)\}_{\tau=1}^{T(r,y_n)} \right\}_{y_n \in Y_n}$ is close to $\left\{ \frac{T(r,y_n)}{T} J_n(m) \mathbf{1}_{y_n} \right\}_{y_n}$ (the frequency of $J_n(m)\mathbf{1}_{y_n}$ using player n's true signal observation):

$$\left\| \frac{1}{T^{\frac{1}{2}}} \sum_{y_n \in Y_n} \sum_{\tau=1}^{T(r,y_n)} \mathbf{\Omega}_{n,\tau}^{J}(m) - \frac{1}{T^{\frac{1}{2}}} \sum_{y_n \in Y_n} \sum_{\tau=1}^{T(r,y_n)} J_n(m) \mathbf{1}_{y_n} \right\| \leq \frac{\varepsilon}{4}.$$

Ex post (after conditioning $\{a_t, y_t\}_{t \in T(r)}$), this is true with probability $1 - \exp(-O(T^{\frac{1}{2}}))$ by the law of large numbers.

If player i is in players $-(j, n)$ and at least one of the above four conditions is not satisfied, then player $i-1$ has $\theta_{i-1}(j \to_m n) = B$.

48.4 Summary of the Properties of $g[n-1](m)$, $f[n](m)$ and $\theta_{i-1}(j \to_m n)$

In summary, we can show the following lemma:

Lemma 45 *For sufficiently large T, for any $j \in I$ and $n \in -j$, the above communication protocol satisfies the following:*

1. *$g[n-1](m) = E$ with probability $1 - 2\eta - \exp(-O(T^{\frac{1}{2}}))$ for any $m \in \{G, B\}$.*
2. *Given any $m \in \{G, B\}$ and any $\mathbf{y}_n(r)$, player n puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the event that $f[n](m) = m$ or $g[n-1](m) = E$.*
3. *Given $m \in \{G, B\}$, any $f[n](m)$ happens with probability at least $\exp(-O(T^{\frac{1}{2}}))$.*
4. *The probability of $g[n-1](m)$ being equal to E does not react to player n's strategy by more than $\exp(-O(T^{\frac{1}{2}}))$.*
5. *For $i \in -(j, i)$, whenever player n does not have $f[n](m) = m$, $\theta_{i-1}(j \to_m n) = B$.*
6. *For each $i \in I$, the distribution of $\theta_{i-1}(j \to_m n)$ is independent of player i's strategy with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$.*

Proof.

1. This follows from the law of large numbers.
2. If $f[n](m) = m$, then we are done. Suppose not. Note that the definition of $g[n-1](m)$ implies that $g[n-1](m) = m$ only if $z_j(m) = m$ and (175) and (176) are satisfied. Therefore, $g[n-1](m) = m$ only if $z_j(m) = m$ and $\mathbf{y}_{n-1} \in \mathcal{H}_{n-1}[\varepsilon](m)$.

   $f[n](m) \neq m$ implies that either

   (a) $\mathbf{y}_n(r, T_n(r)) \in \mathcal{H}_{n-1,n}[\varepsilon](m)$ is not the case, or

   (b) player i infers $f[n](m)$ from the likelihood using $\mathbf{y}_n(r, T_n(r))$ (neglecting $y_{n,t_n(r)}$)

   is the case. If (a) is the case, then by Hoeffding's inequality, player n believes that $\mathbf{y}_{n-1} \notin \mathcal{H}_{n-1}[\varepsilon](m)$ given m with probability $1 - \exp(-O(T^{\frac{1}{2}}))$. If (b) is the case, then by Lemma 37, player n believes that $z_j(m) \neq m$ given m with probability $1 - \exp(-O(T^{\frac{1}{2}}))$.

Note that, by Assumption 3 (full support), neglecting $\left(a_{n,t_n(r)}, y_{n,t_n(r)}\right)$ does not affect the posterior so much.

3. Given m, any $(y_t)_{t \in T(r)}$ can occur with probability at least

$$\left\{ \min_{y,a} q(y \mid a) \right\}^{T^{\frac{1}{2}}}.$$

   Assumption 3 (full support) implies that this probability is $\exp(-O(T^{\frac{1}{2}}))$.

