Coordination Failure in Repeated Games with Private Monitoring

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Abstract

Players coordinate continuation play in repeated games with public monitoring. This paper asks the robustness of such equilibrium play with respect to private-monitoring perturbations that are ex-ante close to the public-monitoring structure. We show that, in two-player games with full support of public signals, no perfect public equilibrium is robust to private-monitoring perturbations if it induces a "regular" 2×2 coordination game in the continuation play. This non-robustness result does not apply to belief-free equilibria, which violate the regularity condition. Indeed, we show that, in two-player games with an individual rank condition on public signals, every interior belief-free equilibrium is robust to private-monitoring perturbations. We also argue by means of an example that the non-robustness result is sensitive to the assumption that every private signal must be interpreted as some public signal with probability 1, and not with probability close to 1.

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1 Introduction

In repeated games with public monitoring, there is common knowledge about past public signals, and thus the players can take an action profile that is not a static equilibrium by coordinating continuation strategies contingent on public histories. In many cases, however, players observe slightly different signals due to private shocks. The purpose of this paper is to investigate whether and how equilibrium construction in public-monitoring repeated games depends on the common knowledge assumption about past histories. More specifically, we ask if players can maintain coordination approximately if the monitoring structure is perturbed so that there is no longer common knowledge about past signals.

We formalize this question as follows. Fix a repeated game with public monitoring. Perturb the game so that players observe noisy private signals. Each player has an "interpretation" of signals, which is a function that maps her private signals to public signals. A private-monitoring structure is close to the public-monitoring structure under a profile of interpretations if, conditional on each action profile, the probability that all players observe private signals that are interpreted as a common public signal y under private monitoring is close to the probability that they observe y under public monitoring. Note that we measure proximity between the two monitoring structures only in the ex-ante sense, and we do not impose any restriction on a player's interim beliefs about the opponents' signals conditional on her own signals. For example, Hörner and Olszewski (2006) analyze repeated games with almost perfect monitoring, which is close to perfect monitoring in the ex-ante sense. Another example is the Cournot oligopoly game, where firms produce almost homogeneous outputs, and "market price" θ is drawn from a continuous distribution. Each firm i does not observe θ directly, but observes a firm-specific price:

$$\omega_i = \theta + \varepsilon_i.$$

This monitoring structure is close to the public monitoring where firms observe binary public signals about high or low prices, $\{\theta \geq \bar{\theta}\}$ or $\{\theta < \bar{\theta}\}$, as long as noise terms ε_i's concentrate around 0.[1]

Given the above class of perturbations, we say that an equilibrium of a public-monitoring repeated game is robust to private-monitoring perturbations if, for every private-monitoring perturbation close to the original public-monitoring structure under a profile of interpretations, there exists an equilibrium of the perturbed game close to the translation of the original equilibrium into private strategies via the same or similar profile of interpretations.

In two-player games with full support of public signals, we show that no perfect public equilibrium (henceforth PPE) is robust to private-monitoring perturbations under a certain regularity condition that excludes belief-free equilibria and in particular, repetitions of static Nash equilibria (Theorem 1). The analysis of PPE in games with public monitoring has flourished since Abreu, Pearce, and Stacchetti (1990) and Fudenberg, Levine, and Maskin (1994). Theorem 1 reveals that such analysis depends critically on the common knowledge assumption on past histories. Except for PPEs that violate the regularity condition, no PPE can be translated into equilibria in all perturbed games; either a perturbed game has no equilibrium that is close to PPEs, or the way to translate a PPE depends on fine details of perturbations.

The proof of Theorem 1 goes as follows. Suppose, for simplicity, there are two public signals $\bar{y}$ and $\underline{y}$. Take any PPE, and imagine we are at the beginning of period 2. By the definition of PPE, it is optimal for each player to follow the continuation strategy after $\bar{y}$ (resp. $\underline{y}$) if the opponent also takes the continuation strategy corresponding to $\bar{y}$ (resp. $\underline{y}$). Now perturb the monitoring structure. At the beginning of period 2, each player decides an interpretation of her private signal in period 1, and follows the continuation strategy that corresponds to the interpretation. This induces a 2×2 coordination game, where the players simultaneously choose their interpretations and payoffs are given by corresponding

[1] Mailath and Morris (2002, 2006) analyze a closely related notion of almost public monitoring. They, however, exclude almost perfect monitoring by assuming full support of the original monitoring structure. They also exclude the Cournot example because their notion of closeness measures distances among conditional probabilities from the interim perspective. See Subsection 1.1.

continuation payoffs in the repeated game. Then, under a regularity condition, either $(\bar{y}, \bar{y})$ or $(\underline{y}, \underline{y})$ is a strictly risk-dominant profile in the 2×2 game, and the profile that is risk-dominant prevails for all private signals in period 1 when each player is uncertain about the opponent's choice of interpretation, as in the contagion argument given by Rubinstein (1989) and Carlsson and van Damme (1993). This makes it impossible to maintain players' incentives to play non-equilibrium actions in period 1.

Thus the basic part of the proof is simple. At the technical level, however, we have two more issues. First, PPE may prescribe players to play mixed actions in period 1, which induce a rather complicated prior over the product set of actions and private signals. Here we use the full-support assumption and show that the contagion argument applies to the signal space independently of the realization of the action profile. Second, since signals do not carry payoff-relevant information in the repeated game, a dominance region, where a particular interpretation of signals is a dominant choice in the 2×2 game, does not exist in our setting. For this reason, we show the existence of a region with non-trivial size where the players follow the risk-dominant interpretation with large probability. These issues are addressed in Lemma 4.

Note that Theorem 1 depends on the regularity condition, which allows us to identify which interpretation, $\bar{y}$ or $\underline{y}$, is strictly risk-dominant in the 2×2 game. The regularity condition is satisfied in many PPEs, but violated in all belief-free equilibria, where each player is indifferent between the two interpretations independently of the opponent's interpretation. Indeed, we show that, in two-player games with an individual rank condition on public signals, every interior belief-free equilibrium is robust to private-monitoring perturbations (Theorem 2).

Belief-free equilibria are arguably one of the most important findings in the literature on private-monitoring repeated games, and have been extensively investigated in the last decade.[2] Given the widespread use of belief-free equilibria, Theorem 2 is probably a "folklore

[2]Belief-free equilibria are first constructed by Piccione (2002) in the repeated prisoners' dilemma, and simplified by Ely and Välimäki (2002). Kandori and Obara (2006) consider belief-free equilibria in private

theorem" and hardly surprising. For example, Theorem 2 can be seen as a generalization of Ely, Hörner, and Olszewski (2005, Lemma 3).[3] Nevertheless, it should be noted that Theorem 2 assures that the notion of robustness, as defined in this paper, is not absurdly restrictive.

Combining Theorems 1 and 2, one can interpret our exercise as a first step toward characterizing belief-free equilibria in terms of their robustness properties. That is, we show that our notion of robustness is almost necessary and almost sufficient for an equilibrium to be belief-free (as long as we regard both (i) non-interior belief-free equilibria and (ii) non-regular non-belief-free equilibria as negligible).

Before moving on to the literature review, we point out a subtlety in the definition of robustness. That is, in Theorem 1, we implicitly require that every private signal must be interpreted as some public signal with probability 1. Suppose instead that, after observing private signals that realize with small probability, players can play continuation strategies that do not resemble original continuation strategies contingent on any public signal. Then the proof of Theorem 1 fails and the contagion of risk-dominant interpretation stops near such "uninterpretable" signals. Indeed, we can construct an equilibrium in the repeated prisoners' dilemma with public monitoring that satisfies the regularity condition but can be approximated by equilibria in perturbed games if we allow for uninterpretable signals with small probability. See Section 3.3. This example suggests that our non-robustness result is not as straightforward an application of the contagion argument as it might appear to be at the first glance.

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strategies in the prisoners' dilemma with public-monitoring. Ely, Hörner, and Olszewski (2005) give a general definition of belief-free equilibrium and characterize the equilibrium payoff set in two-player games. Yamamoto (2009) extends the characterization to general n-player games. Hörner and Olszewski (2006) use "block belief-free" equilibria to show the folk theorem in games with almost-perfect monitoring.

[3]Note, however, that Ely, Hörner, and Olszewski (2005, Lemma 3) analyze the limit case with almost perfect monitoring and sufficiently patient players. They also focus on the robustness of belief-free equilibria in bang-bang strategies.

1.1 Literature Review

Most closely related to our paper are Mailath and Morris (2002, 2006).[4] In those papers, Mailath and Morris require that private-monitoring perturbations be small not only in the ex-ante sense but also in the interim sense. That is, conditional on that a player observes a private signal interpreted as a public signal, the probability that the other players observe private signals interpreted as the same public signal is close to 1. Observing that the interim closeness renders almost-common knowledge about relevant histories of finite length, Mailath and Morris (2002) extend Monderer and Samet (1989) to dynamic environments and show that a strict PPE is robust to almost-public-monitoring perturbations if the PPE has bounded recall. Also, Mailath and Morris (2006) argue that bounded recall is "essentially" necessary for robustness. In our setting, however, since private-monitoring perturbations are small only in the ex-ante sense, conditional on that a player observes a private signal interpreted as a public signal, the probability that the others observe private signals interpreted as the same public signal may be bounded away from 1. Given the larger class of perturbations, our non-robustness result applies to a wider class of equilibrium strategies. In particular, even strict PPEs with bounded recall are not necessarily robust in our sense.

In repeated games with private monitoring, Pęski (2009a) shows that there is no equilibrium other than repetitions of static equilibria if (i) the private-monitoring structure is sufficiently "connected" in the sense that a player's belief about the opponents' signals conditional on her signal changes continuously in her signal, (ii) the continuation strategies depend only on finite partitions of the past histories, and (iii) there are smooth i.i.d. payoff shocks.[5] There are several differences from our non-robustness result, but the most notable one is that we do not use i.i.d. payoff shocks.

Fudenberg and Olszewski (2011) analyze repeated games with a long-run player against a sequence of short-run players under asynchronous monitoring, where each player observes underlying public signals of actions at random and privately known times. They show that

[4] See also Hörner and Olszewski (2009) and Mailath and Olszewski (2011).

[5] See also Pęski (2009b).

the best "cutoff" equilibrium payoff can be strictly lower under asynchronous monitoring than under synchronous monitoring. To see this result, suppose that the long-run player observes a signal near but above the cutoff. Since short-run players who observe signals at a different timing than the long-run player observe signals with a different noise, the long-run player believes with probability close to 1/2 that the next short-run player has observed a signal below the cutoff and switched to the punishment phase. This diminishes the long-run player's incentive to maintain the cooperation phase. This intuition is similar to the contagion argument we use in our non-robustness result.

In the class of repeated games with perfect monitoring, Ely (2002) introduces the notion of weak robustness, and shows in his Theorem 1 that, there is no strict equilibrium that is weakly robust, except for repetitions of static equilibria.[6] This non-robustness result relies on conditionally independent monitoring, and is logically independent of our non-robustness result, which assumes full-support public monitoring, and can apply to mixed-strategy (hence non-strict) equilibria.

To close the Introduction, we must mention that, although belief-free equilibria are robust to private-monitoring perturbations, they may not be robust to other kinds of perturbations. Applying the purification exercise à la Harsanyi (1973) to the repeated prisoners' dilemma, Bhaskar, Mailath, and Morris (2008) argue that the belief-free equilibrium constructed by Ely and Välimäki (2002) is not robust to smooth i.i.d. payoff perturbations.[7]

The rest of the paper is organized as follows. Section 2 defines repeated games with public monitoring and their private-monitoring perturbations. Section 3 gives three examples, which motivate us to formalize the notion of robustness to private-monitoring perturbations in Section 4. Sections 5 and 6 show Theorems 1 (non-robustness) and 2 (robustness), respectively.

[6] See also Matsushima (1991).

[7] See also Bhaskar (1998) and Bhaskar, Mailath, and Morris (2010).

2 Framework

2.1 Repeated Games with Public Monitoring

We begin our analysis by describing a repeated game with public monitoring. Let $I = \{1, \ldots, n\}$ be the set of players with $n \geq 2$. For each $i \in I$, A_i is a finite set of actions available to player i in each period. If the players play an action profile $\mathbf{a} = (a_1, \ldots, a_n) \in A := A_1 \times \cdots \times A_n$, then a public signal y is drawn from a finite set Y with probability $\rho(y \mid \mathbf{a})$, and each player i receives ex-post payoff $u_i^*(a_i, y)$, which induces ex-ante expected payoff $u_i(\mathbf{a}) := \sum_{y \in Y} \rho(y \mid \mathbf{a}) u_i^*(a_i, y)$.

The repeated game with public monitoring is the infinite repetition of this game. A history of player i at the beginning of period t is given by $\tilde{h}_{i,t} = (a_{i,1}, y_1, \ldots, a_{i,t-1}, y_{t-1}) \in \tilde{H}_{i,t} := (A_i \times Y)^{t-1}$. A public component is given by $\tilde{h}_t = (y_1, \ldots, y_{t-1}) \in \tilde{H}_t := Y^{t-1}$, which is commonly known at the beginning of period t. A strategy of player i is a function $\tilde{s}_i$ from $\tilde{H}_i := \bigcup_t \tilde{H}_{i,t}$ to ΔA_i. The continuation strategy of $\tilde{s}_i$ after $\tilde{h}_i \in \tilde{H}_i$ is denoted by $\tilde{s}_i \mid \tilde{h}_i$. Let $U_i(\tilde{\mathbf{s}})$ denote the discounted expected payoff of player i with discount factor δ when players obey a profile $\tilde{\mathbf{s}} = (\tilde{s}_1, \ldots, \tilde{s}_n)$. A strategy profile $\tilde{\mathbf{s}}$ is a *sequential equilibrium* if, for each i and $\tilde{h}_i \in \tilde{H}_i$, the continuation strategy $\tilde{s}_i \mid \tilde{h}_i$ is a best response to the distribution of $\tilde{s}_{-i} \mid \tilde{h}_{-i} := (\tilde{s}_1 \mid \tilde{h}_1, \ldots, \tilde{s}_{i-1} \mid \tilde{h}_{i-1}, \tilde{s}_{i+1} \mid \tilde{h}_{i+1}, \tilde{s}_n \mid \tilde{h}_n)$ given some belief over $\tilde{h}_{-i}$ (conditional on $\tilde{h}_i$) that is consistent with $\tilde{\mathbf{s}}$.

A strategy is *public* if it only depends on the public component of the history: $\tilde{s}_i \colon \tilde{H} := \bigcup_t \tilde{H}_t \to \Delta A_i$. A public-strategy profile $\tilde{\mathbf{s}}$ is a *perfect public equilibrium* (*PPE*) if, for each $\tilde{h} \in \tilde{H}$, the induced continuation strategy profile $\tilde{\mathbf{s}} \mid \tilde{h} = (\tilde{s}_1 \mid \tilde{h}, \ldots, \tilde{s}_n \mid \tilde{h})$ is a Nash equilibrium of the repeated game, i.e., $U_i(\tilde{\mathbf{s}} \mid \tilde{h}) \geq U_i(\tilde{s}'_i, \tilde{\mathbf{s}}_{-i} \mid \tilde{h})$ for all $i \in I$, $\tilde{h} \in \tilde{H}$, and $\tilde{s}'_i$.

The public-monitoring structure (Y, ρ) has *full support* if $\rho(y \mid \mathbf{a}) > 0$ for all $\mathbf{a} \in A$ and $y \in Y$.

2.2 Private-Monitoring Perturbations

The objective of this paper is, given an equilibrium of the repeated game with public monitoring, to check the robustness of the equilibrium to private-monitoring perturbations. For this purpose, we consider private-monitoring perturbations as follows.

