The Real Indeterminacy of the Unemployment Rate in a Dynamic General Equilibrium Model

Takuo Sugaya[*][†]
Princeton University

November 4, 2008

Abstract

We construct a dynamic general equilibrium model with goods and labor markets, where money is a medium of exchange. We show that there is a continuum of stationary equilibria with differing excess demands, which gives rise to the real indeterminacy of the unemployment rate. In addition, we show that cleared markets do not necessarily entail higher welfare than excess demand.

[*]tsugaya@princeton.edu

[†]The author is thankful to Akihiko Matsui and Kazuya Kamiya for their helpful guidance at various stages of the research. I am also grateful to Michihiro Kandori, Hisashi Nakamura, Kazushi Shintoyo, and Satoru Takahashi for their valuable comments and to participants of the 2007 Spring meeting of the Japanese Economic Association at Osaka City University and of the seminars at the University of Tokyo and at Osaka University. Paul Scott offered an excellent proofreading service. This work builds upon the research supported by Grant-in-Aid for JSPS Fellows. The usual disclaimer of course applies.

1 Introduction

Long run unemployment rate is one of the major concerns in economics. To address this problem, the present paper constructs an infinite- and discrete-time dynamic general equilibrium model with goods and labor markets, where money is a medium of exchange. Each period is divided into two subperiods, day and night. During the day, agents search for a job and work at the firms in the endogenous job destruction setting. After paid their wage by fiat money, they go to goods markets at night. In the goods markets, the firms entrust their goods to the auctioneer and get the sales. The goods are distributed to the agents via the Vickrey auction with the cash-in-advance constraint where payments are done in terms of fiat money.

In this setting, we show that there is a continuum of stationary equilibria with different excess demands, which gives rise to the real indeterminacy of unemployment rate. The underlying logic is as follows. Since the amount of money the buyer pays is equal to that the seller accepts even in the presence of excess demand, the traded volume is always balanced, which means, mathematically, the violation of the rank condition for a locally unique equilibrium. Thus, we can find a stationary equilibrium corresponding to any degree of excess demand.

More specifically, we confine our attention to stationary equilibria, where, for some positive number p, all trades occur with p (as we will see, since the labor market is competitive and the marginal productivity of the agent is equal to 1, the wage is equal to the price) and the stationary distribution of the money holdings and employment status of the agents is on $\{(0,e)$, $(0,u)$, (p,e), (p,u),..., (Np,e), $(Np,u)\}$, where (m,e) is the state of being employed with mp money and (m,u) is the state of being unemployed with mp money, and Np is the endogenous upper bound of the money holdings. Let us denote the measure of agents with (mp,e) by $h_{m,e}$. In addition, let $I_{m,e}$ and $O_{m,e}$ be the measures of the inflow into and outflow from (m,e), respectively. $h_{m,u}$, $I_{m,u}$, and $O_{m,u}$ are analogously defined. Then, the conditions for the stationary distribution are $I_{m,e} = O_{m,e}$ and $I_{m,u} = O_{m,u}$ for all $m = 0,...,N$ and $\sum_m (h_{m,e} + h_{m,u}) = 1$. Since all the agents belonging to the outflow from some state must belong to the inflow into the other states, the total population is constant; $\sum_m (I_{m,e} + I_{m,u}) = \sum_m (O_{m,e} + O_{m,u})$ is an identity. Furthermore, the total amount of money the agents pay in the auction is redistributed among the agents through the firms in the form of the wage, the total money supply is unchanged, that is, $\sum_m m (I_{m,e} + I_{m,u}) = \sum_m m (O_{m,e} + O_{m,u})$ is also an identity. Therefore, we have $2N + 2$

unknowns, $2N + 3$ conditions, and 2 identities, which means there always exists one degree of freedom. Hence, the real indeterminacy occurs in the goods markets according to the degree of excess demand, where the degree of excess demand is defined as the probability that the agents with p money who strictly prefer consumption cannot consume.