4. Ex ante, $g[n-1](m) = E$ with probability $1 - 2\eta - \exp(-O(T^{\frac{1}{2}}))$ regardless of m. Therefore, the probability of $g[n-1](m)$ being equal to E does not react to player n's strategy before round r by more than $\exp(-O(T^{\frac{1}{2}}))$.

   In addition, the distribution of $\mathbf{\Omega}^H_{n-1,t}$ is independent of player n's strategy in period t and (175) is satisfied ex post (conditional on $\{a_t, y_t\}_{t \in T(r)}$) with probability $1 - \exp(-O(T^{\frac{1}{2}}))$ by the law of large numbers. Therefore, the probability of $g[n-1](m)$ being equal to E does not react to player n's strategy in round r by more than $\exp(-O(T^{\frac{1}{2}}))$.

5. Follows from the triangle inequality.

6. For player $i \in \{j, n\}$, $\theta_{i-1}(j \to_m n) = G$ always. If $i \notin \{j, n\}$, then ex ante, $\theta_{i-1}(j \to_m n) = G$ with probability $1 - 2\eta - \exp(-O(T^{\frac{1}{2}}))$ regardless of m. Therefore, the distribution of $\theta_{i-1}(j \to_m n)$ is not changed by more than $\exp(-O(T^{\frac{1}{2}}))$ by player i's strategy before round r.

   In addition, by Notation 2, the distribution of $\mathbf{\Omega}^G_{n,\tau}$ and $\mathbf{\Omega}^J_{n,\tau}(m)$ is independent of player i's strategy in round r and Cases 2 and 4 in the definition of $\theta_{i-1}(j \to_m n)$ is satisfied ex post (conditional on $\{a_t, y_t\}_{t \in T(r)}$) with probability $1 - \exp(-O(T^{\frac{1}{2}}))$ by the law of large numbers. Therefore, the distribution of $\theta_{i-1}(j \to_m n)$ is independent of player i's strategy in round r with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$.

■

49 Equilibrium Strategies

In this section, we define $\sigma_i(x_i)$ and π_i^{main}.

49.1 States

The states $\lambda_i(l+1)$, $\hat{\lambda}_{i-1}(l+1)$, $d_i(l+1)$, $d_j(l+1)(i)$, $c_i(l+1)$, $\theta_i(l)$, $\theta_i(\lambda_j(l+1))$ and $\theta_i(d_j(l+1))$ are defined as in the Supplemental Material 3 except that x is replaced with $x(i)$ defined in Section 46.5.

If we replace the noisy cheap talk with messages via actions, then we use $f[i](m)$ (when player i is a receiver) and $g[i](m)$ (when player $i+1$ is a receiver) defined in Section 48. In addition, each player i makes player $i+1$ indifferent between any action profile sequence if the following events happen:

- In the coordination block, $\theta_i(\text{c}) = B$ happens.
- In a round where player $j \in I$ sends a message m to player $n \in -j$, $\theta_i(j \to_m n) = B$ happens.

49.2 Player i's Action

49.2.1 With the Noisy Cheap Talk

In the coordination block, the players play the game as explained in Section 46. For the other blocks, $\sigma_i(x_i)$ prescribes the same action with x replaced with $x(i)$ except for the report and re-report blocks. See Sections 51 and 52 for the strategy in the report and re-report blocks.

49.2.2 Without the Noisy Cheap Talk

When player $j \in I$ sends a message m to player $n \in -j$, then the strategies are determined in Section 48.

49.3 Reward Function

In this subsection, we explain player $i-1$'s reward function on player i, $\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}} : \delta)$. In general, the total reward $\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}} : \delta)$ is the summation of rewards for each round r:

$$\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}} : \delta) = \sum_{l=1}^{L} \sum_{t \in T(l)} \pi_i^{\delta}(t, \alpha_{-i,t}, y_{i-1,t}) + \sum_r \pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, r : \delta).$$

Note that we add (106) to ignore discounting only for the review rounds. As we will see, for the round where the players communicate, we use reward function that take discounting into account directly.

We define $\pi_i^{\text{main}}(x, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, r : \delta)$ for each r.