A perturbed game is a repeated game with private monitoring. In each period, if the players play an action profile $\mathbf{a} \in A$, a profile $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ of private signals is drawn from a product $\Omega := \Omega_1 \times \cdots \times \Omega_n$ of measurable spaces according to joint probability $\pi(\cdot \mid \mathbf{a})$. Each player i observes $\omega_i \in \Omega_i$, interprets it as $f_i(\omega_i) \in Y$, and receives ex-post payoff $u_i^*(a_i, f_i(\omega_i))$, where f_i is a measurable function from Ω_i to Y. We call f_i player i's signal interpretation. Let $\mathbf{f} = (f_1, \ldots, f_n)$.[8]

A private history of player i at the beginning of period t is given by $h_{i,t} = (a_i^{t-1}, \omega_i^{t-1}) \in H_{i,t} := (A_i \times \Omega_i)^{t-1}$, which consists of player i's past actions $a_i^{t-1} = (a_{i,1}, \ldots, a_{i,t-1})$ and private signals $\omega_i^{t-1} = (\omega_{i,1}, \ldots, \omega_{i,t-1})$. A private strategy s_i of player i is a measurable function from $\bigcup_t H_{i,t}$ to ΔA_i.

For notational convenience, given a history $h_{i,t} = (a_i^{t-1}, \omega_i^{t-1})$ in the private-monitoring perturbation and an interpretation function $f_i \colon \Omega_i \to Y$, let $f_i(h_{i,t}) = (a_{i,1}, f_i(\omega_{i,1}), ..., a_{i,t-1}, f_i(\omega_{i,t-1}))$ be the corresponding history in the original game.

We define the closeness of private monitoring to public monitoring in the following sense.

Definition 1. For $\varepsilon \geq 0$, a private-monitoring structure $(\Omega, \pi, \mathbf{f})$ is *ex-ante ε-close to the public-monitoring structure* (Y, ρ) if

$$|\pi(\{\boldsymbol{\omega} \in \Omega : f_i(\omega_i) = y \text{ for all } i \in I\} \mid \mathbf{a}) - \rho(y \mid \mathbf{a})| \leq \varepsilon$$

for all $\mathbf{a} \in A$ and $y \in Y$.

This definition says that the probability that every player observes a signal interpreted as y in the private-monitoring structure is close to the probability that every player observes

[8] One can extend all the results in this paper to non-stationary perturbations, where the monitoring structure $(\Omega_t, \pi_t, \mathbf{f}_t)$ is indexed by t.

y in the public-monitoring structure. Note that the definition of ex-ante closeness does not impose any restriction on a player's interim beliefs about the opponents' signals conditional on her own signals. This is in contrast with the approach by Mailath and Morris (2002, 2006), who require both ex-ante closeness (Definition 1) and interim closeness, that is,

$$\pi(\{\boldsymbol{\omega}_{-i} \in \Omega_{-i} : f_j(\omega_j) = y \text{ for all } j \neq i\} \mid \mathbf{a}, \omega_i) \geq 1 - \varepsilon$$

for all $i \in I$, $\mathbf{a} \in A$, and $\omega_i \in f_i^{-1}(y)$ whenever the conditional probability is well defined.[9,10]

3 Examples

In this section, we give three examples to illustrate how private-monitoring perturbations prevent players from coordinating their future play. These examples also motivate the formal definition of robustness in Section 4.

3.1 One-Period Punishment

The first example is the prisoners' dilemma game with two signals, taken from Mailath and Morris (2002, Section 3.1). Let $I = \{1, 2\}$, $A_1 = A_2 = \{C, D\}$, and $Y = \{\underline{y}, \overline{y}\}$. Each signal realizes with probability

$$\rho(\overline{y} \mid \mathbf{a}) = \begin{cases} p & \text{if } \mathbf{a} = CC, \\ q & \text{if } \mathbf{a} = CD \text{ or } DC, \\ r & \text{if } \mathbf{a} = DD, \end{cases} \qquad \rho(\underline{y} \mid \mathbf{a}) = 1 - \rho(\overline{y} \mid \mathbf{a}).$$

[9] Mailath and Morris (2006) extend the definition of interpretations so that f_i is a measurable function from Ω_i to $Y \cup \{\emptyset\}$. Here, $f_i(\omega_i) = \emptyset$ means that ω_i is an "uninterpretable" signal with no corresponding public signal. We will discuss the issue of uninterpretable signals in Section 3.3.

[10] Mailath and Morris (2002) assume that $\Omega_i = Y$ and f_i is the identity function for every $i \in I$. In this case, ex-ante closeness implies interim closeness if (Y, ρ) has full support.

We assume that $0 < p < 1$ so that both signals occur with positive probabilities when the players play CC. The ex post utility function $u_i^*(a_i, y)$ is represented by the following matrix:

	$\overline{y}$	$\underline{y}$
C	$\frac{3-p-2q}{p-q}$	$-\frac{p+2q}{p-q}$
D	$\frac{3(1-r)}{q-r}$	$-\frac{3r}{q-r}$

Then, the ex-ante stage-game payoffs are given by

	C	D
C	$2, 2$	$-1, 3$
D	$3, -1$	$0, 0$

We consider the following *public* strategy:

$$\tilde{s}_i(h_t) = \begin{cases} C & \text{if } t = 1 \text{ or } y_{t-1} = \overline{y}, \\ D & \text{if } t \geq 2 \text{ and } y_{t-1} = \underline{y}. \end{cases}$$

Under this strategy, players punish each other for one period immediately after $\underline{y}$ is observed. We assume $r \geq q$ and $\delta \geq (3p - 2q - r)^{-1}$ so that the profile $\tilde{\mathbf{s}} = (\tilde{s}_1, \tilde{s}_2)$ is a *perfect public equilibrium.*

Given the repeated prisoners' dilemma and the one-period-punishment strategy above, we define the following 2×2 game:

$$G(\tilde{\mathbf{s}}) = \begin{array}{c|cc} & \overline{y} & \underline{y} \\ \hline \overline{y} & U_1(\tilde{s}_1 \mid \overline{y}, \tilde{s}_2 \mid \overline{y}), U_2(\tilde{s}_1 \mid \overline{y}, \tilde{s}_2 \mid \overline{y}) & U_1(\tilde{s}_1 \mid \overline{y}, \tilde{s}_2 \mid \underline{y}), U_2(\tilde{s}_1 \mid \overline{y}, \tilde{s}_2 \mid \underline{y}) \\ \underline{y} & U_1(\tilde{s}_1 \mid \underline{y}, \tilde{s}_2 \mid \overline{y}), U_2(\tilde{s}_1 \mid \underline{y}, \tilde{s}_2 \mid \overline{y}) & U_1(\tilde{s}_1 \mid \underline{y}, \tilde{s}_2 \mid \underline{y}), U_2(\tilde{s}_1 \mid \underline{y}, \tilde{s}_2 \mid \underline{y}) \end{array},$$

where each player i chooses an action from $\{\overline{y}, \underline{y}\}$, and she receives payoff $U_i(\tilde{\mathbf{s}}_i \mid y_i, \tilde{\mathbf{s}}_j \mid y_j)$ when she plays y_i and the opponent $j \neq i$ plays y_j. Note that the values along the diagonal are the players' continuation payoffs in the public-monitoring repeated prisoners' dilemma after corresponding public histories, whereas the values off the diagonal are the payoffs when the players fail to coordinate on their future play, which play no clear role in the analysis of public monitoring without perturbations. Since $\tilde{\mathbf{s}}$ is a symmetric PPE, $G(\tilde{\mathbf{s}})$ is a symmetric coordination game with two pure-strategy equilibria $(\overline{y}, \overline{y})$ and $(\underline{y}, \underline{y})$. We assume that one strictly risk-dominates the other, i.e.,

$$U_i(\tilde{s}_i \mid \overline{y}, \tilde{s}_j \mid \overline{y}) + U_i(\tilde{s}_i \mid \overline{y}, \tilde{s}_j \mid \underline{y}) \neq U_i(\tilde{s}_i \mid \underline{y}, \tilde{s}_j \mid \overline{y}) + U_i(\tilde{s}_i \mid \underline{y}, \tilde{s}_j \mid \underline{y})$$

for $i \neq j$.

We perturb the monitoring structure as follows. Each action profile $\mathbf{a} \in A$ generates a real number θ, which is uniformly distributed on $[p-1, p]$ if $\mathbf{a} = CC$, on $[q-1, q]$ if $\mathbf{a} = CD$ or DC, and on $[r-1, r]$ if $\mathbf{a} = DD$. As in Carlsson and van Damme (1993), each player i observes a noisy signal ω_i about θ:

$$\omega_i = \theta + \xi_i,$$

where ξ_1 and ξ_2 are uniformly distributed on $[-\varepsilon/2, \varepsilon/2]$ independently of θ. Player i interprets the signal ω_i as $\overline{y}$ if $\omega_i \geq 0$, and as $\underline{y}$ if $\omega_i < 0$.[11] Note that this perturbation is ex-ante ε-close to the original public-monitoring structure. Call this perturbation the *global-game perturbation* (with ε).

We will show that the perturbed game has no equilibrium that approximates the one-period-punishment strategy. For the sake of concreteness, we focus on the following threshold strategy: in period 1, play C; in period $t \geq 2$, play C if and only if $\omega_{i,t-1} \geq \bar{\omega}$.

Claim 1. *There exist $\eta > 0$ and $\bar{\varepsilon} > 0$ such that, if $0 < \varepsilon \leq \bar{\varepsilon}$ and $|\bar{\omega}| \leq \eta$, then the threshold strategy with threshold $\bar{\omega}$ is not an equilibrium of the perturbed game.*

Proof. Pick sufficiently small $\eta, \bar{\varepsilon} > 0$. Suppose that $(\overline{y}, \overline{y})$ strictly risk-dominates $(\underline{y}, \underline{y})$ in $G(\tilde{\mathbf{s}})$. (A similar argument holds if $(\underline{y}, \underline{y})$ strictly risk-dominates $(\overline{y}, \overline{y})$ in $G(\tilde{\mathbf{s}})$.) Consider player i's private history at the end of period 1 such that she observes $\omega_{i,1}$ below but sufficiently close to $\bar{\omega}$. She believes that the opponent j observes $\omega_{j,1}$ above $\bar{\omega}$ with probability close to 1/2, and hence expects that the opponent follows $\tilde{s}_j \mid \overline{y}$ and $\tilde{s}_j \mid \underline{y}$ with almost equal

[11] Here, players receive continuous signals. One can instead use discrete signals drawn from finite or countably infinite signal spaces, similar to those in Rubinstein's (1989) e-mail game.

probabilities in the future. Since $(\overline{y}, \overline{y})$ risk-dominates $(\underline{y}, \underline{y})$ in $G(\tilde{\mathbf{s}})$, player i strictly prefers $\tilde{s}_i \mid \overline{y}$ to $\tilde{s}_i \mid \underline{y}$, which implies that the threshold strategy is not an equilibrium. ∎

It is clear from the proof that Claim 1 does not depend on particular choices of payoffs or strategies. The proof only uses the fact that the induced game $G(\tilde{\mathbf{s}})$ has a strictly risk-dominant equilibrium. In Section 5, we extend Claim 1 to other PPEs and show the non-robustness of such equilibria even if we allow for non-threshold strategies in perturbed games.

Note that, since the one-period-punishment strategy has bounded recall, it is robust to private-monitoring perturbations that are both ex-ante and interim close to the public-monitoring (Mailath and Morris, 2002). Thus, Claim 1 exemplifies that the notion of robustness is sensitive to the class of perturbations to which robustness is tested.

3.2 The Ely-Välimäki Equilibrium

Consider the repeated prisoners' dilemma with the same ex-ante payoff matrix as before, but assume perfect monitoring this time. The following strategy is adapted from Ely and Välimäki (2002):

$$\tilde{s}_i(\tilde{h}_t) = \begin{cases} \alpha C + (1-\alpha)D & \text{if } t = 1 \text{ or } a_{j,t-1} = C, \\ \beta C + (1-\beta)D & \text{if } t \geq 2 \text{ and } a_{j,t-1} = D. \end{cases}$$

In this strategy, if a player plays C, it costs her 1 in the current period, but induces the opponent to play C in the next period with additional probability $\alpha - \beta$. Therefore, if $\alpha - \beta = 1/(3\delta)$, then she is indifferent between C and D, hence this strategy becomes a PPE.

This equilibrium is *belief-free*: for each i and t, whether player j will take $\alpha C + (1-\alpha)D$ or $\beta C + (1-\beta)D$ in period t, as long as the ex ante probability that player j will play $\alpha C + (1-\alpha)D$ in period $t+1$ is 1 after observing $a_{i,t} = C$ and the ex ante probability that player j will play $\beta C + (1-\beta)D$ in period $t+1$ is 1 after observing $a_{i,t} = D$, player i is indifferent between C and D in period t. That is, player i has (weak) incentives to follow the equilibrium without coordinating her play with player j.

We have the following.

Claim 2. *If $\alpha < 1$, $\beta > 0$, and $\alpha - \beta = 1/(3\delta)$, then the Ely-Välimäki equilibrium $\tilde{\mathbf{s}} = (\tilde{s}_1, \tilde{s}_2)$ is robust to private-monitoring perturbations in the following sense. For every $T < \infty$, there exists $\varepsilon > 0$ such that, for every private-monitoring perturbation $(\Omega, \pi, \mathbf{f})$ that is ex-ante ε-close to (Y, ρ), the perturbed game admits an equilibrium $\mathbf{s} = (s_1, s_2)$ such that*

$$s_i(h_{i,t}) = \tilde{s}_i(f_i(h_{i,t}))$$

for all $t \leq T$, $i \in I$, and $h_{i,t} \in H_{i,t}$.

Sketch of Proof. As explained above, in the Ely-Välimäki equilibrium, each player faces the trade-off between payoffs in the current and the next periods. This trade-off is independent of whether the opponent is expected to play C with probability α or β (or an in-between probability in expectation) in the current period as long as the ex ante probability of taking $\alpha C + (1-\alpha)D$ is 1 after observing $a_{i,t} = C$ and the ex ante probability of taking $\beta C + (1-\beta)D$ is 1 after observing $a_{i,t} = D$. Therefore, even in the perturbed game, the player remains approximately indifferent between C and D regardless of her beliefs about the opponent's continuation strategies.

We need a slightly more careful argument, however. In the perturbed game, since the marginal probability of $f_j(\omega_j)$ can be slightly different. In that case, ex-ante payoffs are also slightly different from those in the original game. Also, a player's action leads to rewards or punishments in the next period with probabilities slightly different from those under perfect monitoring. Therefore, to keep the player being exactly indifferent between C and D, her continuation payoffs must be adjusted appropriately. We make this adjustment by modifying strategies after period T. This is possible as long as ε is small and we have enough slacks in all the variables, which are guaranteed by assuming $\alpha < 1$ and $\beta > 0$ (and hence $\delta > 1/3$). ∎

For a complete proof, see Theorem 2, which generalizes Claim 2.

The intuition behind the robustness is straightforward: as we have seen in Claim 1, the coordination of the future play based on the past signals is difficult in the perturbed game.

The belief-free equilibrium, on the other hand, does not require the coordination based on the past signals and all what is important is the ex ante probability. Therefore, it is robust to private-monitoring perturbations that are ex-ante close to the public monitoring structure.

However, as the second paragraph of the sketch of the proof indicates, for Claim 2, we need to allow for small departures from the original equilibrium to make adjustments in continuation payoffs. This motivates us to introduce (η, T)-closeness in the next section, which basically requires that the equilibrium should be close until period T but the continuation play after period T can be completely different.