In the equilibrium with excess demand, some agents who strictly prefer consumption are rationed. In the Vickrey auction with equilibrium price p, while the probability of winning is 1 for those with bids more than p, the probability is less than 1 for those with bids equal to p if there exists excess demand. In such a situation, even if one has p units of money and strictly prefers consuming, the cash-in-advance constraint prevents her from bidding more than p and getting goods for sure. This rationing in the goods markets changes the incremental value of currency; that is, even if money holdings are constant at p, the probability of consumption varies as excess demand. This alters the unemployment rate through agents' incentives to work, which induces the real indeterminacy of the unemployment rate. Since the unemployment rate alters the amounts of consumed goods, social welfare is also indeterminate.

The two related literatures are worth mentioning. Firstly, our results and approaches are different from the traditional discussion of the long run unemployment rate. Cooper and John (1988) pointed out that, for the results of inefficiency, misspecification of prices, irrational expectations, wage and price rigidities, increasing returns, or strategic complementarities are necessary. However, in our model, the real indeterminacy of the long run unemployment rate does not depend on these features.

Secondly, the real indeterminacy of stationary equilibria is mentioned in recent papers in monetary economics, such as Green and Zhou (1998, 2002), Matsui and Shimizu (2005), Zhou (1999), Kamiya and Shimizu (forthcoming), and Kamiya and Shimizu (2006). Lagos and Wright (2005) resolved this indeterminacy by introducing a coexisting Walrasian market. However, in the Walrasian market, the market clearing condition is imposed by definition. Therefore, it is worthwhile to investigate whether the introduction of a large centralized auction market without a priori market clearing condition can resolve the indeterminacy. In addition, to the best of our knowledge, this paper is the first one that introduces money as a medium of exchange to the analyses of unemployment. Since, in the real economy, the reward for labor takes the form of fiat money, it is important to investigate whether the real indeterminacy in the goods markets spills over to the labor markets.

The rest of the paper is organized as follows. Section 2 sets forth a model. Section 3 defines the stationary equilibria we focus on. In Section 4, we solve the model and offer the intuitive explanation of the factors of the real indeterminacy. Section 5 concludes.

2 The Model

2.1 The Natural Environment

We consider a dynamic general equilibrium model with following features. There is a $[0,1]$ continuum of nonatomic private agents who live forever. In addition to the agents, there exist competitive firms. The number of the firms is exogenously given by $k \in \mathbf{N}$. We assume that k can be arbitrarily large and that the firms are competitive. Thus, without loss of generality, we assume $k = 1$.

Time is discrete and infinite. Each period is divided into two subperiods, day and night, following Lagos and Wright (2005). The agents supply labor to the firm during the day and consume goods at night. The instantaneous stochastic disutility x is resulting from working. Goods are indivisible and the agents can consume at most one unit per a day. The consumption of one unit of the goods brings c instantaneous utility to the agents. The time discount factor is $\beta \in (0,1)$ and we assume agents do not discount utility at night during the day.[1] Therefore, the expected discounted lifetime utility is given by

$$E \sum_{t=0}^{\infty} \beta^t \left[-I_t^e x_t + I_t^c c \right],$$

where I_t^e and I_t^c are (random) indicator functions that equal one if the agents are employed and consume at t, respectively, and zero otherwise. The specific process in each subperiod is discussed below.

2.2 What Occurs during the Day

The process during the day is analogous to an endogenous job destruction model. There are two employment states for the agents; employed and unemployed. Only unemployed agents search for jobs in McColl (1979)'s type search model. One interpretation of McColl's type search is that the

[1] This assumption is without loss of generality.

productivity of a job is the same while the disutility of a job differs. We adopt this interpretation and the productivity is normalized to 1.

An unemployed agent finds a working opportunity with probability $\theta(U)$, where U denotes the aggregate unemployed rate. We assume that $\theta(U)$ is continuous in U with $\{\theta : \exists U \in [0,1] \text{ such that } \theta = \theta(U)\} \subseteq (0,1]$ being convex and compact. If $\theta(U) = 1$ for all U, we say the labor markets are centralized since the unemployed can find an opportunity for sure. If she finds one, the idiosyncratic initial disutility $x \in \mathbf{R}$ is drawn from the cumulative distribution F. If she wants to work under this disutility, she applies for the opportunity. If she is not satisfied, she skips the opportunity.