49.3.1 With the Noisy Cheap Talk

In the coordination block, for round r where player j sends message x_{i-1} to player $n^*(i)$, player $i-1$ gives

$$\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, r : \delta) = \sum_{t \in T(r)} \delta^{t-1} \pi_i^{x_{i-1}}(j, a_{i-1,t}, y_{i-1,t})$$

to make player i indifferent between any action profile sequence.[102] Note that we take discounting into account.

In the coordination block, for round r where player $n^*(j)$ sends message w_j to player $N(j)$, player $i-1$ gives

$$\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, r : \delta) = \sum_{t \in T(r)} \delta^{t-1} \pi_i^{x_{i-1}}(n^*(j) \to N(j), a_{i-1,t}, y_{i-1,t}).$$

In the main blocks, the reward function is the same as in the Supplemental Material 3

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[102] $\pi_i^{x_{i-1}}(j, a_{i-1,t}, y_{i-1,t})$ is defined in Lemma 38. Here, we use the assumption that the same a_j^G, a_j^B in Assumption 13 satisfy Assumption 15 for each j.

If not, assume that $\left(a_j^G, a_j^B\right)_j$ in Assumption 13 satisfy 6 and 7 in Assumption 15.

except that x replaced with $x(i-1)$ and that if $\theta_{i-1}(\mathrm{c}) = B$ happens, then player $i-1$ uses

$$\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, l) = \sum_{t \in T(l)} \pi_i^{x_{i-1}}(\alpha_{-i,t}, y_{i-1,t})$$

for all the review rounds to make player i indifferent between any action profile.[103]

49.3.2 Without the Noisy Cheap Talk

For round r corresponding to a review round, the reward function is the same as in the case with the noisy cheap talk except that if there is round $\tilde{r} \leq r-1$ (before r) such that player j sends a message m to player n in round $\tilde{r}$ and $\theta_{i-1}(j \rightarrow_m n) = B$ happens, then player $i-1$ uses

$$\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, l) = \sum_{t \in T(l)} \pi_i^{x_{i-1}}(\alpha_{-i,t}, y_{i-1,t})$$

to make player i indifferent between any action profile.

For round r where player j sends a message, player $i-1$ gives

$$\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, r : \delta) = \sum_{t \in T(r)} \delta^{t-1} \pi_i^{x_{i-1}}(j, a_{i-1,t}, y_{i-1,t})$$

defined in Lemma 38. Again, we take discounting into account.

50 Almost Optimality of the Strategy

We want to verity (8), (4) and (5) are satisfied. First, by definition in Section 49.3, (5) is satisfied.

Second, since the length of the rounds other than the review rounds is $T^{\frac{1}{2}}$, the payoff from the review rounds approximately determines the payoff from the review phase for sufficiently large δ (and so sufficiently large T). Therefore, we neglect the payoffs from the rounds other

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[103] Since $\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, l)$ does not depend on δ, with abuse of notation, we omit δ from $\pi_i^{\text{main}}(x_{i-1}, h_{i-1}^{\text{main}}, h_{i-1}^{\text{rereport}}, r : \delta)$ with r corresponding to the lth review round.

than the review rounds.

Third, we consider (8) and (4) in the case with the noisy cheap talk. Suppose that $x(j) = x(i)$ for all $i, j \in I$ at the end of the coordination block. Then, (8) and (4) are shown as in the case with the perfect cheap talk.

This implies that the premises of Lemma 44 are satisfied. Therefore, (i) the incentive in the coordination block is satisfied and (ii) we can concentrate on the case with $x(j) = x(i)$ for all $i, j \in I$.

(i) and (ii) implies (8). In addition, by the law of large numbers, $x(j) = x$ for all $j \in -i$ in the coordination block with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$. Therefore, (4) is satisfied at the beginning of the review phase.

Finally, we consider (8) and (4) in the case without the noisy cheap talk. Again, suppose that $x(j) = x(i)$ for all $i, j \in I$ at the end of the coordination block. Then, (8) and (5) are verified as in the case with the perfect cheap talk except for the following two points:

- Player $i-1$ makes player i indifferent between any action profile sequence because of $g[i-1](m) = E$ or $\theta_{i-1}(j \rightarrow_m n) = B$ with higher probability. However, the probability of $g[i-1](m) = E$ or $\theta_{i-1}(j \rightarrow_m n)$ is bounded by $O(\eta)$. Hence, re-taking η sufficiently small as we do in (141), we can deal with this problem as in the two-player case.