3.3 Threshold Strategies with Buffers

In Claim 1, we require that each player always interpret her private signal as some public signal and follow the same continuation strategy as that in the original game via the interpretation. Instead, if we weakened the notion of robustness so that a small "buffer area" were allowed between the region where the signals are interpreted as $\bar{y}$ and the region where the signals are interpreted as $\underline{y}$, and continuation strategies after signals in that buffer area were not restricted at all, then the proof we used in Claim 1 based on the contagion argument would fail. This opens the possibility that there may be an equilibrium in the perturbed game that approximates the original equilibrium. In what follows, we argue that it is indeed the case by constructing an equilibrium in the repeated prisoners' dilemma with public monitoring (but different from the one-period-punishment strategy) such that robustness is sensitive to whether we allow for such a "buffer area" or not.[12,13]

Let us explain the "buffer area" more specifically. Fix some strategy profile $\tilde{\mathbf{s}}$ in a game with public monitoring. Take $\eta > 0$ arbitrarily. Consider the global-game perturbation with $\varepsilon > 0$, as we introduced in Subsection 3.1.[14] Suppose we only require that player i

[12]The following example uses a private-strategy equilibrium and not a PPE.

[13]The reader is recommended to skip this subsection on first reading, for it is long and will not be used in the rest of the paper.

[14]Although we restrict ourselves to global-game perturbations for the sake of concreteness, the argument

interpret ω_i as $\bar{y}$ if $\omega_i \geq \varepsilon/2$; $\underline{y}$ if $\omega_i \leq -\varepsilon/2$. Given these interpretations, her strategy s_i in the perturbed game satisfies $|s_i(h_{i,t}) - \tilde{s}_i(f_i(h_{i,t}))| < \eta$ for all $t \geq 1$ and $h_{i,t} \in H_{i,t}$ such that $\omega_{i,\tau} \notin (-\varepsilon/2, \varepsilon/2)$ for all $\tau \leq t-1$.[15] On the other hand, if there exists t with $\omega_{i,t} \in (-\varepsilon/2, \varepsilon/2)$, we do not restrict the continuation strategy after any history that contains such $\omega_{i,t}$.

The next lemma shows two important features of the buffer area in the global-game perturbation.

Lemma 1. *There exists $\bar{\varepsilon} > 0$ such that, for each i and for any $\varepsilon < \bar{\varepsilon}$, the global-game perturbation with ε satisfies the following conditions with $\Omega_i^\varepsilon = \mathbb{R} \setminus (-\varepsilon/2, \varepsilon/2)$.*

1. *For any $\omega_i \in \Omega_i^\varepsilon$, conditional on that player j's signal belongs to Ω_j^ε, the probability that player j's interpretation is the same as player i's interpretation $f_i(\omega_i)$ is sufficiently high, i.e.,* $\Pr\left(\omega_j \in f_j^{-1}(f_i(\omega_i)) \mid \mathbf{a}, \omega_i, \Omega_j^\varepsilon\right) \geq 1-\varepsilon$ *for any $\mathbf{a} \in A$ and $\omega_i \in \Omega_i^\varepsilon$.*
2. *The probability that player i's signal belongs to Ω_i^ε is high, i.e.,* $\Pr\left(\omega_i \in \Omega_i^\varepsilon \mid \mathbf{a}\right) \geq 1-\varepsilon$ *for any $\mathbf{a}$.*

Condition 1 of Lemma 1 means that, if player i observes a signal ω_i not in the buffer area and knows that the opponent j also observes some signal not in the buffer area (but does not know which signal), then she believes with large probability that the opponent j interprets his signal in the same way as player i interprets ω_i. Notice that, if we required $\Pr\left(\omega_j \in f_j^{-1}(f_i(\omega_i)) \mid \mathbf{a}, \omega_i\right) \geq 1-\varepsilon$, then Condition 1 would be equivalent to Condition 2 of Definition 2 in Mailath and Morris (2006), implying that there is almost common knowledge that the players interpret their signals in the same way. However, our Condition 1 is significantly weaker than requiring almost common knowledge about players' interpretations. In our Condition 1, player i *a priori* conditions on the event that j's signal belongs to Ω_j^ε, but

in this subsection can apply to a general private-monitoring structure $(\Omega, \pi, \mathbf{f})$ as long as it satisfies Lemma 1 for some $\Omega_1^\varepsilon \subseteq \Omega_1$ and $\Omega_2^\varepsilon \subseteq \Omega_2$.

[15] Here we use the max-norm $|\alpha_i - \alpha_i'| = \max_{a_i \in A_i} |\alpha_i(a_i) - \alpha_i'(a_i)|$ for $\alpha_i, \alpha_i' \in \Delta(A_i)$.

player i does not necessarily put large probability on that event. Indeed, if $\omega_i = \pm\varepsilon/2$, then he puts probability only 1/2 that $\omega_j \in \Omega_j^\varepsilon$.

Our objective is to construct an equilibrium in a game with public monitoring such that, by properly specifying continuation strategies after observing signals in the buffer area $(-\varepsilon/2, \varepsilon/2)$, we can separate the regions above $\varepsilon/2$ and below $-\varepsilon/2$ and players outside the buffer area can coordinate their strategies within each of these regions of the global-game perturbation. (Continuation strategies after signals in the buffer area are completely differently from the original equilibrium, but their incentive constraints must be satisfied in the perturbed game.) Formally, we will construct an equilibrium $\tilde{\mathbf{s}}$ in a game with public monitoring such that, for every $\eta > 0$, there exists $\bar{\varepsilon} > 0$ such that, for every global-game perturbation with $\varepsilon < \bar{\varepsilon}$, the perturbed game admits an equilibrium $\mathbf{s}$ such that

$$|s_i(h_{i,t}) - \tilde{s}_i(f_i(h_{i,t}))| < \eta$$

for all $i \in I$, $t \geq 1$, and $h_{i,t} \in H_{i,t}$ with $\omega_{i,\tau} \in \Omega_i^\varepsilon$ for all $\tau \leq t-1$.

The construction of a desired equilibrium goes as follows. Consider the two-player prisoner's dilemma with public monitoring as in Section 3.1. We assume[16]

$$p > q > r, \tag{1}$$

$$2 - \tfrac{1-p}{p-q} > \max\left\{\tfrac{1}{p-q}, \tfrac{1-r}{q-r}\right\} \tag{2}$$

$$2 + \tfrac{p}{p-q} > \tfrac{1+r}{q-r}. \tag{3}$$

In each odd period $t = 1, 3, \ldots$, each player i determines a state $x_i \in \{G, B\}$. (How the state is determined will be discussed later.) In each odd period t, player i takes C if $x_i = G$; D if $x_i = B$. In the next (even) period $t+1$, she takes C if she observes $y_t = \bar{y}$; D if she observes $y_t = \underline{y}$. With a slight abuse of notations, let $\tilde{s}_i(x_i)$ denote her action plan in periods t and $t+1$ described.

Let $p_j[x_j](\tilde{h}_j) = p_j[x_j](a_{j,t}, y_t, a_{j,t+1}, y_{t+1})$ be the probability that player j's next state will be G in period $t+2$ if her current state is x_j in period t and she observes $\tilde{h}_j =$

[16]It is possible to construct a robust equilibrium even without this assumption.

$(a_{j,t}, y_t, a_{j,t+1}, y_{t+1})$ for the last two periods. We determine $p_j[x_j](\tilde{h}_j)$ as follows.

1. Conditional on $x_j = G$, player i is indifferent between C and D in period t; strictly prefers C in period $t+1$ if she observes $y_t = \bar{y}$; strictly prefers D in period $t+1$ if she observes $y_t = \underline{y}$. That is, if player j plays $\tilde{s}_j(G)$ followed by transition probabilities $p_j[G](\tilde{h}_j)$, both $\tilde{s}_i(G)$ and $\tilde{s}_i(B)$ are best responses for player i in period t, and player i has strict incentives in period $t+1$.

2. Conditional on $x_j = B$, player i is indifferent between C and D in both periods t and $t+1$. Any strategy is a best response for player i in period t if player j plays $\tilde{s}_j(B)$ followed by transition probabilities $p_j[B](\tilde{h}_j)$.

We specify $p_j[x_j](\tilde{h}_j)$ so that conditional on $x_j \in \{G, B\}$, both $\tilde{s}_i(G)$ and $\tilde{s}_i(B)$ are optimal and attain $v_i(x_j)$:

$$v_i(x_j) = \max_{\tilde{s}_i} \mathbb{E}\left[(1-\delta^2)\left(u_i(\mathbf{a}_t) + \delta u_i(\mathbf{a}_{t+1})\right)\right. \\ \left. +\delta^2 \left(p_j[x_j](\tilde{h}_j) v_i(G) + \left(1 - p_j[x_j](\tilde{h}_j)\right) v_i(B)\right) \mid \tilde{s}_i, \tilde{s}_j(x_j)\right], \tag{4}$$

where the maximization is taken over all behavioral strategies for the two periods t and $t+1$. We also want to have $v_i(G) > v_i(B)$, i.e., $\tilde{s}_j(G)$ is a "good" strategy for player i; $\tilde{s}_j(B)$ is a "bad" strategy for player i.

For sufficiently large δ, it is equivalent to consider the following "reward functions" $\theta_j[x_j](\tilde{h}_j)$ instead of $p_j[x_j](\tilde{h}_j)$ such that $\theta_j[x_j](\tilde{h}_j)$ is uniformly bounded with respect to δ,

$$\begin{aligned} v_i(G) &= \frac{1}{1+\delta} \max_{\tilde{s}_i} \mathbb{E}\left[u_i(\mathbf{a}_t) + \delta u_i(\mathbf{a}_{t+1}) + \theta_j[G](\tilde{h}_j) \mid \tilde{s}_i, \tilde{s}_j(G)\right], \\ v_i(B) &= \frac{1}{1+\delta} \max_{\tilde{s}_i} \mathbb{E}\left[u_i(\mathbf{a}_t) + \delta u_i(\mathbf{a}_{t+1}) + \theta_j[B](\tilde{h}_j) \mid \tilde{s}_i, \tilde{s}_j(B)\right], \\ 0 &\geq \theta_j[G](\tilde{h}_j), \\ 0 &\leq \theta_j[B](\tilde{h}_j). \end{aligned}$$

Then,

$$p_j[G](\tilde{h}_j) = 1 + \frac{1-\delta}{\delta^2 \left(v_i(G) - v_i(B)\right)} \theta_j[G](\tilde{h}_j),$$
$$p_j[B](\tilde{h}_j) = \frac{1-\delta}{\delta^2 \left(v_i(G) - v_i(B)\right)} \theta_j[B](\tilde{h}_j) \tag{5}$$

satisfy (4). Here, $\theta_j[G] \leq 0$ describes the amount of punishments player j gives to player i by transiting from the good state to the bad state; $\theta_j[B] \geq 0$ describes the amount of rewards player j gives to player i by transiting from the bad state to the good state.

We further specify those reward functions as follows.

1. At state $x_j = G$:

   (a) $\theta_j[G](\tilde{h}_j)$ is additively separable and independent of $(a_{j,t}, a_{j,t+1})$, i.e., $\theta_j[G](\tilde{h}_j) = \theta_j[G](y_t) + \delta\theta_j[G](y_t, y_{t+1})$, where $\theta_j[G](y_t)$ is the reward for period t and $\theta_j[G](y_t, y_{t+1})$ is the reward for period $t+1$ after y_t.

   (b) In period $t+1$:

      i. After $y_t = \bar{y}$, we have $\theta_j[G](\bar{y}, \bar{y}) = 0$ (no punishment after the signal indicating C) and $\theta_j[G](\bar{y}, \underline{y})$ is sufficiently small so that player i strictly prefers C to D:

$$2 + (1-p)\,\theta_j[G]\left(\bar{y}, \underline{y}\right) > 3 + (1-q)\,\theta_j[G]\left(\bar{y}, \underline{y}\right). \tag{6}$$

      ii. After $y_t = \underline{y}$, $\theta_j[G](\underline{y}, y_{t+1}) = 0$ for all y_{t+1} (no reward) so that player i strictly prefers D to C.

   (c) In period t, $\theta_j[G](\underline{y})$ and $\theta_j[G](\bar{y})$ are determined so that player i is indifferent between C and D:

$$v_i(G) = \frac{1}{1+\delta}\left(2 + p\left(\theta_j[G](\bar{y}) + \delta\left(2 + (1-p)\,\theta_j[G]\left(\bar{y}, \underline{y}\right)\right)\right) + (1-p)\,\theta_j[G](\underline{y})\right)$$
$$= \frac{1}{1+\delta}\left(3 + q\left(\theta_j[G](\bar{y}) + \delta\left(2 + (1-q)\,\theta_j[G]\left(\bar{y}, \underline{y}\right)\right)\right) + (1-q)\,\theta_j[G](\underline{y})\right). \tag{7}$$

2. At state $x_j = B$:

    (a) $\theta_j[B](\tilde{h}_j)$ is additively separable, independent of $a_{j,t}$, and the second term is independent of y_t, i.e., $\theta_j[B](\tilde{h}_j) = \theta_j[B](y_t) + \delta\theta_j[B](a_{j,t+1}, y_{t+1})$.

    (b) In period $t+1$, we have $\theta_j[B](a_{j,t+1}, \underline{y}) = 0$ (no reward after the signal indicating D) and $\theta_j[B](a_{j,t+1}, \bar{y})$ is determined so that conditional on $a_{j,t+1}$, player i is indifferent between C and D:

$$\theta_j[B](a_{j,t+1}, \bar{y}) = \begin{cases} \frac{1}{p-q} & \text{if } a_{j,t+1} = C, \\ \frac{1}{q-r} & \text{if } a_{j,t+1} = D. \end{cases} \tag{8}$$

    (c) In period t, $\theta_j[B](\underline{y})$ and $\theta_j[B](\bar{y})$ are determined so that player i is indifferent between C and D. By (8), we have

$$\begin{aligned} v_i(B) &= \frac{1}{1+\delta}\left(-1 + q\left(\theta_j[B](\bar{y}) + \delta\left(2 + \frac{p}{p-q}\right)\right)\right. \\ &\qquad \left. + (1-q)\left(\theta_j[B](\underline{y}) + \delta\left(-1 + \frac{q}{q-r}\right)\right)\right) \\ &= \frac{1}{1+\delta}\left(r\left(\theta_j[B](\bar{y}) + \delta\left(2 + \frac{p}{p-q}\right)\right)\right. \\ &\qquad \left. + (1-r)\left(\theta_j[B](\underline{y}) + \delta\left(-1 + \frac{q}{q-r}\right)\right)\right). \end{aligned} \tag{9}$$

First, consider state $x_j = G$. (6) is satisfied if and only if $\theta_j[G]\left(\bar{y}, \underline{y}\right) < -\frac{1}{p-q}$. Suppose $\theta_j[G]\left(\bar{y}, \underline{y}\right) \approx -\frac{1}{p-q}$ and δ is sufficiently large. If we calculated the payoffs of taking C and of taking D in period t respectively in (7) assuming $\theta_j[G](\underline{y}) = \theta_j[G](\bar{y}) = 0$, then, (2) would imply that player i strictly prefers C to D in period t. Hence, player j punishes player i when $y_t = \bar{y}$ to make player i indifferent between C and D by letting, for example, $\theta_j[G](\bar{y}) \approx -\delta\left(2 - \frac{1-p}{p-q}\right) + \frac{1}{p-q}$ and $\theta_j[G](\underline{y}) = 0$, which implies $v_i(G) = 2 + \frac{p}{p-q}$.