On the other hand, an employed agent also faces risk. We assume an idiosyncratic shock arrives at a job with probability λ. At the arrival of the shock, the disutility x is newly drawn from the cumulative distribution function G, which is independent of the previous disutility. After observing her disutility, she decides whether to reapply for the job or to quit. If she quits, she becomes unemployed and starts to search for a job from the next day.

We assume following three assumptions. First, there exists some $\varepsilon > 0$ such that $F(\varepsilon) = G(\varepsilon) = 0$. This assumption means that working has some positive cost as Zhou (1999). As Zhou, it will be shown that this assumption ensures the existence of an endogenous upper bound on money holdings. Second, we assume $F\left(\frac{c}{1-\beta}\right) = G\left(\frac{c}{1-\beta}\right) = 1$ without loss of generality.[2] Third, there exists δ such that $F(\delta)(1-\lambda)\beta c - \{1+\beta(1-\lambda)\}\int_{\varepsilon}^{\delta} x' dF(x') > 0$. This assumption means that there exists a possible state where the disutility of working is low compared to the instantaneous utility from consumption. This ensures that those without money apply for the job with some positive probability.

After the decision of the agents, the firm decides which agents to hire and rehire. Figure 1 summarizes the discussion.

We assume the labor market clears. Therefore, if an agent applies or reapplies, she will be employed. Thus, Figure 2 summarizes the transitions of the states of the agents during the day, where $(\tilde{m}, i)$ denotes the state at the beginning of night, $\tilde{m}$ is money holdings, $i \in \left[\varepsilon, \frac{c}{1-\beta}\right]$ means she was employed with disutility i during the previous day, and $i = u$ means she was unemployed

[2] If $F\left(\frac{c}{1-\beta}\right) < 1$, redefine the new cumulative distribution function $\tilde{F}$ as $\begin{cases} \tilde{F}\left(\frac{c}{1-\beta}\right) = 1 \\ \tilde{F}(x) = F(x) \text{ for } x < \frac{c}{1-\beta} \end{cases}$. We can similarly redefine $\tilde{G}$ from G. If the shock is more than $\frac{c}{1-\beta}$, the agents do not want to work since each agent can receive at most c utility per day.