- When player $j \in -i$ sends a message m to player $n \in -(i, j)$, player i can manipulate the distribution of $f[n](m)$. However, Lemma 45 implies that player i cannot manipulate $\theta_{i-1}(j \rightarrow_m n)$. $f[n](m)$ matters for player i's continuation payoff if and only if $\theta_{i-1}(j \rightarrow_m n) = G$. Hence, the relevant events for player i are

    - $f[n](m) = m$ and $\theta_{i-1}(j \rightarrow_m n) = G$, or
    - $f[n](m) \neq m$ or $\theta_{i-1}(j \rightarrow_m n) = B$.

  Since $f[n](m) \neq m$ implies $\theta_{i-1}(j \rightarrow_m n) = B$, the relevant histories for player i are

    - $\theta_{i-1}(j \rightarrow_m n) = G$, or

- $\theta_{i-1}(j \rightarrow_m n) = B$.

  Since player i cannot manipulate $\theta_{i-1}(j \rightarrow_m n)$ by Lemma 45, player i does not have an incentive to manipulate $f[n](m)$.

To verify the incentives in the coordination block, we consider the premises of Lemmas 43 and 44 in the case without the noisy cheap talk.

The premise 1 of Lemmas 43 and premise 1-(a) of Lemma 44 are satisfied by Lemma 45.

As we have mentioned above, when player $j \in -i$ sends a message m to player $n \in -(i,j)$, player i does not have an incentive to manipulate $f[n](m)$. Therefore, the premise 2 of Lemmas 43 and premise 1-(b) of Lemma 44 are satisfied.

We are left to verify the premise 1-(c) of Lemma 44: Player i's value is almost the same between $x_n(j) = G$ for all $j \in I$ and $x_n(j) = B$ for all $j \in I$ regardless of $\{x_{n'}(j)\}_{j \in I, n' \neq n}$. To formally show this, we proceed backward from player N's state. There are following two cases:

- Suppose that $x_{n'}(j) \neq x_{n'}(j')$ happens for some $n' \in \{1, ..., N-1\}$, $j \in I$ and $j' \in -j$. Then, by definition of $\{\theta_{i-1}(j'' \rightarrow_m n'')\}_{j'',n''}$ and 2 of Lemma 44,[104] player i puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the event that $\theta_{i-1}(\mathrm{c}) = B$ in the coordination rounds for $x_{n'}$ or that there exist $j'' \in I$ and $n'' \in -j''$ such that $\theta_{i-1}(j'' \rightarrow_m n'') = B$ happens when player $j'' \in N(n')$ sends a message m to player n'' in the coordination round 3 for $x_{n'}$. Therefore, if $x_{n'}(j) \neq x_{n'}(j')$ happens for some $n' \in \{1, ..., N-1\}$, $j \in I$ and $j' \in -j$, then player i is almost indifferent between any action profile sequence, which implies player i's value is almost constant.

- Suppose that $x_{n'}(j) = x_{n'}(j')$ for all $n \in \{1, ..., N-1\}$ and $j, j' \in I$. Then, if either $x_N(j) = G$ for all $j \in I$ or $x_N(j) = B$ for all $j \in I$ is the case, then we have verified that (4) holds with x replaced with $x(j)$. Since $i \neq N+1$, player i's value is almost the same between $x_N(j) = G$ for all $j \in I$ and $x_N(j) = B$ for all $j \in I$.

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[104] 2 of Lemma 44 does not use the premises 1-(a), 1-(b) and 1-(c).

Therefore, 1-(c) of Lemma 44 holds for $n = N$. This implies that each player follows the equilibrium path in the coordination rounds for x_N. Hence, at the end of the coordination rounds for x_{N-1}, each player i believes that $x_N(j) = x_N$ for all $j \in I$ or $\theta_{i-1}(j \rightarrow_m n) = B$ in the coordination round 3 for x_N between some $j \in I$ and $n \in -j$ with probability no less than $1 - \exp(-O(T^{\frac{1}{2}}))$. Hence, the same argument as for $n = N$ holds for $n = N - 1$. By induction, we are done.