We can give player i strict incentives to play C after $y_t = \bar{y}$ at state $x_j = G$ by setting

$$\begin{aligned}
\theta_j[G]\left(\bar{y}, \bar{y}\right) &= 0, \\
\theta_j[G]\left(\bar{y}, \underline{y}\right) &= -\frac{1}{p-q} - \frac{\kappa}{\delta\left(1-p\right)}, \\
\theta_j[G]\left(\underline{y}, \bar{y}\right) &= \theta_j[G]\left(\underline{y}, \underline{y}\right) = 0, \\
\theta_j[G](\bar{y}) &= -\delta\left(2 - \frac{1-p}{p-q}\right) + \kappa + \frac{1}{p-q}, \\
\theta_j[G](\underline{y}) &= 0
\end{aligned}$$

for sufficiently small $\kappa > 0$, where, thanks to (2), $\theta_j[G](\bar{y}) < 0$ for sufficiently large δ. We have

$$v_i(G) = \frac{1}{1+\delta}\left(2 + \frac{p}{p-q}\right). \tag{10}$$

Second, consider state $x_j = B$. Suppose δ is sufficiently large. If we calculated the payoffs of taking C and of taking D in period t respectively in (9) assuming $\theta_j[B](\underline{y}) = \theta_j[B](\bar{y}) = 0$, then (3) would imply that player i strictly prefers C to D in period t. Hence, player j rewards player i when $y_t = \underline{y}$ to make player i indifferent between C and D by letting, for example, $\theta_j[B](\underline{y}) = \delta\left(2 + \frac{p}{p-q}\right) - \frac{1+\delta r}{q-r}$ and $\theta_j[B](\bar{y}) = 0$. In addition, (3) implies $\theta_j[B](\underline{y}) > 0$ for sufficiently large δ. We have

$$v_i(B) = \frac{1}{1+\delta}\left(\delta\left(2 + \frac{p}{p-q}\right) - \frac{1-r}{q-r}\right). \tag{11}$$

By (10) and (11) together with (1), we have $v_i(G) > v_i(B)$.

Finally, by setting the initial states $x_1 = x_2 = G$ in period 1, we can completely specify an entire profile $\tilde{\mathbf{s}} = (\tilde{s}_1, \tilde{s}_2)$ of equilibrium strategies for sufficiently large δ.[17]

Claim 3. *For sufficiently large δ, the above equilibrium $\tilde{\mathbf{s}}$ is robust to global-game perturbations with buffer areas. That is, for every $\eta > 0$, there exists $\bar{\varepsilon} > 0$ such that, for every global-game perturbation with $\varepsilon < \bar{\varepsilon}$, the perturbed game admits an equilibrium $\mathbf{s}$ such that*

$$|s_i(h_{i,t}) - \tilde{s}_i(f_i(h_{i,t}))| < \eta$$

[17] We take sufficiently large δ so that $p_j[x_j](\tilde{h}_j)$ belongs to $[0,1]$ when it is defined from $\theta_j[x_j](\tilde{h}_j)$ via (5).

for all $i \in I$, $t \geq 1$, *and* $h_{i,t} \in H_{i,t}$ *with* $\omega_{i,\tau} \in \Omega_i^\varepsilon = \mathbb{R} \setminus (-\varepsilon/2, \varepsilon/2)$ *for all* $\tau \leq t-1$.

Proof. See Appendix. ∎

Note that, although both Claims 2 and 3 establish robustness to private-monitoring perturbations, underlying logics behind these claims are entirely different. In Claim 2, the key was the belief-freeness property of the Ely-Välimäki equilibrium such that players do not need to coordinate their future play. In contrast, players do need to coordinate in the equilibrium used in Claim 3. For example, in period 2, players coordinate to play CC if they observe $\bar{y}$ and DD if they observe $\underline{y}$. Also, since (6) is satisfied with strict inequality, the induced 2×2 game $G(\tilde{\mathbf{s}})$ is a coordination game with two strict equilibria, and generically one of them strictly risk-dominates the other. This implies that, by applying the contagion argument used in Claim 1, one can show that this equilibrium is not robust to global-game perturbations if we do not allow for a buffer area. Thus it is not the belief-freeness but the possibility of a buffer area that makes this equilibrium robust.

The proof of Claim 3 is involved, but the equilibrium we construct in the perturbed game has the following structure. The equilibrium is recursive every odd period as in the original (public-monitoring) game. In each odd period t, each player i decides a state $x_i \in \{G, B\}$, and then follows an action plan in the next two periods (periods t and $t+1$), denoted by $s_i(x_i)$. More specifically, in period t, player i plays C if $x_i = G$; D if $x_i = B$. In period $t+1$, she takes C if she observes $\omega_{i,t} \in \Omega_i^\varepsilon \cap f_i^{-1}(\bar{y})$ (i.e., $\omega_{i,t} \geq \varepsilon/2$); D if she observes $\omega_{i,t} \in \Omega_i^\varepsilon \cap f_i^{-1}(\underline{y})$ (i.e., $\omega_{i,t} \leq -\varepsilon/2$); her action will be specified later if $\omega_{i,t} \notin \Omega_i^\varepsilon$ (i.e., $\omega_{i,t} \in (-\varepsilon/2, \varepsilon/2)$).

To distinguish it from the reward function in the original game, for each state $x_i \in \{G, B\}$ and each private history $h_i = (a_{i,t}, \omega_{i,t}, a_{i,t+1}, \omega_{i,t+1})$, we write $\theta_i^\varepsilon[x_i](h_i)$ with superscript ε for the reward function in the perturbed game. Player i's incentives are controlled by player j's strategy $s_j(x_j)$ and the reward function $\theta_j^\varepsilon[x_j](h_j)$ so that the following conditions are satisfied. (Details are deferred to Appendix.)

1. At state $x_j = G$:

(a) $\theta_j^\varepsilon[G](h_j)$ is additively separable: $\theta_j^\varepsilon[G](h_j) = \theta_j^\varepsilon[G](f_j(\omega_{j,t})) + \delta\theta_j^\varepsilon[G](\omega_{j,t}, a_{j,t+1}, f_j(\omega_{j,t+1}))$.

(b) In period $t+1$,

i. After player j observes $\omega_{j,t} \in \Omega_j^\varepsilon$ (i.e., $|\omega_{j,t}| \geq \varepsilon/2$), we set $\theta_j^\varepsilon[G](\omega_{j,t}, a_{j,t+1}, f_j(\omega_{j,t+1})) = \theta_j[G](f_j(\omega_{j,t}), f_j(\omega_{j,t+1}))$.

ii. After player j observes $\omega_{j,t} \notin \Omega_j^\varepsilon$ (i.e., $\omega_{j,t} \in (-\varepsilon/2, \varepsilon/2)$), we determine $\theta_j^\varepsilon[G](\omega_{j,t}, a_{j,t+1}, f_j(\omega_{j,t+1}))$ so that player i is indifferent between C and D in period $t+1$, conditional on $a_{j,t+1}$.

(c) In period t, we determine $\theta_j^\varepsilon[G](f_j(\omega_{j,t}))$ so that player i is indifferent between C and D in period t.

2. At state $x_j = B$:

(a) $\theta_j^\varepsilon[B](h_j)$ is additively separable: $\theta_j^\varepsilon[B](h_j) = \theta_j^\varepsilon[B](f_j(\omega_{j,t})) + \delta\theta_j^\varepsilon[B](a_{j,t+1}, f_j(\omega_{j,t+1}))$.

(b) In period $t+1$, we have $\theta_j^\varepsilon[B](a_{j,t+1}, f_j(\omega_{j,t+1})) = \theta_j[B](a_{j,t+1}, f_j(\omega_{j,t+1}))$ so that player i is indifferent between C and D in period $t+1$, conditional on $a_{j,t+1}$.

(c) In period t, we determine $\theta_j^\varepsilon[B](f_j(\omega_{j,t}))$ so that player i is indifferent between C and D in period t.

Finally, we set player i's action in period $t+1$ so that, after she observes $\omega_{i,t} \notin \Omega_i^\varepsilon$, she plays a best response (if there are multiple best responses, we choose one arbitrarily) to player j's strategy $s_j(G)$ at state $x_j = G$, given i's belief about $\omega_{j,t}$.

Now we want to show that the above strategy constitutes an equilibrium in the perturbed game. We first verify player i's incentive at the beginning of period $t+1$, after observing $(a_{i,t}, \omega_{i,t})$. Since any action is optimal if $x_j = B$, regardless of player i's belief about x_i, player i plays a best response to $s_j(G)$. Further, Condition 1-(b)-ii ensures that player i plays a best response to $s_j(G)$ conditional on that $\omega_{j,t} \in \Omega_j^\varepsilon$. Conditional on the event that $x_j = G$ and $\omega_{j,t} \in \Omega_j^\varepsilon$, Condition 1-(b)-i together with Lemma 1 implies that after observing $\omega_{i,t} \in \Omega_i^\varepsilon$, player i believes that player j observes $\omega_{j,t} \in f_j^{-1}(f_i(\omega_{i,t}))$, i.e., player j's interpretation of

$\omega_{j,t}$ is the same as player i's interpretation of $\omega_{i,t}$, with large probability. Note that player i has a strict incentive in the original equilibrium $\tilde{\mathbf{s}}$, and we use the same reward function after $\omega_{j,t} \in \Omega_j^\varepsilon$. Thus, in the perturbed game, if ε is sufficiently small, player i strictly prefers C after observing $\omega_{i,t} \in f_i^{-1}(\bar{y})$ and D after observing $\omega_{i,t} \in f_i^{-1}(\underline{y})$.

What remains to be checked is player i's incentive in period t. Note that, due to Lemma 1, the probability that player j will observe $\omega_{j,t} \notin \Omega_j^\varepsilon$ is small. Thus, player i's continuation payoffs in the perturbed game are close to those in the original game in expectation at the beginning of period t, for each state x_j. Hence, we can take a reward function $\theta_j^\varepsilon[x_j](f_j(\omega_{j,t})) \approx \theta_j[x_j](f_j(\omega_{j,t}))$ such that player i is indifferent between C and D in period t.

Therefore, for each $x_i, x_j \in \{G, B\}$, the above two-period strategy $s_i(x_i)$ is optimal for player i against $s_j(x_j)$, given the reward function $\theta_j^\varepsilon[x_j](h_j)$. Moreover, since the reward function $\theta_j^\varepsilon[x_j](h_j)$ is close to that in the original game as long as $\omega_{i,t} \notin \Omega_i^\varepsilon$, $s_i(h_{i,t})$ is close to $\tilde{s}_i(f_i(h_{i,t}))$ if $\omega_{i,\tau} \in \Omega_i^\varepsilon = \mathbb{R} \setminus (-\varepsilon/2, \varepsilon/2)$ for all $\tau \leq t-1$.

A similar equilibrium construction is used in Hörner and Olszewski (2006), who consider "block strategies" $s_i(G)$ and $s_i(B)$ such that conditional on $s_j(G)$, both $s_i(G)$ and $s_i(B)$ are optimal (but there may be non-optimal strategies), and that conditional on $s_j(B)$, any strategy is optimal regardless of player i's beliefs about player j's private histories.[18] Since Hörner and Olszewski (2006) assume almost-perfect monitoring and the canonical signal space $\Omega_i = A_j$, as long as player i observes a signal sequence that occurs with positive probability in player j's equilibrium strategy, player i believes with high probability that player j observes the signal that corresponds to the realization of player i's strategy. Thus there is almost common knowledge between the players about their histories that occur with positive probability in equilibrium. In contrast, in our equilibrium, even if y occurs with positive probability in the original equilibrium and $\omega_{i,t} \in f_i^{-1}(y)$, if $\omega_{i,t}$ is close to 0 in the global-game perturbation, player i puts beliefs only 1/2 on the event that player j interprets her signal as y, which destroys the almost common knowledge property about interpretations

[18] Hörner and Olszewski (2006) also consider games with more than two players.

of private histories.

4 Definition of Robustness

An equilibrium in the repeated game with public monitoring is robust to private-monitoring perturbations if, for any $T < \infty$, whenever private-monitoring perturbations are sufficiently small in the ex-ante sense, then there exists an equilibrium of the perturbed game that is close to the original equilibrium for the first T periods. The formal definition is given below.

Definition 2. Fix a private-monitoring perturbation $(\Omega, \pi, \mathbf{f})$, $\eta \geq 0$, and $T \leq \infty$.

1. For each $t \leq T-1$, a profile $\mathbf{g}_t = (g_{1,t}, \ldots, g_{n,t})$ of measurable functions from $H_{i,t+1}$ to Y is *η-close to* $\mathbf{f}$ if

$$\pi(\{\boldsymbol{\omega} \in \Omega : g_{i,t}(\omega_{i,t} \mid a_i^{t-1}, a_{i,t}, \omega_i^{t-1}) = f_i(\omega_{i,t})\} \mid a_{i,t}, \mathbf{a}_{-i,t}) \geq 1 - \eta$$

   for all $i \in I$, $(a_{i,t}, \mathbf{a}_{-i,t}) \in A$, and $(a_i^{t-1}, \omega_i^{t-1}) \in H_{i,t}$.

2. A profile $\mathbf{s} = (s_1, \ldots, s_n)$ of strategies in the perturbed game is *(η, T)-close to* $\tilde{\mathbf{s}}$ *via* $\{\mathbf{g}_t\}_{t=1}^{T-1}$ if

$$|s_i(h_{i,t}) - \tilde{s}_i(g_i(h_{i,t}))| \leq \eta$$

   for all $i \in I$, $t \leq T$, and $h_{i,t} = (a_i^{t-1}, \omega_i^{t-1}) \in H_{i,t}$, where

$$g_i(h_{i,t}) = (a_{i,1}, g_{i,1}(\omega_{i,1} \mid a_{i,1}), a_{i,2}, g_{i,2}(\omega_{i,2} \mid a_{i,1}, a_{i,2}, \omega_{i,1}), \ldots, a_{i,t-1}, g_{i,t-1}(\omega_{i,t-1} \mid a_i^{t-1}, \omega_i^{t-2})).$$

Definition 3. A sequential equilibrium $\tilde{\mathbf{s}}$ of a public-monitoring repeated game is *robust to private-monitoring perturbations* if, for every $\eta > 0$ and $T < \infty$, there exists $\varepsilon > 0$ such that, for every private-monitoring perturbation $(\Omega, \pi, \mathbf{f})$ that is ex-ante ε-close to (Y, ρ), there exist a sequence $\{\mathbf{g}_t\}$ of interpretations and a Nash equilibrium $\mathbf{s}$ of the perturbed game such that each $\mathbf{g}_t$ is η-close to $\mathbf{f}$ and $\mathbf{s}$ is (η, T)-close to $\tilde{\mathbf{s}}$ via $\{\mathbf{g}_t\}$.

In the above definition of robust equilibria, we are allowed to choose $\{\mathbf{g}_t\}$ to construct an equilibrium in the perturbed game that approximates the original equilibrium. For this reason, we call $\{\mathbf{g}_t\}$ "endogenous" interpretations. We require that those endogenous interpretations be equal to exogenous interpretations $\mathbf{f}$ with large probability for the first $T-1$ periods, but can be different with small probability or after period T. In this sense, the notion of endogenous interpretations generalizes threshold strategies in Section 3.1, where the endogenous threshold $\bar{\omega}$ can be different from but must be close to the exogenous threshold 0. (Of course, we allow for more flexible interpretations than thresholds. For example, $g_{i,t}$ may depend on her entire private history $(\omega_i^{t-1}, a_i^{t-1}, a_{i,t})$.) If instead the strategy profile translated via the exogenous interpretations $\mathbf{f}$ were required to be an equilibrium in the perturbed game, that is, if exogenous and endogenous interpretations were required to be equal with probability 1, then one could prove almost immediately that there is no robust strict equilibrium other than repetitions of static equilibria.[19]

Note that we require the closeness between exogenous and endogenous interpretations only for the first $T-1$ periods. It is because we will make small adjustments in continuation strategies after T to deal with small changes in the marginal distribution, as we discussed in Section 3.2. Note also that we require that s_i and $\tilde{s}_i$ prescribe similar mixed actions for *all* private histories $h_{i,t}$ (via endogenous interpretations), and not just for private histories that realize with high probability. In this sense, in the rest of the paper, we employ the notion of robustness stronger than the one we used in Section 3.3, which allowed for small buffer areas.

Suppose that the perturbed game has an equilibrium that approximates the original equilibrium. Then player i's equilibrium payoff in the perturbed game is close to her equilibrium payoff in the original game. Moreover, conditional on that each player j takes a_j and interprets his signal as y_j, player i's continuation payoff in the perturbed game is close

[19]One could prove such a non-robustness result by considering a perturbation such that player i receives a signal ω_i with small probability, where she interprets ω_i as y but, conditional on ω_i, she believes with probability 1 that the other players interpret their signals as $y' \neq y$.

to $U_i(\tilde{s}_1 \mid a_1, y_1, \ldots, \tilde{s}_n \mid a_n, y_n)$. It is obvious from the above definitions, but worth stating explicitly.

Lemma 2. *Fix a strategy profile* $\tilde{\mathbf{s}}$ *in the repeated game with public monitoring. For every* $\gamma > 0$*, there exist* $\overline{\eta} > 0$, $\overline{T} < \infty$*, and* $\overline{\varepsilon} > 0$ *such that, if a private-monitoring perturbation* $(\Omega, \pi, \mathbf{f})$ *is ex-ante* $\overline{\varepsilon}$*-close to* (Y, ρ)*, a profile* $\mathbf{g}_t$ *of endogenous interpretations is* $\overline{\eta}$*-close to* $\mathbf{f}$ *for every* $t \leq \overline{T} - 1$*, and a profile* $\mathbf{s}$ *of strategies in the perturbed game is* $(\overline{\eta}, \overline{T})$*-close to* $\tilde{\mathbf{s}}$ *via* $\{\mathbf{g}_t\}$*, then player i's continuation payoff in the perturbed game conditional on* $\mathbf{a}_1$ *and* $\boldsymbol{\omega}_1$ *is within distance* γ *from* $U_i(\tilde{s}_1 \mid a_{1,1}, g_1(\omega_{1,1} \mid a_{1,1}), \ldots, \tilde{s}_n \mid a_{n,1}, g_n(\omega_{n,1} \mid a_{n,1}))$.

5 A Non-Robustness Result

In this section, we extend Claim 1 and provide a sufficient condition for non-robustness. For simplicity, assume that the public-monitoring repeated game has two players $I = \{1, 2\}$ and two public signals $Y = \{\overline{y}, \underline{y}\}$, and the public-monitoring structure has full support. We concentrate on a public perfect equilibrium (Extensions to general stage games and private strategies are discussed at the end of this section.) As in Section 3.1, a PPE $\tilde{\mathbf{s}}$ induces the following 2×2 game:

$$G(\tilde{\mathbf{s}}) := \begin{array}{c|cc} & \overline{y} & \underline{y} \\ \hline \overline{y} & U_1(\tilde{s}_1 \mid \overline{y}, \tilde{s}_2 \mid \overline{y}), U_2(\tilde{s}_1 \mid \overline{y}, \tilde{s}_2 \mid \overline{y}) & U_1(\tilde{s}_1 \mid \overline{y}, \tilde{s}_2 \mid \underline{y}), U_2(\tilde{s}_1 \mid \overline{y}, \tilde{s}_2 \mid \underline{y}) \\ \underline{y} & U_1(\tilde{s}_1 \mid \underline{y}, \tilde{s}_2 \mid \overline{y}), U_2(\tilde{s}_1 \mid \underline{y}, \tilde{s}_2 \mid \overline{y}) & U_1(\tilde{s}_1 \mid \underline{y}, \tilde{s}_2 \mid \underline{y}), U_2(\tilde{s}_1 \mid \underline{y}, \tilde{s}_2 \mid \underline{y}) \end{array}$$

with two pure-strategy equilibria $(\overline{y}, \overline{y})$ and $(\underline{y}, \underline{y})$. Since the action of player i in period 1, $\mathbf{a}_{i,1}$, does not affect the continuation strategy of player i in the public strategy except for the effect on the interpretation through g_i, we can omit the dependence of the continuation payoff on the action profile in period 1, $\mathbf{a}_1$.

A regularity condition is imposed on the 2×2 coordination game $G(\tilde{\mathbf{s}})$ in the following sense.

Definition 4. A 2×2 coordination game

	α_2	β_2
α_1	u_1, u_2	v_1, v_2
β_1	w_1, w_2	x_1, x_2

with two pure-strategy equilibria (α_1, α_2) and (β_1, β_2) is *regular* if (α_1, α_2) and (β_1, β_2) have different Nash products in the normalized game

	α_2	β_2
α_1	$u_1 - w_1, u_2 - v_2$	$0, 0$
β_1	$0, 0$	$x_1 - v_1, x_2 - w_2$

,

i.e., $(u_1 - w_1)(u_2 - v_2) \neq (x_1 - v_1)(x_2 - w_2)$.

The regularity condition is easily satisfied in many pure-strategy PPE. For example, if $\tilde{\mathbf{s}}$ is a strict PPE under some parameters and not a repetition of static equilibria, then $\tilde{\mathbf{s}}$ is regular under almost all nearby parameters in the monitoring structure, payoffs, and discounting. Also the regularity condition is consistent with mixed-strategy PPEs. On the other hand, the regularity condition exclude belief-free equilibria (we will discuss further in Section 6) and in particular, repetitions of static Nash equilibria.

In Theorem 1, we show that in any repeated game with two players, two public signals, and full support, if $\tilde{\mathbf{s}}$ induces a regular coordination game $G(\tilde{\mathbf{s}})$, then $\tilde{\mathbf{s}}$ is not robust to private-monitoring perturbations. This result generalizes Claim 1, and is consistent with Claims 2 and 3 as we exclude the Ely-Välimäki equilibrium by the regularity condition, and employ the notion of robustness that does not allow for "buffer areas".

Theorem 1. *If $\tilde{\mathbf{s}}$ is a PPE of a public-monitoring repeated game with two players, two public signals, and full support, and $G(\tilde{\mathbf{s}})$ is regular, then $\tilde{\mathbf{s}}$ is not robust to private-monitoring perturbations.*

Proof. Suppose that $\tilde{\mathbf{s}}$ is a regular PPE robust to private-monitoring perturbations. For each $\varepsilon > 0$, we will construct a private-monitoring perturbation $(\Omega, \pi, \mathbf{f})$ that is ex-ante ε-close to (Y,ρ) such that the perturbed game has no equilibrium close to $\tilde{\mathbf{s}}$, contradictory to our assumption.

By the regularity of $G(\tilde{\mathbf{s}})$, after changing the labels of $\overline{y}$ and $\underline{y}$ if necessary, there exist $p_1, p_2 > 0$ with $p_1 + p_2 < 1$ such that $(\overline{y}, \overline{y})$ is a strict (p_1, p_2)-dominant equilibrium in $G(\tilde{\mathbf{s}})$, i.e.,

$$p_i U_i(\tilde{s}_i \mid \overline{y}, \tilde{s}_j \mid \overline{y}) + (1-p_i) U_i(\tilde{s}_i \mid \overline{y}, \tilde{s}_j \mid \underline{y}) > p_i U_i(\tilde{s}_i \mid \underline{y}, \tilde{s}_j \mid \overline{y}) + (1-p_i) U_i(\tilde{s}_i \mid \underline{y}, \tilde{s}_j \mid \underline{y}) \tag{12}$$

for all $i \in I$.

The following lemma shows that the similar risk dominance property holds for the coordination game where we replace public signals with the endogenous interpretations.

Lemma 3. *There exist $\overline{\eta} > 0$, $\overline{T} < \infty$, and $\overline{\varepsilon} > 0$ such that the following is true. Suppose that $(\Omega, \pi, \mathbf{f})$ is ex-ante $\overline{\varepsilon}$-close to (Y, ρ), $\mathbf{g}_t$ is $\overline{\eta}$-close to $\mathbf{f}$ for every $t \leq \overline{T} - 1$, and $\mathbf{s}$ is an equilibrium of the perturbed game that is $(\overline{\eta}, \overline{T})$-close to $\tilde{\mathbf{s}}$ via $\{\mathbf{g}_t\}$. Then, for every $i \in I$ and $a_{i,1} \in \operatorname{supp} s_i(\emptyset)$, if $\omega_{i,1}$ satisfies $\pi(\{\omega_{j,1} \in \Omega_j : g_{j,1}(\omega_{j,1} \mid a_{j,1}) = \overline{y}\} \mid a_{i,1}, a_{j,1}, \omega_{i,1}) > p_i$ for all $a_{j,1} \in \operatorname{supp} s_j(\emptyset)$, then $g_{i,1}(\omega_{i,1} \mid a_{i,1}) = \overline{y}$.*

Proof of Lemma 3. Pick any $i \in I$ and $a_{i,1} \in \operatorname{supp} s_i(\emptyset)$. If player i believes that, regardless of $a_{j,1} \in \operatorname{supp} s_j(\emptyset)$, player j interprets $g_{j,1}(\omega_{j,1} \mid a_{j,1}) = \overline{y}$ with probability no less than p_i, then, by Lemma 2 and (12), player i receives a higher payoff if she interprets $\omega_{i,1}$ as $\overline{y}$ and follows $\tilde{s}_i \mid \overline{y}$ than $\tilde{s}_i \mid \underline{y}$. ∎

For the rest of the proof, we take the global-game perturbation as in Section 3.1. More precisely, let $\Omega_1 = \Omega_2 = \mathbb{R}$ and $\pi = \pi^\varepsilon$, where, conditional on each $\mathbf{a} \in A$, $\pi^\varepsilon(\cdot \mid \mathbf{a})$ denotes the joint distribution of $\boldsymbol{\omega} = (\omega_1, \omega_2)$ induced by the uniform distribution of (θ, ξ_1, ξ_2) on $[-\rho(\underline{y} \mid \mathbf{a}_1), \rho(\overline{y}, \mid \mathbf{a}_1)] \times [-\varepsilon/2, \varepsilon/2]^2$ and $\omega_i = \theta + \xi_i$ for each $i \in I$. Let $f_i(\omega_i) = \overline{y}$ if $\omega_i \geq 0$ and $f_i(\omega_i) = \underline{y}$ if $\omega_i < 0$.

Since (Y, ρ) has full support, let $B := \min_{\mathbf{a}, y} \rho(y \mid \mathbf{a}) > 0$. The following lemma follows from the characteristics of the perturbation in period 1.

Lemma 4. *There exist $\overline{\overline{\eta}}, \overline{\overline{\varepsilon}}, d, C_1, C_2, D > 0$ such that the following is true.*

(i) For every $\eta < \overline{\overline{\eta}}$ and $\varepsilon < \overline{\overline{\varepsilon}}$, if $g_{1,1}$ satisfies

$$\pi^{\varepsilon}(\{\boldsymbol{\omega}_1 \in \mathbb{R}^2 : g_{1,1}(\omega_{1,1} \mid a_{1,1}) = f_1(\omega_{1,1})\} \mid a_{1,1}, a_{2,1}) \geq 1 - \eta \tag{13}$$

for all $(a_{1,1}, a_{2,1}) \in A$, then there exists $x_0 \in [D\varepsilon, B - D\varepsilon]$ such that

$$\pi^{\varepsilon}(\{\omega_{1,1} \in \mathbb{R} : g_{1,1}(\omega_{1,1} \mid a_{1,1}) = \overline{y}\} \mid a_{1,1}, a_{2,1}, \omega_{2,1}) > p_2$$

for all $(a_{1,1}, a_{2,1}) \in A$, and $\omega_{2,1} \in [x_0 - C_2\varepsilon, x_0 + C_2\varepsilon]$.

(ii) For every $\varepsilon > 0$ and $x \in [-B + D\varepsilon, B - D\varepsilon]$, we have

$$\pi^{\varepsilon}(\{\omega_{j,1} \in [x - C_j\varepsilon, x + C_j\varepsilon]\} \mid \mathbf{a}_1, \omega_{i,1}) > p_i$$

for all $i \in I$, $\mathbf{a}_1 \in A$, and $\omega_{i,1} \in [x - (C_i + d)\varepsilon, x + C_i\varepsilon]$.

Proof of Lemma 4. See Appendix. ∎

Fix any $\eta < \min\{B, \overline{\eta}, \overline{\overline{\eta}}\}$ and $T > \overline{T}$. By the robustness of $\tilde{\mathbf{s}}$, there exists $\varepsilon_{\eta,T} > 0$ such that, for every $\varepsilon < \min\{\varepsilon_{\eta,T}, \overline{\varepsilon}, \overline{\overline{\varepsilon}}\}$, there exist a sequence $\{\mathbf{g}_t\}$ of endogenous interpretations and an equilibrium $\mathbf{s}$ of the perturbed game such that each $\mathbf{g}_t$ is η-close to $\mathbf{f}$, and $\mathbf{s}$ is (η, T)-close to $\tilde{\mathbf{s}}$ via $\{\mathbf{g}_t\}$. This implies (13) is satisfied. By Lemma 4 (i), there exists $x_0 \in [D\varepsilon, B - D\varepsilon]$ such that player 2 with private signal $\omega_{2,1} \in [x_0 - C_2\varepsilon, x_0 + C_2\varepsilon]$ believes that player 1 interprets $\omega_{1,1}$ as $\overline{y}$ with probability more than p_2. By Lemma 3, player 2 interprets $\omega_{2,1}$ as $\overline{y}$ on the equilibrium path. By Lemma 4 (ii), player 1 with private signal $\omega_{1,1} \in [x_0 - (C_1 + d)\varepsilon, x_0 + C_1\varepsilon]$ believes that player 2 interprets $\omega_{2,1}$ as $\overline{y}$ with probability more than p_1, and, by Lemma 3, player 1 interprets $\omega_{1,1}$ as $\overline{y}$ on the equilibrium path. Once again, by Lemma 4 (ii), player 2 with private signal $\omega_{2,1} \in [x_0 - (C_2 + 2d)\varepsilon, x_0 + C_2\varepsilon]$ believes that player 1 interprets $\omega_{1,1}$ as $\overline{y}$ with probability more than p_2, and, by Lemma 3, player 2 interprets $\omega_{2,1}$ as $\overline{y}$ on the equilibrium path. By induction, each player i interprets $\omega_{i,1} \in [-B + (D + d)\varepsilon, x_0 + C_i\varepsilon]$ as $\overline{y}$ on the equilibrium path. Since this is true for all $\varepsilon < \min\{\varepsilon_{\eta,T}, \overline{\varepsilon}, \overline{\overline{\varepsilon}}\}$, this is a contradiction. ∎

Theorem 1 extends easily to games with more than two public signals as follows. Pick an arbitrary pair of the public signals from Y and name them $\overline{y}$ and $\underline{y}$. If a public signal $y \neq \bar{y}, \underline{y}$ realizes, both players observe y directly. If $\bar{y}$ or $\underline{y}$ realizes, then the players observe perturbed signals. We require that players' endogenous interpretations after observing such private signals be either $\overline{y}$ or $\underline{y}$.

Corollary 1. *If* $\tilde{\mathbf{s}}$ *is a PPE of a public-monitoring repeated game with two players and full support and there exist two signals* $\bar{y}$ *and* $\underline{y}$ *such that* $G(\tilde{\mathbf{s}})$ *is regular, then* $\tilde{\mathbf{s}}$ *is not robust to private-monitoring perturbations.*

Theorem 1 also extends to games with more than two players under an additional condition. (Here we assume two public signals or use the trick as above.) For a PPE $\tilde{\mathbf{s}}$, let $G(\tilde{\mathbf{s}})$ be the n-player game, where each player i's action set is $\{\overline{y}, \underline{y}\}$, and her payoff is $U_i(\tilde{s}_1 \mid y_1, \ldots, \tilde{s}_n \mid y_n)$ when players play $(y_1, \ldots, y_n) \in \{\overline{y}, \underline{y}\}^n$. The induced game $G(\tilde{\mathbf{s}})$ is *supermodular* if $U_i(\tilde{s}_i \mid \overline{y}, (\tilde{s}_j \mid y_j)_{j \neq i}) - U_i(\tilde{s}_i \mid \underline{y}, (\tilde{s}_j \mid y_j)_{j \neq i})$ is weakly increasing as each player $j \neq i$ switches from $y_j = \underline{y}$ to $\overline{y}$. A supermodular game $G(\tilde{\mathbf{s}})$ is *regular* if either $(\overline{y}, \ldots, \overline{y})$ or $(\underline{y}, \ldots, \underline{y})$ is a contagious equilibrium in an open neighborhood of $G(\tilde{\mathbf{s}})$.[20] Under these definitions, Theorem 1 generalizes to games with more than two players.