Therefore, all the premises in Lemmas 43 and Lemma 44 are satisfied. This implies that

1. It is almost optimal for player i to follow the equilibrium strategy in the coordination block.

2. For any i, for any t in the main blocks, for any h_i^t, player i puts a belief no less than $1 - \exp(-O(T^{\frac{1}{2}}))$ on the event that $x(j) = x(i)$ for all $j \in -i$ or "$\theta_{i-1}(j \rightarrow_m n) = B$ or $\theta_{i-1}(\mathrm{c}) = B$ happens in the coordination block."

Note that 1 implies the almost optimality of $\sigma_i(x_i)$ in the coordination block and that 2 implies the almost optimality of $\sigma_i(x_i)$ in the main blocks. Hence, (8) is verified.

Since we have verified (4) for $x(j) = x(i)$ for all $i, j \in I$, we are left to show (4) at the beginning of the review phase. Compared to the case with the noisy cheap talk, we need to deal with the fact that $g[n-1](m) = E$ and $\theta_{i-1}(j \rightarrow_m n) = B$ can happen when player j sends a message m to player n in the coordination block with higher probability. However, since the ex ante probability of $\theta_{i-1}(j \rightarrow_m n) = B$ for some $j \in I$, $n \in -j$ and m is bounded by $O(\eta)$, re-taking η smaller if necessary, we are done.

51 Report Block

We are left to construct the report and re-report blocks to attain the exact optimality of the equilibrium strategies. In this section, we explain the report block.

Contrary to the two-player case, we directly construct the report block without public randomization or any cheap talk.

51.1 Structure of the Report Block

The report block proceeds as follows:

1. Player N sends the messages about h_N^{main}.
2. Player $N-1$ sends the messages about h_{N-1}^{main}.

   $\vdots$

3. Player 3 sends the messages about h_3^{main}.
4. As in the two-player case, players 1 and 2 coordinate on which of them will send messages:
   (a) Each player takes a^G and each player i observes her private signal y_i.
   (b) If player 2 observes $y_2 \in Y_{2,1}^2$, then player 2 sends the message that $y_2 \in Y_{2,1}^2$ to player 1. Otherwise, that is, if player 2 observes $y_2 \in Y_{2,2}^2$, then player 2 sends the message that $y_2 \in Y_{2,2}^2$ to player 1.
5. If player 2 has sent the message $y_2 \in Y_{2,1}^2$, then player 2 sends the meaningful messages about h_2^{main}. If player 2 has sent the message $y_2 \in Y_{2,2}^2$, then player 2 takes a_2^G for the periods where player 2 would send the messages about h_2^{main} otherwise.
6. Player 1 sends the message about h_1^{main}.
7. The players play the round for conditional independence.

We explain each step in the sequel.

51.2 Player $i \geq 3$ sends h_i^{main}

Since there is a chronological order for the rounds and r is a generic serial number of rounds, the notations $\#_i^r$, $\#_i^r(k)$, $T(r,k)$ and $\{a_{i,t}, y_{i,t}\}_{t\in T(r,k)}$ defined in the Supplemental Material 3 is still valid.

Player i sends the messages about h_i^{main} in the same way as player 2 sends the messages in the Supplemental Material 4 with two players.

That is, for each round r,

1. First, player i reports $\#_i^r$.
2. Second, player i reports $\{\#_i^r(k)\}_{k\in\{1,\dots,K\}}$. See Section 44.2.2 for the definition of K.
3. Third, players i and $i-1$ coordinate on $k(r)$ as players 2 and 1 coordinate on $k(r)$ in Section 44.2.2.
4. Fourth, player i sends $\{a_{i,t}, y_{i,t}\}_{t\in T(r,k(r,i))}$. $k(r,i)$ is the result of the coordination on $k(r)$.