We point out that Theorem 1 holds for equilibria in private (non-public) strategies as long as the players play a pure action profile in period 1 and the 2×2 coordination game induced by continuation payoffs after that action profile is regular.

Corollary 2. *If* $\tilde{\mathbf{s}}$ *is a sequential equilibrium of a public-monitoring repeated game with two players and full support,* $\tilde{\mathbf{s}}(\emptyset)$ *is pure, and there exist two signals* $\bar{y}$ *and* $\underline{y}$ *such that* $G(\tilde{\mathbf{s}})$ *is regular after* $\tilde{\mathbf{s}}(\emptyset)$*, then* $\tilde{\mathbf{s}}$ *is not robust to private-monitoring perturbations.*

This result shows that the robustness result in Subsection 3.3 is not because the equilibrium in the original game is in private strategies, but because we use a flexible notion of

[20] An equilibrium of a static game G is *contagious* if it is selected in a global game with some noise distribution. Frankel, Morris, and Pauzner (2003) show that every supermodular game has at least one (and only one, generically) contagious equilibrium.

closeness of strategies: in Subsection 3.3, we use a small "buffer area" on which players play completely different actions between the original and perturbed games; in Corollary 2, we use the notion of robustness (Definitions 2 and 3) that requires players to play similarly on any history.

6 A Robustness Result

In this section, we show that belief-free equilibria are robust to private-monitoring perturbations. Once again, we focus on two-player games.

Definition 5 (Ely, Hörner, and Olszewski, 2005)**.** A pair of strategies $\tilde{\mathbf{s}} = (\tilde{s}_1, \tilde{s}_2)$ in the repeated game with public monitoring is a *belief-free equilibrium* if, for every $i \in I$ and $\tilde{h}_{i,t} \in \tilde{H}_{i,t}, \tilde{h}_{j,t} \in \tilde{H}_{j,t}$, $U_i(\tilde{s}_i \mid \tilde{h}_{i,t}, \tilde{s}_j \mid \tilde{h}_{j,t}) \geq U_i(\tilde{s}'_i, \tilde{s}_j \mid \tilde{h}_{j,t})$ for all $\tilde{s}'_i$.

The Ely-Välimäki equilibrium we discussed in Section 3.2 is a belief-free equilibrium. Note that no belief-free public equilibrium induces a regular 2×2 game since both $\tilde{s}_{i,t} \mid \bar{y}$ and $\tilde{s}_{i,t} \mid \underline{y}$ are the best responses to both $\tilde{s}_{j,t} \mid \bar{y}$ and $\tilde{s}_{j,t} \mid \underline{y}$

We also define belief-free equilibria for the perturbed game. Namely, a pair $\mathbf{s} = (s_1, s_2)$ of strategies in the perturbed game is a *belief-free equilibrium* if, for every $i \in I$, $h_{i,t} \in H_{i,t}$, and $h_{j,t} \in H_{j,t}$, $s_i \mid h_{i,t}$ is a best response for player i against $s_j \mid h_{j,t}$ in the perturbed game starting in period t.

6.1 Unperturbed Marginal Distributions

Suppose that a private-monitoring perturbation $(\Omega, \pi, \mathbf{f})$ does not alter *marginal* distributions of interpreted signals. For example, $Y = \Omega_1 = \Omega_2 = \{\overline{y}, \underline{y}\}$ with identity interpretations $\mathbf{f}$, and $\pi(\cdot \mid \mathbf{a})$ is given by

	$\overline{y}$	$\underline{y}$
$\overline{y}$	$\rho(\overline{y} \mid \mathbf{a}) - \varepsilon$	ε
$\underline{y}$	ε	$\rho(\underline{y} \mid \mathbf{a}) - \varepsilon$

.

Then every belief-free public equilibrium remains a belief-free equilibrium in the perturbed game (under the translation via exogenous interpretations $\mathbf{f}$ for the entire horizon). This follows immediately from the definition of belief-free equilibria.

Proposition 1. *Let* $\tilde{\mathbf{s}}$ *be a belief-free equilibrium in* (Y, ρ)*. If* $\pi(\{\boldsymbol{\omega} \in \Omega : f_i(\omega_i) = y\} \mid \mathbf{a}) = \rho(y \mid \mathbf{a})$ *for all* $i \in I$*,* $\mathbf{a} \in A$*, and* $y \in Y$*, then the pair of private strategies,* $\mathbf{s} = (s_1, s_2)$*, defined by*

$$s_i(h_{i,t}) = \tilde{s}_i(f_i(h_{i,t}))$$

for all $t \geq 1$*,* $i \in I$*, and* $h_{i,t} = (a_i^{t-1}, \omega_i^{t-1}) \in H_{i,t}$*, is a belief-free equilibrium of the perturbed game.*

6.2 General Perturbations

With a slight perturbation on the marginal distributions of interpreted signals, a belief-free equilibrium may not be an equilibrium in the perturbed game with the straightforward interpretation for all t. As we saw in Section 3.2, however, we can restore players' incentives by modifying future actions.

A mixed action $\alpha_i \in \Delta(A_i)$ of player i has *individual full rank in* (Y, ρ) if the collection $(\rho(\cdot \mid \alpha_i, a_j))_{a_j \in A_j}$ of $|Y|$-dimensional vectors is linearly independent. If some action of player i has individual full rank, then $|Y| \geq |A_j|$.

For a strategy $\tilde{s}_j$, let $V_i(\tilde{s}_j) := \max_{\tilde{s}_i} U_i(\tilde{s}_i, \tilde{s}_j)$. A *regime* is a product of nonempty subsets of A_1 and A_2, $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. For a nonempty subset $\mathcal{A}_i$ of A_i, we say that player i's payoff v_i is $\mathcal{A}_i$*-enforced by* $\alpha_j \in \Delta(A_j)$ *and* $w_i \colon A_j \times Y \to \mathbb{R}$ if we have

$$v_i \geq (1-\delta) u_i(a_i, \alpha_j) + \delta \mathbb{E}\left[w_i(a_j, y) \mid a_i, \alpha_j\right]$$

for all $a_i \in A_i$ with equality if $a_i \in \mathcal{A}_i$, where the expectation is taken over (a_j, y) generated by (a_i, α_j) and ρ.

Definition 6. A belief-free equilibrium $\tilde{\mathbf{s}}$ is *compatible with a sequence* $\{\mathcal{A}_t\}$ *of regimes* if the following conditions are satisfied for all $t \geq 1$, $i \in I$, $\tilde{h}_{i,t} \in \tilde{H}_{i,t}$, and $\tilde{h}_{j,t} \in \tilde{H}_{j,t}$:

1. $\operatorname{supp} \tilde{s}_i(\tilde{h}_{i,t}) \subseteq \mathcal{A}_{i,t}$.

2. $V_i(\tilde{s}_j \mid \tilde{h}_{j,t})$ is $\mathcal{A}_{i,t}$-enforced by $\tilde{s}_j(\tilde{h}_{j,t})$ and $\{V_i(\tilde{s}_j \mid (h_{j,t}, a_{j,t}, y_t))\}_{a_{j,t}\in A_j, y_t \in Y}$.

Every belief-free equilibrium is compatible with some sequence of regimes.[21] For notational simplicity, we focus on a fixed regime: $\mathcal{A}_t = \mathcal{A}$ for all $t \geq 1$.[22] For a given regime $\mathcal{A}$, the set of all belief-free equilibria that are compatible with $\mathcal{A}$ satisfies the following properties.

exchangeability If $(\tilde{s}_1, \tilde{s}_2)$ and $(\tilde{s}'_1, \tilde{s}'_2)$ are belief-free equilibria compatible with $\mathcal{A}$, then so are $(\tilde{s}_1, \tilde{s}'_2)$ and $(\tilde{s}'_1, \tilde{s}_2)$ (Ely, Hörner, and Olszewski, 2005, Proposition 1).

recursion If $(\tilde{s}_1, \tilde{s}_2)$ is a belief-free equilibrium compatible with $\mathcal{A}$, then so is $(\tilde{s}_1 \mid \tilde{h}_{1,t}, \tilde{s}_2 \mid \tilde{h}_{2,t})$ for every $\tilde{h}_{1,t} \in \tilde{H}_{1,t}, \tilde{h}_{2,t} \in \tilde{H}_{2,t}$.

Let $W(\mathcal{A})$ be the set of payoff profiles sustained by belief-free equilibria compatible with $\mathcal{A}$. By the exchangeability, $W(\mathcal{A})$ is a product of two compact intervals, $W(\mathcal{A}) = [\underline{w}_1(\mathcal{A}), \overline{w}_1(\mathcal{A})] \times [\underline{w}_2(\mathcal{A}), \overline{w}_2(\mathcal{A})]$ (Ely, Hörner, and Olszewski, 2005, Corollary 1).

A product set $W = W_1 \times W_2$ is *self-$\mathcal{A}$-generating* if, for every $i \in I$ and $v_i \in W_i$, v_i is $\mathcal{A}_i$-enforced by some $\alpha_j \in \Delta(\mathcal{A}_j)$ and $w_i \colon A_j \times Y \to W_i$. By the recursion, $W(\mathcal{A})$ is self-$\mathcal{A}$-generating. Also, every bounded self-$\mathcal{A}$-generating product set is a subset of $W(\mathcal{A})$. We say that $W(\mathcal{A})$ is *properly self-$\mathcal{A}$-generating* if, for every $i \in I$, $\underline{w}_i(\mathcal{A})$ is $\mathcal{A}_i$-enforced by some $\underline{\alpha}_j \in \Delta(\mathcal{A}_j)$ and $\underline{w}_i \colon A_j \times Y \to [\underline{w}_i(\mathcal{A}), \overline{w}_i(\mathcal{A}))$ (excluding the right endpoint), and $\overline{w}_i(\mathcal{A})$ is $\mathcal{A}_i$-enforced by some $\overline{\alpha}_j \in \Delta(\mathcal{A}_j)$ and $\overline{w}_i \colon A_j \times Y \to (\underline{w}_i(\mathcal{A}), \overline{w}_i(\mathcal{A})]$ (excluding the left endpoint). Note that $W(\mathcal{A})$ with a nonempty interior is properly self-$\mathcal{A}$-generating if δ is sufficiently large (Ely, Hörner, and Olszewski, 2005, p. 391).

We say that a belief-free equilibrium $\tilde{\mathbf{s}} = (\tilde{s}_1, \tilde{s}_2)$ compatible with $\mathcal{A}$ is *interior* if $\underline{w}_i(\mathcal{A}) < V_i(\tilde{s}_j \mid \tilde{h}_{j,t}) < \overline{w}_i(\mathcal{A})$ for all $i \in I$ and $\tilde{h}_{j,t} \in \tilde{H}_{j,t}$.

[21]Our definition of regimes slightly generalizes those of Ely, Hörner, and Olszewski (2005) and of Yamamoto (2009). According to our definition, a belief-free public equilibrium may be compatible with multiple sequences of regimes, and Ely, Hörner, and Olszewski (2005) and Yamamoto (2009) provide explicit ways of constructing such regime sequences.

[22]Extensions to time-varying regimes are straightforward.

In Theorem 2, we show that in a two-player repeated game with individual full rank such that $W(\mathcal{A})$ is properly self-$\mathcal{A}$-generating, any interior $\mathcal{A}$-compatible belief-free equilibrium is robust to private-monitoring perturbations. This generalizes Claim 2 in Subsection 3.2 because for regime $\mathcal{A} = \{C, D\} \times \{C, D\}$, $W(\mathcal{A}) = [0, 2] \times [0, 2]$ is properly self-$\mathcal{A}$-generating for $\delta > 1/3$, and the Ely-Välimäki equilibrium with $\alpha < 1$ and $\beta > 0$ is interior.

Theorem 2. *If* $\tilde{\mathbf{s}}$ *is an interior* $\mathcal{A}$*-compatible belief-free equilibrium of a public-monitoring repeated game with two players, every pure action has individual full rank in* (Y, ρ)*, and* $W(\mathcal{A})$ *is properly self-*$\mathcal{A}$*-generating, then* $\tilde{\mathbf{s}}$ *is robust to private-monitoring perturbations.*

Proof. Fix $T < \infty$. We will show that there exists $\varepsilon > 0$ such that, if a private-monitoring perturbation $(\Omega, \pi, \mathbf{f})$ is ex-ante ε-close to (Y, ρ), then the perturbed game admits a belief-free equilibrium that is $(0, T)$-close to $\tilde{\mathbf{s}}$ via $\mathbf{f}$.

For each $i \in I$, $\mathbf{a} \in A$, and $y \in Y$, define

$$\pi_i(y \mid \mathbf{a}) := \pi(\{\boldsymbol{\omega} \in \Omega : f_i(\omega_i) = y\} \mid \mathbf{a}).$$

Since $(\Omega, \pi, \mathbf{f})$ is ex-ante ε-close to (Y, ρ), we have $|\pi_i(y \mid \mathbf{a}) - \rho(y \mid \mathbf{a})| \leq (|Y| - 1)\varepsilon$.

Since $\tilde{\mathbf{s}}$ is an interior belief-free equilibrium compatible with $\mathcal{A}$, for every $t \geq 1$, $i \in I$, $\tilde{h}_{i,t} \in \tilde{H}_{i,t}$, and $\tilde{h}_{j,t} \in \tilde{H}_{j,t}$, we have

$$\begin{aligned} V_i(\tilde{s}_j \mid \tilde{h}_{j,t}) &\geq (1 - \delta) u_i(a_{i,t}, \tilde{s}_j(\tilde{h}_{j,t})) + \delta \mathbb{E}_\rho[V_i(\tilde{s}_j \mid (\tilde{h}_{j,t}, a_{j,t}, y_t)) \mid a_{i,t}, \tilde{s}_j(\tilde{h}_{j,t})] \\ &= \mathbb{E}_\rho[(1 - \delta) u_i^*(a_{i,t}, y_t) + \delta V_i(\tilde{s}_j \mid (\tilde{h}_{j,t}, a_{j,t}, y_t)) \mid a_{i,t}, \tilde{s}_j(\tilde{h}_{j,t})] \end{aligned}$$

for all $a_{i,t} \in A_i$ with equality if $a_{i,t} \in \mathcal{A}_i$, and $\underline{w}_i(\mathcal{A}) < V_i(\tilde{s}_j \mid \tilde{h}_{j,t}) < \overline{w}_i(\mathcal{A})$, where the expectation is taken over $(a_{j,t}, y_t)$ generated by $(a_{i,t}, \tilde{s}_j(\tilde{h}_{j,t}))$ and ρ.

Fix $i \in I$. We define $w_i \colon \tilde{H}_{j,t} \to (\underline{w}_i(\mathcal{A}), \overline{w}_i(\mathcal{A}))$ for each $t \leq T + 1$ recursively as follows. Let $w_i(\emptyset) = V_i(\tilde{s}_j) \in (\underline{w}_i(\mathcal{A}), \overline{w}_i(\mathcal{A}))$. For each $t = 1, \ldots, T$ and $\tilde{h}_{j,t} \in \tilde{H}_{j,t}$, given that $w_i(\tilde{h}_{j,t}) \in (\underline{w}_i(\mathcal{A}), \overline{w}_i(\mathcal{A}))$, it follows from the individual full rank that, for sufficiently small $\varepsilon > 0$, there exists $\{w_i(\tilde{h}_{j,t}, a_j, y)\}_{a_j \in A_j, y \in Y}$ such that

$$w_i(\tilde{h}_{j,t}) \geq \mathbb{E}_\pi[(1 - \delta) u_i^*(a_{i,t}, f_j(\omega_{j,t})) + \delta w_i(\tilde{h}_{j,t}, a_{j,t}, f_j(\omega_{j,t})) \mid a_{i,t}, \tilde{s}_j(\tilde{h}_{j,t})]$$

for all $a_{i,t} \in A_i$ with equality if $a_{i,t} \in \mathcal{A}_i$, and $w_i(\tilde{h}_{j,t}, a_{j,t}, y_t) \in (\underline{w}_i(\mathcal{A}), \overline{w}_i(\mathcal{A}))$ for all $a_{j,t} \in A_j$ and $y_t \in Y$, where the expectation is taken over $(a_{j,t}, \omega_{j,t})$ generated by $(a_{i,t}, \tilde{s}_j(\tilde{h}_{j,t}))$ and π. Further, $|w_i(\tilde{h}_{j,t}, a_{j,t}, y_t) - V_i(\tilde{h}_{j,t}, a_{j,t}, y_t)|$ goes to 0 as ε goes to 0 and $|w_i(\tilde{h}_{j,t}) - V_i(\tilde{h}_{j,t})|$ goes to 0.

The following lemma shows that the set of belief-free equilibrium payoffs compatible with $\mathcal{A}$ is lower hemicontinuous with respect to private-monitoring perturbations.

Lemma 5. *If every pure action has individual full rank in (Y, ρ) and $W(\mathcal{A})$ is properly self-$\mathcal{A}$-generating, then there exists $\bar{\gamma} > 0$ such that, if $0 < \gamma < \bar{\gamma}$, then there exists $\varepsilon > 0$ such that $[\underline{w}_1(\mathcal{A}) + \gamma, \overline{w}_1(\mathcal{A}) - \gamma] \times [\underline{w}_1(\mathcal{A}) + \gamma, \overline{w}_1(\mathcal{A}) - \gamma]$ is self-$\mathcal{A}$-generating in any perturbed game that is ex-ante ε-close to (Y, ρ).*

Proof of Lemma 5. See Appendix. ∎

For each $\tilde{h}_{1,T+1} \in \tilde{H}_{1,T+1}$, $\tilde{h}_{2,T+1} \in \tilde{H}_{2,T+1}$, since $w_1(\tilde{h}_{2,T+1}) \in (\underline{w}_1(\mathcal{A}), \overline{w}_1(\mathcal{A}))$ and $w_2(\tilde{h}_{1,T+1}) \in (\underline{w}_2(\mathcal{A}), \overline{w}_2(\mathcal{A}))$, by Lemma 5, for sufficiently small $\varepsilon > 0$, there exists a belief-free equilibrium $(\sigma_1^{\tilde{h}_{1,T+1}}, \sigma_2^{\tilde{h}_{2,T+1}})$ of the perturbed game starting in period $T+1$ that sustains $(w_1(\tilde{h}_{2,T+1}), w_2(\tilde{h}_{1,T+1}))$. Note that, by the exchangeability of belief-free equilibria, we can assume without loss of generality that the choice of player 1's strategy is independent of $\tilde{h}_{2,T+1}$, and that the choice of player 2's strategy is independent of $\tilde{h}_{1,T+1}$.

For each $t \geq 1$, $i \in I$, and $h_{i,t} = (a_i^{t-1}, \omega_i^{t-1}) \in H_{i,t}$, let

$$s_i(h_{i,t}) = \begin{cases} \tilde{s}_i(f_i(h_{i,t})) & \text{if } t \leq T \\ \sigma_i^{f_i(h_{i,T+1})}({}_T h_{i,t}) & \text{if } t \geq T+1, \end{cases}$$

where ${}_T h_{i,t}$ is the truncation of $h_{i,t}$ by removing the first T periods. Then $\mathbf{s} = (s_1, s_2)$ is a belief-free equilibrium of the perturbed game that is $(0, T)$-close to $\tilde{\mathbf{s}}$ via $\mathbf{f}$. ∎

We can strengthen the positive result by allowing for a larger class of perturbations. In this direction, Theorem 2 extends easily to perturbed games that differ from the original game not only in the monitoring structure, but also in payoffs.

On the contrary, we can ask what happens if we use a more restrictive class of private-monitoring strategies. For example, one can require that an equilibrium in the private-monitoring repeated game be (η, T)-close to the original equilibrium via exogenous interpretations with $\eta = 0$ or with $T = \infty$. Here, $\eta = 0$ means that each player's action is exactly the same as her action in the original equilibrium, and $T = \infty$ means that the two strategy profiles prescribe similar actions uniformly over all histories. Proposition 1 used $\eta = 0$ and $T = \infty$, and the current proof of Theorem 2 shows $\eta = 0$. We conjecture that Theorem 2 would hold for $\eta > 0$ but $T = \infty$ under additional regularity conditions.

A Appendix

A.1 Proof of Claim 3

We construct an equilibrium $\mathbf{s} = (s_1, s_2)$ in the perturbed game with $\varepsilon > 0$ that is close to the original equilibrium $\tilde{\mathbf{s}}$.

As explained in the main text, the strategy is recursive every other period. In period 1, both players are in state G, i.e., $x_1 = x_2 = G$. In each subsequent odd period $t = 3, \ldots$, each player i determines a state $x_i \in \{G, B\}$, and takes C if $x_i = G$; D if $x_i = B$. In period $t+1$, she takes C if she observes $\omega_{i,t} \in \Omega_i^\varepsilon \cap f_i^{-1}(\bar{y})$; D if she observes $\omega_{i,t} \in \Omega_i^\varepsilon \cap f_i^{-1}(\underline{y})$; her action will be specified later if $\omega_{i,t} \notin \Omega_i^\varepsilon$. With a slight abuse of notations, let $s_i(x_i)$ denote her action plan in periods t and $t+1$ described.

We write $p_j^\varepsilon[x_j](h_j) = p_j^\varepsilon[x_j](a_{j,t}, \omega_{j,t}, a_{j,t+1}, \omega_{j,t+1})$ with superscript ε for the probability that player j's next state will be G in period $t+2$ if her current state is x_j in period t and she observes $h_j = (a_{j,t}, \omega_{j,t}, a_{j,t+1}, \omega_{j,t+1})$ for the last two periods.

We specify $p_j^\varepsilon[x_j](h_j)$ so that conditional on $x_j \in \{G, B\}$, both $s_i(G)$ and $s_i(B)$ are optimal and attain the value $v_i^\varepsilon(x_j)$:

$$v_i^\varepsilon(x_j) = \max_{s_i} \mathbb{E}\left[(1-\delta^2)\left(u_i(\mathbf{a}_t) + \delta u_i(\mathbf{a}_{t+1})\right)\right.$$
$$\left. +\delta^2 \left(p_j[x_j](h_j) v_i^\varepsilon(G) + \left(1 - p_j^\varepsilon[x_j](h_j)\right) v_i^\varepsilon(B)\right) \mid s_i, s_j(x_j)\right]. \tag{14}$$

For sufficiently large δ, it is equivalent to consider the reward functions $\theta_j^\varepsilon[x_j](h_j)$ that is uniformly bounded with respect to δ with

$$\begin{aligned}
v_i^\varepsilon(G) &= \frac{1}{1+\delta} \max_{s_i} \mathbb{E}\left[u_i(\mathbf{a}_t) + \delta u_i(\mathbf{a}_{t+1}) + \theta_j^\varepsilon[G](h_j) \mid s_i, s_j(G)\right], \\
v_i^\varepsilon(B) &= \frac{1}{1+\delta} \max_{s_i} \mathbb{E}\left[u_i(\mathbf{a}_t) + \delta u_i(\mathbf{a}_{t+1}) + \theta_j^\varepsilon[B](h_j) \mid s_i, s_j(B)\right], \\
v_i^\varepsilon(G) &> v_i^\varepsilon(B), \\
&0 \geq \theta_j^\varepsilon[G](h_j), \\
&0 \leq \theta_j^\varepsilon[B](h_j).
\end{aligned}$$

Then,

$$\begin{aligned}
p_j[G](h_j) &= 1 + \frac{(1-\delta)}{\delta^2\left(v_i^\varepsilon(G) - v_i^\varepsilon(B)\right)} \theta_j^\varepsilon[G](h_j), \\
p_j[B](h_j) &= \frac{(1-\delta)}{\delta^2\left(v_i^\varepsilon(G) - v_i^\varepsilon(B)\right)} \theta_j^\varepsilon[B](h_j)
\end{aligned} \tag{15}$$

satisfy (14).

We specify $\theta_j^\varepsilon[x_j](h_j)$ as follows:

1. At state $x_j = G$:

   (a) $\theta_j^\varepsilon[G](h_j)$ is additively separable: $\theta_j^\varepsilon[G](h_j) = \theta_j^\varepsilon[G](f_j(\omega_{j,t})) + \delta\theta_j^\varepsilon[G](\omega_{j,t}, a_{j,t+1}, f_j(\omega_{j,t+1}))$.

   (b) In period $t+1$,

      i. After player j observes $\omega_{j,t} \in \Omega_j^\varepsilon$ (i.e., $|\omega_{j,t}| \geq \varepsilon/2$), we set $\theta_j^\varepsilon[G](\omega_{j,t}, a_{j,t+1}, f_j(\omega_{j,t+1})) = \theta_j[G](f_j(\omega_{j,t}), f_j(\omega_{j,t+1}))$.

      ii. After player j observes $\omega_{j,t} \notin \Omega_j^\varepsilon$ (i.e., $\omega_{j,t} \in (-\varepsilon/2, \varepsilon/2)$), we determine $\theta_j^\varepsilon[G](\omega_{j,t}, a_{j,t+1}, f_j(\omega_{j,t+1}))$ so that player i is indifferent between C and D in period $t+1$, conditional on $a_{j,t+1}$. Further, player i's expected payoff at the beginning of $t+1$ is constant regardless of $\mathbf{a}_{t+1}$:

$$\mathbb{E}\left[u_i(\mathbf{a}_{t+1}) + \theta_j^\varepsilon[G](\omega_{j,t}, a_{j,t+1}, f_j(\omega_{j,t+1})) \mid \mathbf{a}_{t+1}\right] = \text{constant}. \tag{16}$$

Specifically, let

$$\theta_j^\varepsilon[G](\omega_{j,t}, a_{j,t+1}, \bar{y}) = \begin{cases} -\left(2 - \frac{1-p}{p-q} - \frac{1-r}{q-r}\right) & \text{if } a_{j,t+1} = C, \\ 0 & \text{if } a_{j,t+1} = D. \end{cases}$$

$$\theta_j^\varepsilon[G](\omega_{j,t}, a_{j,t+1}, \underline{y}) = \begin{cases} -\left(2 - \frac{1-p}{p-q} - \frac{1-r}{q-r}\right) - \frac{1}{p-q} & \text{if } a_{j,t+1} = C, \\ -\frac{1}{q-r} & \text{if } a_{j,t+1} = D. \end{cases}$$

Then, from (2), $\theta_j^\varepsilon[G](\omega_{j,t}, a_{j,t+1}, f_j(\omega_{j,t+1})) \le 0$ and

$$\mathbb{E}\left[u_i(\mathbf{a}_{t+1}) + \theta_j^\varepsilon[G](\omega_{j,t}, a_{j,t+1}, f_j(\omega_{j,t+1})) \mid \mathbf{a}_{t+1}\right] = -\frac{1-r}{q-r}.$$

Note that we have not yet specified player j's strategy after observing $\omega_{j,t} \notin \Omega_j^\varepsilon$. However, (16) guarantees that the specification of player j's action $a_{j,t+1}$ after observing $\omega_{j,t} \notin \Omega_j^\varepsilon$ does not change player i's continuation payoff. Therefore,

$$v_i^\varepsilon[G](a_{i,t}, \omega_{i,t}) = \max_{a_{i,t+1}} \mathbb{E}\left[u_i(\mathbf{a}_{t+1}) + \theta_j^\varepsilon[G](\omega_{j,t}, a_{j,t+1}, f_j(\omega_{j,t+1})) \mid a_{i,t}, a_{i,t+1}, \omega_{i,t}, s_j(G)\right]$$

is uniquely defined without specifying $s_j(G)$ after observing $\omega_{j,t} \notin \Omega_j^\varepsilon$.

Here is a good place to fully specify $s_i(x_i)$ for $x_i \in \{G, B\}$. As mentioned in the main text, we choose player i's action $a_{i,t+1}$ in period $t+1$ that maximizes

$$\mathbb{E}\left[u_i(\mathbf{a}_{t+1}) + \theta_j^\varepsilon[G](\omega_{j,t}, a_{j,t+1}, f_j(\omega_{j,t+1})) \mid a_{i,t}, a_{i,t+1}, \omega_{i,t}, s_j(G)\right].$$

(If there are multiple maximizers, choose one arbitrarily.)

(c) In period t, we determine $\theta_j^\varepsilon[G](f_j(\omega_{j,t}))$ so that player i is indifferent between C and D in period t. Specifically, we choose $\theta_j^\varepsilon[G](\underline{y})$ and $\theta_j^\varepsilon[G](\bar{y})$ that satisfy

$$\begin{aligned} v_i^\varepsilon(G) &= \frac{1}{1+\delta}\left(2 + p\theta_j^\varepsilon[G](\bar{y}) + (1-p)\,\theta_j^\varepsilon[G](\underline{y}) + \delta\mathbb{E}\left[v_i^\varepsilon[G](C, \omega_{i,t}) \mid (a_{i,t}, a_{j,t}) = (C, C)\right]\right) \\ &= \frac{1}{1+\delta}\left(3 + q\theta_j^\varepsilon[G](\bar{y}) + (1-q)\,\theta_j^\varepsilon[G](\underline{y}) + \delta\mathbb{E}\left[v_i^\varepsilon[G](D, \omega_{i,t}) \mid (a_{i,t}, a_{j,t}) = (D, C)\right]\right). \end{aligned} \tag{17}$$

Since the equation (17) converges to the equation (7) as $\varepsilon \to 0$, it is easy to see that we can find a solution for (17) for each ε such that

$$\theta_j^{\varepsilon}[G](y) \uparrow \theta_j[G](y) \le 0 \tag{18}$$

as $\varepsilon \to 0$ for $y = \underline{y}, \bar{y}$.

2. At state $x_j = B$:

   (a) $\theta_j^{\varepsilon}[B](h_j)$ is additively separable: $\theta_j^{\varepsilon}[B](h_j) = \theta_j^{\varepsilon}[B](f_j(\omega_{j,t})) + \delta\theta_j^{\varepsilon}[B](a_{j,t+1}, f_j(\omega_{j,t+1}))$.

   (b) In period $t+1$, we have $\theta_j^{\varepsilon}[B](a_{j,t+1}, f_j(\omega_{j,t+1})) = \theta_j[B](a_{j,t+1}, f_j(\omega_{j,t+1}))$ so that player i is indifferent between C and D in period $t+1$, conditional on $a_{j,t+1}$.

   (c) In period t, we determine $\theta_j^{\varepsilon}[B](f_j(\omega_{j,t}))$ so that player i is indifferent between C and D in period t. Specifically, we choose $\theta_j^{\varepsilon}[B](\underline{y})$ and $\theta_j^{\varepsilon}[B](\bar{y})$ that satisfy

$$\begin{aligned}
v_i^{\varepsilon}(B) = \frac{1}{1+\delta}\Bigg(& -1 + q\theta_j^{\varepsilon}[B](\bar{y}) + (1-q)\theta_j^{\varepsilon}[B](\underline{y}) \\
& + \delta\left(\Pr(a_{j,t+1} = C \mid (a_{i,t}, a_{j,t}) = (C,D), s_j(B))\left(2 + \frac{p}{p-q}\right)\right. \\
& \left. + \Pr(a_{j,t+1} = D \mid (a_{i,t}, a_{j,t}) = (C,D), s_j(B))\left(-1 + \frac{q}{q-r}\right)\right)\Bigg) \\
= \frac{1}{1+\delta}\Bigg(& r\theta_j^{\varepsilon}[B](\bar{y}) + (1-r)\theta_j^{\varepsilon}[B](\underline{y}) \\
& + \delta\left(\Pr(a_{j,t+1} = C \mid (a_{i,t}, a_{j,t}) = (D,D), s_j(B))\left(2 + \frac{p}{p-q}\right)\right. \\
& \left. + \Pr(a_{j,t+1} = D \mid (a_{i,t}, a_{j,t}) = (D,D), s_j(B))\left(-1 + \frac{q}{q-r}\right)\right)\Bigg)
\end{aligned} \tag{19}$$

   It is easy to find a solution for (19) for each ε such that

$$\theta_j^{\varepsilon}[B](y) \downarrow \theta_j[B](y) \ge 0 \tag{20}$$

   as $\varepsilon \to 0$ for $y = \underline{y}, \bar{y}$.