In Steps 1, 2 and 4, player i sends a message as player 2 does in the Supplemental Material 4 and player $i-1$ interprets the message as player 1 does in the Supplemental Material 4: Player i takes $a_i \in \{a_i^G, a_i^B\}$, player $i-1$ takes $\bar{\alpha}_{i-1}$ and players $-(i-1,i)$ take $a_{-(i-1,i)}^G$. Player $i-1$ constructs $\Phi_{i-1} \in \{0,1\}$ from $\phi_{i-1}(a_{i-1}, y_{i-1})$ as player 1 constructs $\Phi_1 \in \{0,1\}$ from $\phi_1(a_1, y_1)$ in the Supplemental Material 4. Then, player $i-1$ infers player i's message from Φ_{i-1} as player 1 infers player 2's message from Φ_1. Then, from Lemma 39, player i cannot infer Φ_{i-1} from player i's signals.

In Step 3, the coordination between player i and $i-1$ is the same as in Section 44.2.2 with j replaced with $i-1$. Assumption 16 implies that this is a well defined procedure.

51.3 Player 2 sends h_2^{main}

Player 2 sends the messages about h_2^{main} as player $i \geq 3$ if and only if player 2 observed $y_2 \in Y_{2,1}^2$ in Step 4 of Section 51.1. If player 2 observes $y_2 \in Y_{2,2}^2$, then player 2 takes a_2^G for periods where player 2 would send $\#_2^r$, $\{\#_2^r(k)\}_{k\in\{1,\dots,K\}}$ and $\{a_{2,t}, y_{2,t}\}_{t\in T(r,k(r,2))}$ otherwise. In addition, the coordination on $k(r)$ between players 2 and 1 is the same as in the Supplemental Material 4 (with the other players $-(1,2)$ taking $a_{-(i,2)}^G$). Assumption 16 implies that this is a well defined procedure.

As for the case with $i \geq 3$, player 2 takes $a_2 \in \{a_2^G, a_2^B\}$, player 1 takes $\bar{\alpha}_1$ and players $-(1,2)$ take $a_{-(1,2)}^G$. Player 1 constructs $\Phi_1 \in \{0,1\}$ from $\phi_1(a_1, y_1)$ to infer player 2's message. Lemma 39 guarantees that player 2 cannot infer Φ_1 from player 2's signals.

51.4 Player 1 sends h_1^{main}

Player 1 sends the messages about h_1^{main} to player N as player $i \geq 3$. As in the two-player case, player 1 takes $a_1 \in \{a_1^G, a_1^B\}$ and players -1 take a_{-1}^G.

After that, player 1 sends the histories in the report block to player N as player 1 does to player 2 in the round for conditional independence in Section 44.4.1. Again, this set of periods is called "the round for conditional independence." In this round, player 1 takes some action $a_1 \in A_1$ and players -1 take a_{-1}^G. Player N infers this message from y_N. By 7 of Assumption 15, player N can statistically identify player 1's action.[105]

From the history in the round for conditional independence, player N constructs Φ_N. Compared to the two-player case, player 2 is replaced with player N.

51.5 Reward Function π_i^{report}

First, for each i, player $i-1$ gives the reward for player i that cancels out the instantaneous utility. When player $n \in -1$ sends the message about h_n^{main}, player $i-1$ gives

$$\delta^{t-1} \pi_i^{x_{i-1}}(\text{report}, n, a_{i-1}, y_{i-1})$$

to player i. (166) implies the payoff of each player i is constant for any action.

When player 1 reports h_1^{main}, player 1 takes $\left\{a_1^G, a_1^B\right\}$ and players -1 take a_{-i}^G.[106] Hence, by 7 of Assumption 15, for each player i, player $i-1$ can cancel out the differences in player i's instantaneous utilities by

$$\delta^{t-1} \pi_i^{x_{i-1}}(1, a_{i-1}, y_{i-1}).$$

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[105] Since we use Assumption 15, $\left\{a_1^G, a_1^B\right\}$ and a_{-i}^G here are actions defined in Assumption 15, not in Assumption 17. Note that, for notational simplicity, we use the same notations for different assumptions.

[106] Remember that $\left\{a_1^G, a_1^B\right\}$ and a_{-i}^G here are actions defined in Assumption 15.

Next, we consider the reward in the round for conditional independence. As we will see, player 1 sends in the re-report block what action player 1 takes in each period in the round for conditional independence. Hence, for player $i \in -1$, player $i-1$ will know a_1 from the re-report block and player $i-1$ knows that players $-(1,i)$ take $a^G_{-(1,i)}$. For player $i=1$, since players -1 take a^G_{-1}, player $i-1=N$ knows a_{-i} without the messages in the re-report block. For each i, player $i-1$ gives

$$\delta^{t-1}\pi_i^{x_{i-1}}(a_{-i}, y_{i-1})$$

to cancel out the difference of player i's instantaneous utilities.