Now we show that the above strategy constitutes an equilibrium in the perturbed game. First, we verify player i's incentive in period $t+1$ after observing $(a_{i,t}, \omega_{i,t})$. Since any action is optimal if $x_j = B$ by 2-(b) (note that $\theta_j^\varepsilon[B](f_j(\omega_{j,t}))$ is a "sunk cost" in period $t+1$), regardless of player i's belief about x_j, player i plays a best response to $s_j(G)$. Further, Condition 1-(b)-ii ensures that player i plays a best response to $s_j(G)$ conditional on that $\omega_{j,t} \in \Omega_j^\varepsilon$, regardless of player i's belief about $\omega_{j,t} \notin \Omega_j^\varepsilon$. Conditional on the event that $x_j = G$ and $\omega_{j,t} \in \Omega_j^\varepsilon$, Condition 1-(b)-i together with Lemma 1 implies that after observing $\omega_{i,t} \in \Omega_i^\varepsilon$, player i believes with large probability that player j observes $\omega_{j,t} \in f_j^{-1}(f_i(\omega_{i,t}))$, i.e., player j's interpretation of $\omega_{j,t}$ is the same as player i's interpretation of $\omega_{i,t}$. Note that player i has a strict incentive in the original equilibrium $\tilde{\mathbf{s}}$, and after $\omega_{j,t} \in \Omega_j^\varepsilon$, we use the same reward function as in the original game. Thus, in the perturbed game, if ε is sufficiently small, player i strictly prefers C after observing $\omega_{i,t} \in \Omega_i^\varepsilon \cap f_i^{-1}(\bar{y})$ and D after observing $\omega_{i,t} \in \Omega_i^\varepsilon \cap f_i^{-1}(\underline{y})$.

Second, we verify player i's incentive in period t. By 1-(c) and 2-(c), player i is indifferent between C and D, regardless of player i's belief about x_j.

Finally, we show that the above equilibrium is close to the original equilibrium. Clearly $s_i(\emptyset) = \tilde{s}_i(\emptyset) = C$ in period 1. Take any odd period t. Let x_i be player i's state in period t. In period t, player i takes C if $x_i = G$ and D if $x_i = D$ as in the original game. In period $t+1$, player i takes C if player i observes $\omega_{i,t} \in \Omega_i^\varepsilon \cap f_i^{-1}(\bar{y})$ and D if player i observes $\omega_{i,t} \in \Omega_i^\varepsilon \cap f_i^{-1}(\underline{y})$. Therefore, as long as $\omega_{i,t} \in \Omega_i^\varepsilon$, given player i's state in period t, player i's action plan in the next two periods is the same as in the original equilibrium. Now consider state transitions. To show that state transitions are close to each other in the original and perturbed equilibria, from (5) and (15), it suffices to show that rewards are close to each other if $\omega_{i,t} \in \Omega_i^\varepsilon$. Note that $\theta_i^\varepsilon[G](\omega_{i,t}, a_{i,t+1}, f_i(\omega_{i,t+1}))$ and $\theta_i^\varepsilon[B](a_{i,t+1}, f_i(\omega_{i,t+1}))$ are the same as $\theta_i[G](f_i(\omega_{i,t}), f_i(\omega_{i,t+1}))$ and $\theta_i[B](a_{i,t+1}, f_i(\omega_{i,t+1}))$ if $\omega_{i,t} \in \Omega_i^\varepsilon$ as desired. In addition, by (18) and (20), $\theta_i^\varepsilon[x_i](f_i(\omega_{i,t})) \to \theta_i[x_i](f_i(\omega_{i,t}))$ as $\varepsilon \to 0$ for $x_i \in \{G, B\}$ and $\omega_{i,t} \in \Omega_i^\varepsilon \subseteq \Omega_i$.

Since player i's action in odd period t can pin down player i's state x_i in that period,

$s_i(h_{i,t})$ is close to $\tilde{s}_i(f_i(h_{i,t}))$ if $\omega_{i,\tau} \in \Omega_i^\varepsilon = \mathbb{R} \setminus (-\varepsilon/2, \varepsilon/2)$ for all $\tau \leq t-1$.

A.2 Proof of Lemma 4

Let $\zeta := (1-p_1-p_2)/4 > 0$. Let $C_1 := 2 - \sqrt{2(p_1+2\zeta)} \in [1,2)$ and $C_2 := 1$ if $p_1 \leq p_2$, and $C_1 := 1$ and $C_2 := 2 - \sqrt{2(p_2+2\zeta)} \in [1,2)$ if $p_1 > p_2$. Let $D := \max\{C_1, C_2\} + 1$. Let λ denote the Lebesgue measure on $\mathbb{R}$. The following lemmas are useful.

Lemma 6. *There exists $d > 0$ such that the following is true. For every $\varepsilon > 0$ and $x \in [-B + D\varepsilon, B - D\varepsilon]$, we have*

$$\pi^\varepsilon(\omega_{j,1} \in [x - C_j\varepsilon, x + C_j\varepsilon] \mid \mathbf{a}_1, \omega_{i,1}) \geq p_i + \zeta$$

for all $i \in I$, $\mathbf{a}_1 \in A$ and $\omega_{i,1} \in [x - (C_i + d)\varepsilon, x + C_i\varepsilon]$.

Proof. Take $d > 0$ sufficiently small. Then, noting that, for $\omega_{i,1} \in [-B + \varepsilon/2, B - \varepsilon/2]$, the probability density of $\omega_{j,1}$ with respect to the Lebesgue measure conditional on $\mathbf{a}_1$ and $\omega_{i,1}$ is given by

$$\frac{d\pi^\varepsilon(\omega_{j,1} \mid \mathbf{a}_1, \omega_{i,1})}{d\lambda} = \begin{cases} 1/\varepsilon - |\omega_{j,1} - \omega_{i,1}|/\varepsilon^2 & \text{if } \omega_{j,1} \in [\omega_{i,1} - \varepsilon, \omega_{i,1} + \varepsilon], \\ 0 & \text{otherwise,} \end{cases}$$

a simple algebra yields the result. ∎

Lemma 7. *There exist $\bar{\bar{\varepsilon}}, K > 0$ such that, for every $\eta > 0$ and $\varepsilon < \bar{\bar{\varepsilon}}$, if $g_{1,1}$ satisfies*

$$\pi^\varepsilon(\{\boldsymbol{\omega}_1 \in \mathbb{R}^2 : g_{1,1}(\omega_{1,1} \mid a_{1,1}) = f_1(\omega_{1,1})\} \mid a_{1,1}, a_{2,1}) \geq 1 - \eta \tag{21}$$

for all $(a_{1,1}, a_{2,1}) \in A$, then there exists $x_0 \in [D\varepsilon, B - D\varepsilon]$ such that

$$\lambda(\{\omega_{1,1} \in [x_0 - C_1\varepsilon, x_0 + C_1\varepsilon] : g_{1,1}(\omega_{1,1} \mid a_{1,1}) = \underline{y}\}) < K\eta\varepsilon \tag{22}$$

for all $a_{1,1} \in A_1$.

Proof. Fix any $K > 2C_1|A_1|/B$. Suppose that there exist $\eta > 0$, sufficiently small $\varepsilon > 0$, and $g_{1,1}$ such that (21) holds, but (22) does not. Pick any $a_{2,1} \in A_2$. Since the probability density of $\omega_{1,1}$ with respect to the Lebesgue measure conditional on $(a_{1,1}, a_{2,1})$ is 1 on $\omega_{1,1} \in [-B+\varepsilon/2, B-\varepsilon/2]$, by (21),

$$\begin{aligned}
&\lambda(\{\omega_{1,1} \in [D\varepsilon, B - D\varepsilon] : g_{1,1}(\omega_{1,1} \mid a_{1,1}) = \underline{y} \text{ for some } a_{1,1} \in A_1\}) \\
&\leq \sum_{a_{1,1} \in A_1} \lambda(\{\omega_{1,1} \in [D\varepsilon, B - D\varepsilon] : g_{1,1}(\omega_{1,1} \mid a_{1,1}) = \underline{y}\}) \\
&= \sum_{a_{1,1} \in A_1} \pi^{\varepsilon}(\{\omega_{1,1} \in [D\varepsilon, B - D\varepsilon] : g_{1,1}(\omega_{1,1} \mid a_{1,1}) = \underline{y}\} \mid a_{1,1}, a_{2,1}) \\
&\leq |A_1|\eta. \qquad (23)
\end{aligned}$$

Let $M\varepsilon$ be the largest integer smaller than $(B - 2D\varepsilon)/(2C_1\varepsilon)$. Then there exists $\{x_m\}_{m=1}^{M\varepsilon}$ such that $[x_1 - C_1\varepsilon, x_1 + C_1\varepsilon], \ldots, [x_{M\varepsilon} - C_1\varepsilon, x_{M\varepsilon} + C_1\varepsilon]$ are disjoint and included in $[D\varepsilon, B - D\varepsilon]$. Since no x_m satisfies (22),

$$\begin{aligned}
&\lambda(\{\omega_{1,1} \in [D\varepsilon, B - D\varepsilon] : g_{1,1}(\omega_{1,1} \mid a_{1,1}) = \underline{y} \text{ for some } a_{1,1} \in A_1\}) \\
&\geq \sum_{m=1}^{M\varepsilon} \lambda(\{\omega_{1,1} \in [x_m - C_1\varepsilon, x_m + C_1\varepsilon] : g_{1,1}(\omega_{1,1} \mid a_{1,1}) = \underline{y} \text{ for some } a_{1,1} \in A_1\}) \\
&\geq M\varepsilon K\eta\varepsilon. \qquad (24)
\end{aligned}$$

Since $M\varepsilon\varepsilon$ converges to $B/(2C_1)$ as ε goes to 0, (23) and (24) contradict to each other for sufficiently small ε. ∎

Proof of Lemma 4. We will show that Lemma 4 holds for $d > 0$ in Lemma 6, $\overline{\overline{\varepsilon}} > 0$ in Lemma 7, and some $\overline{\overline{\eta}} > 0$. Since Lemma 4 (ii) is a corollary of Lemma 6, what remains to show is Lemma 4 (i).

Pick $K > 0$ as in Lemma 7. Suppose that $\eta > 0$, $\varepsilon < \overline{\overline{\varepsilon}}$, and $g_{1,1}$ satisfies

$$\pi^{\varepsilon}(\{\boldsymbol{\omega}_1 \in \mathbb{R}^2 : g_{1,1}(\omega_{1,1} \mid a_{1,1}) = f_1(\omega_{1,1})\} \mid a_{1,1}, a_{2,1}) \geq 1 - \eta$$

for all $(a_{1,1}, a_{2,1}) \in A$. Pick $x_0 \in [D\varepsilon, B - D\varepsilon]$ as in Lemma 7. For every $\omega_{2,1} \in [x_0 - C_2\varepsilon, x_0 + C_2\varepsilon]$, note that the probability density of $\omega_{1,1}$ with respect to the Lebesgue measure conditional on $\mathbf{a}_1$, $\{\omega_{1,1} : \omega_{1,1} \in [x_0 - C_1\varepsilon, x_0 + C_1\varepsilon]\}$, and $\omega_{2,1}$ is bounded from above by

$$\frac{2}{\varepsilon} \max\left\{1, \frac{1}{(1 + C_1 - C_2)^2}\right\} =: \frac{E}{\varepsilon}.$$

Thus, by Lemma 7,

$$\pi^\varepsilon(\{\omega_{1,1} : g_{1,1}(\omega_{1,1} \mid a_{1,1}) = \overline{y}\} \mid a_{1,1}, a_{2,1}, \{\omega_{1,1} : \omega_{1,1} \in [x_0 - C_1\varepsilon, x_0 + C_1\varepsilon]\}, \omega_{2,1}) \geq 1 - EK\eta.$$

for all $(a_{1,1}, a_{2,1}) \in A$. Pick $\overline{\overline{\eta}} > 0$ such that $(p_1 + \zeta)(1 - EK\overline{\overline{\eta}}) > p_1$. If $\eta < \overline{\overline{\eta}}$, then, by Lemma 6,

$$\begin{aligned}
&\pi^\varepsilon(\{\omega_{1,1} \in \mathbb{R} : g_{1,1}(\omega_{1,1} \mid a_{1,1}) = \overline{y}\} \mid a_{1,1}, a_{2,1}, \omega_{2,1}) \\
&\geq \pi^\varepsilon(\omega_{1,1} \in [x_0 - C_1\varepsilon, x_0 + C_1\varepsilon] \mid a_{1,1}, a_{2,1}, \omega_{2,1}) \\
&\qquad \times \pi^\varepsilon(\{\omega_{1,1} \in \mathbb{R} : g_{1,1}(\omega_{1,1} \mid a_{1,1}) = \overline{y}\} \mid a_{1,1}, a_{2,1}, \{\omega_{1,1} : \omega_{1,1} \in [x_0 - C_1\varepsilon, x_0 + C_1\varepsilon]\}, \omega_{2,1}) \\
&\geq (p_1 + \zeta)(1 - EK\eta) > p_1
\end{aligned}$$

for all $(a_{1,1}, a_{2,1}) \in A$. ∎

A.3 Proof of Lemma 5

Since $W(\mathcal{A})$ is properly self-$\mathcal{A}$-generating, take $\bar{\gamma} > 0$ such that, for every $i \in I$, $\underline{w}_i(\mathcal{A})$ is $\mathcal{A}_i$-enforced by some $\underline{\alpha}_j \in \Delta(A_j)$ and $\overline{w}_i\colon A_j \times Y \to [\underline{w}_i(\mathcal{A}), \overline{w}_i(\mathcal{A}) - (1 + 1/\delta)\bar{\gamma}]$, and $\overline{w}_i(\mathcal{A})$ is $\mathcal{A}_i$-enforced by some $\overline{\alpha}_j \in \Delta(A_j)$ and $\overline{w}_i\colon A_j \times Y \to [\underline{w}_i(\mathcal{A}) + (1 + 1/\delta)\bar{\gamma}, \overline{w}_i(\mathcal{A})]$. For any $\gamma \in (0, \bar{\gamma}]$, by translating continuation payoffs by $\pm\gamma/\delta$, we have that every $v_i \in [\underline{w}_i(\mathcal{A}) + \gamma, \overline{w}_i(\mathcal{A}) - \gamma]$ is $\mathcal{A}_i$-enforced by some $\alpha_j \in \Delta(\mathcal{A}_j)$ and $w_i\colon A_j \times Y \to [\underline{w}_i(\mathcal{A}) + \gamma/\delta, \overline{w}_i(\mathcal{A}) - \gamma/\delta]$. Since each a_j has individual full rank in (Y, ρ), by modifying w_i into w_i', where $|w_i(a_j, y) - w_i'(a_j, y)|$ is of the order of ε, v_i is $\mathcal{A}_i$-enforced by α_j and w_i' in the perturbed game. Taking ε sufficiently small, we have $|w_i(a_j, y) - w_i'(a_j, y)| \leq (1/\delta - 1)\gamma$, and thus $[\underline{w}_1(\mathcal{A}) + \gamma, \overline{w}_1(\mathcal{A}) - \gamma] \times [\underline{w}_2(\mathcal{A}) + \gamma, \overline{w}_2(\mathcal{A}) - \gamma]$ is self-$\mathcal{A}$-generating in the perturbed game.

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