On the top of that, while the players should take a^G to coordinate on $k(r)$ or whether player 2 reports the history, player $i-1$ incentivizes player i to take a^G_i. Since player $i-1$ knows that players $-i$ take $a^G_{-i} \in A_{-i}$, player $i-1$ can construct a strict reward on a^G_i from Lemma 16.

In the report block, when player i sends the message, no player $j \in -i$ has an incentive to manipulate player $(i-1)$'s inference of player i's message since player i's message only affects player $(i-1)$'s reward on player i and we construct the structure of the report block in Section 51.1 and the punishment for telling a lie, $g_j(h^{\text{main}}_{j-1}, h^{\text{rereport}}_{j-1}, \hat{a}_{j,t}, \hat{y}_{j,t})$, so that player j does not have an incentive to learn player i's history from the report block.

Finally, we construct π_i^{report} that makes $\sigma_i(x_i)$ exactly optimal. This step is the same as in Section 36 except for the following:

- $\varphi_{n,t}$ for each round r is defined as follows:
    - If round r corresponds to the coordination round 1 for x_j with some $j \in I$ where player n infers player j's message x_j (that is, player n is $n^*(j)$), then $\varphi_{n,t}$ is $\Psi^j_{n,t}$ defined in (168).
    - If round r corresponds to the coordination round 1 for x_j with some $j \in I$ where player n is not $n^*(j)$, then $\varphi_{n,t}$ is $\{\emptyset\}$.

  - If round r corresponds to the coordination round 2 for x_j with some $j \in I$ where player n receives a message from $n^*(j)$ (that is, player n is in $N(j) \setminus \{n^*(j)\}$), then $\varphi_{n,t}$ is $\boldsymbol{I}_{n,t}(j)$ defined in (169).
  - If round r corresponds to the coordination round 2 for x_j with some $j \in I$ where player n is not in $N(j) \setminus \{n^*(j)\}$, then $\varphi_{n,t}$ is $\{\emptyset\}$.
  - If round r corresponds to the coordination round 3 or the supplemental round, then $\varphi_{n,t}$ is $\{\emptyset\}$.
  - If round r corresponds to the review round, then $\varphi_{n,t}$ is the same as in Section 36.

- $t_{i-1}(r)$ is not defined for a round in the coordination block or supplemental round if player $(i-1)$'s continuation strategy does not depend on player $(i-1)$'s history in that round. In that case, player $i-1$ randomly picks one.

- For a round in the coordination block where player i takes a mixed strategy to send a message, we (i) first cancel out the effect of the history in the round on learning about the best responses from the next rounds, and (ii) second make any action sequence is indifferent ex ante. Since player i believes that player i is almost indifferent between any strategies whenever player i sends a message, this treatment is the same as we incentivize player i to take a mixed minimaxing strategy in the review round.

We are left to deal with the probability that the message does not transmit correctly with probability 1. We deal with this problem in Section 53 after we explain the re-report block.

52 Re-Report Block

As in Section 37, we introduce the re-report block so that, for each player i, player $i-1$ can collect the information necessary to construct π_i from players $-(i-1,i)$.

The basic structure of the re-report block is the same as in Section 37:

- Players $-(N-1,N)$ sends the information to player $N-1$ to construct π_N.
- Players $-(N-2,N-1)$ sends the information to player $N-2$ to construct π_{N-1}.

  $\vdots$

- Players $-(1,2)$ sends the information to player 1 to construct π_2.
- Players $-(N,1)$ sends the information to player N to construct π_1.

When players $-(i-1,i)$ sends the information to player $i-1$, each player takes turns to send the information:

- Player 1 sends the information to player $i-1$ if $1 \in -(i-1,i)$. If $1 \notin -(i-1,i)$, then skip this step.

  $\vdots$

- Player N sends the information to player $i-1$ if $N \in -(i-1,i)$. If $N \notin -(i-1,i)$, then skip this step.

When player $n \in -(i-1,i)$ sends the information about her history, she sends the following information chronologically:

- For each round r, what strategy α_n player n took in round r. Note that this contains the information about what message player n sent if player n sends a message in that round. The cardinality of this message is no more than

$$|A_n| + \underbrace{N-1}_{\substack{\text{the mixed strategy is taken} \\ \text{only if player } n \text{ sends } z_j(m)=M \\ \text{or minimaxes another player}}}.$$

- For each round r, for each (a_n, y_n, φ_n), how many times player n observed (a_n, y_n, φ_n). Note that this contains the information about what was player n's inference of a message if player n receives a message in that round. The cardinality of this message is no more than $T^{O(1)}$.

- Note that the above two pieces of information is sufficient for player $i-1$ to construct $\theta_{i-1}(j \to_m n)$.
- For each round r, what was $t_n(r)$. The cardinality of this message is no more than T.
- At the end of each lth review round, what was the realization of player n's randomization for the construction of some states. The cardinality of this message is a finite fixed number.
- So that player $i-1$ knows $(a_{-i,t})_{t \in T(r,k(r,i))}$ and $\left(y_{n,t}, \varphi_{n,t}\right)_{t \in T(r,k(r,i))}$,
  - first, for each r, player $i-1$ sends the message about $k(r,i)$ to players $-(i-1,i)$.[107] Each player $n \in -(i-1,i)$ infers $k(r,i)$ from their private signals. Let $k_n(r,i)$ be player n's inference. The cardinality of this message is no more than $T^{\frac{3}{4}}$.
  - Second, player n sends the messages about $\left(a_{n,t}, y_{n,t}, \varphi_{n,t}\right)_{t \in T(r,k_n(r,i))}$ to player $i-1$. The cardinality of this message is $\exp(O(T^{\frac{1}{4}}))$.
- If player n is player 1, then player 1 sends the message about player 1's history in the round for conditional independence: $(a_{1,t}, y_{1,t})$ for all t in the round for conditional independence. Since the length of the round for conditional independence is $S \sum_r \left| T\left(r, \hat{k}_n(r,i)\right) \right| = O(T^{\frac{1}{4}})$, the cardinality of this message is $\exp(O(T^{\frac{1}{4}}))$.

Therefore, the cardinality of the whole message is $\exp(O(T^{\frac{1}{4}}))$ and the length of the sequence of binary messages $\{G, B\}$ necessary to encode the information is $O(T^{\frac{1}{4}})$. To send a binary message $m \in \{G, B\}$, player n repeats a_n^m for $T^{\frac{1}{3}}$ times to increase the precision. The other players $-n$ take a_{-n}^G.[108]

By 7 of Assumption 15, player $i-1$ can statistically identify player n's action. Also, for each player j, player $j-1$ can cancel out the differences in player j's instantaneous utilities by

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[107] We assume that player $i-1$ knew player i's inference $k(r,i)$. See Section 53 for how to deal with the small probability that player $i-1$ misinterprets player i's message about $k(r,i)$.

[108] Since we use Assumption 15, $\left\{a_1^G, a_1^B\right\}$ and a_{-i}^G here are actions defined in Assumption 15, not in Assumption 17. Note that, for notational simplicity, we use the same notations for different assumptions.

the reward. The incentive to tell the truth is automatically satisfied since player n's message is used only for the reward on player i with $i \neq n$ except for the round for conditional independence. The incentive in the round for conditional independence is established as in Lemma 31.

53 The Probability of Errors in the Report and Re-Report Blocks

Note that the cardinality of the whole messages in the report and re-report blocks is $\exp(O(T^{\frac{1}{4}}))$. Hence, the length of the sequence of binary messages $\{G, B\}$ that each player takes to send the messages in the report or re-report block is $O(T^{\frac{1}{4}})$.

Since all the messages transmit correctly with probability at least

$$1 - O(T^{\frac{1}{4}}) \exp(-O(T^{\frac{1}{3}})),$$

by the same treatment as in Section 44.3, we can assume as if all the messages would transmit correctly. We do not apply this procedure for the messages in the round for conditional independence. As seen in Lemma 31, the incentive in the round for conditional independence is established taking into account the probability of mis-transmission.