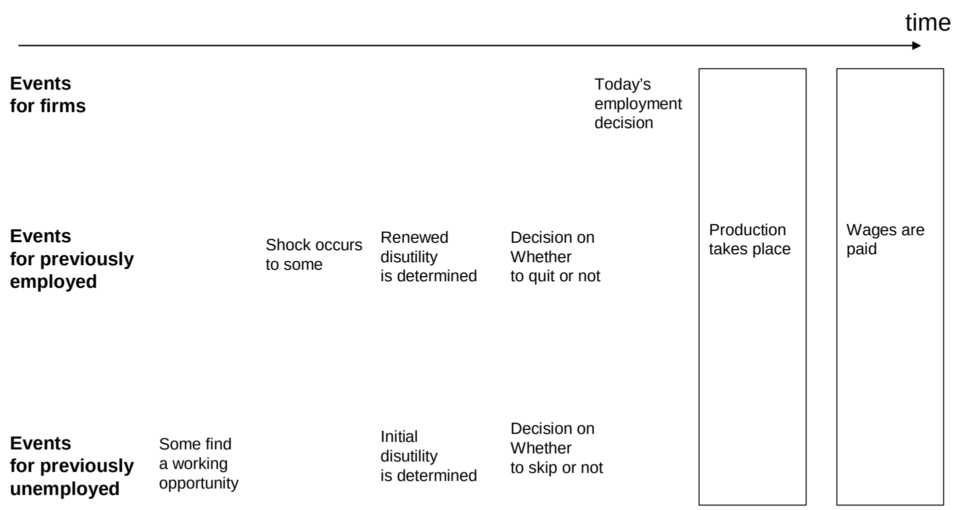


Figure 1: The time table for the day

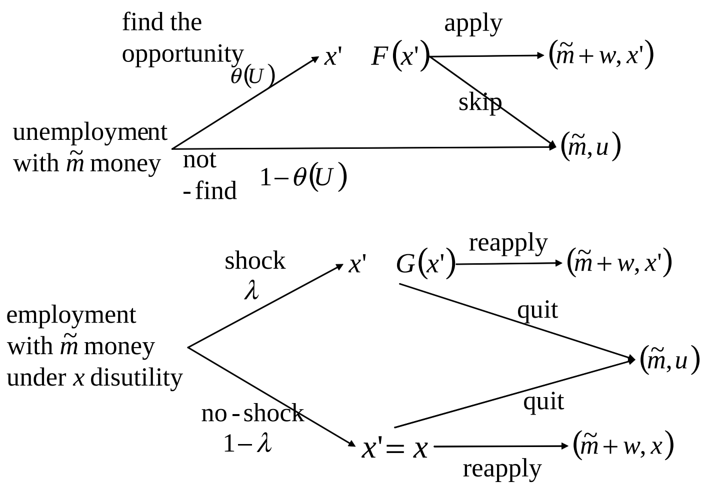


Figure 2: Transitions during the day

during the previous day.

2.3 What Occurs at Night

After being paid their wages by fiat money, the agents go shopping at night. There are several market places where the Vickrey auction is held. The firm entrusts its products to the auctioneer and gets sales. We assume goods are perishable and only goods produced during the last day can be sold at night. The agents search for the markets where the goods they want to consume are sold. We simply assume that the agents cannot consume what they produced at day directly and that they must search for the markets. This assumption approximates well the situation where the division of labor is important and self-consumption is rare.[3] If all the markets are symmetric, we can assume there is one market place without loss of generality. Let $q \in (0, 1]$ denote the probability of the agents' finding the market. $q < 1$ approximates the situation where the shopping has some random-matching nature while $q = 1$ represents an economy with a centralized market.

The agents who can find the market participate in the Vickrey auction with the cash-in-advance constraint, that is, a bid should be less than or equal to the bidder's money holdings. The price is determined as the supremum of the prices such that there is no excess supply with positive measure.

[3] See Diamond (1982) for more discussion.

Thus, the situation where the price is p can be represented by Figure 3. The probability of getting

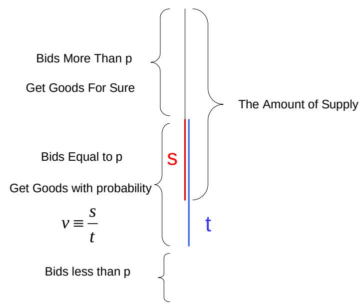


Figure 3: The Vickley auction

one unit of the goods is 0 and 1 if the bid is less than and more than p, respectively. If the bid is p, the probability is

$$v \equiv \frac{\#\text{ of supply} - \#\text{ of bids more than } p}{\#\text{ of bids equal to } p}.$$

Thus, Figure 4 illustrates nighttime transactions with price p.

3 Definition of Stationary Equilibria

What we want to show is the existence of real indeterminacy in the economy described above. Firstly, we define equilibria we concentrate on. After that, we solve the individual decision problem taking the aggregate states as given in the next section. Then, we search for equilibria, where the individual optimal decision is consistent with the aggregate states that the agents take as given.

Here, we are going to consider stationary equilibria in the trading environment just described. Let $p \in \mathbf{R}_{++}$ be the stationary equilibrium price. The domain of possible monetary holdings is $\mathbf{R}_+$ and that of employment status is $\{u\} \cup \left[\varepsilon, \frac{c}{1-\beta}\right]$. Let $\left(\frac{\tilde{m}}{p}, i; p\right)$ be the state of the individual agents, where $\tilde{m} \in \mathbf{R}_+$ and $i \in \{u\} \cup \left[\varepsilon, \frac{c}{1-\beta}\right]$. $\left(\frac{\tilde{m}}{p}, u; p\right)$ means that the agents in question are in the night markets with $\tilde{m}$ money holdings and were unemployed during the previous day when the equilibrium price is p. $\left(\frac{\tilde{m}}{p}, x; p\right)$ with $x \in \left[\varepsilon, \frac{c}{1-\beta}\right]$ means that the agents in question are in the night markets with $\tilde{m}$ money holdings and were employed during the previous day with x disutility. Here, we focus on equilibria where monetary distribution is $\{0, p, 2p, 3p, \cdots\}$. Hence, given p, we

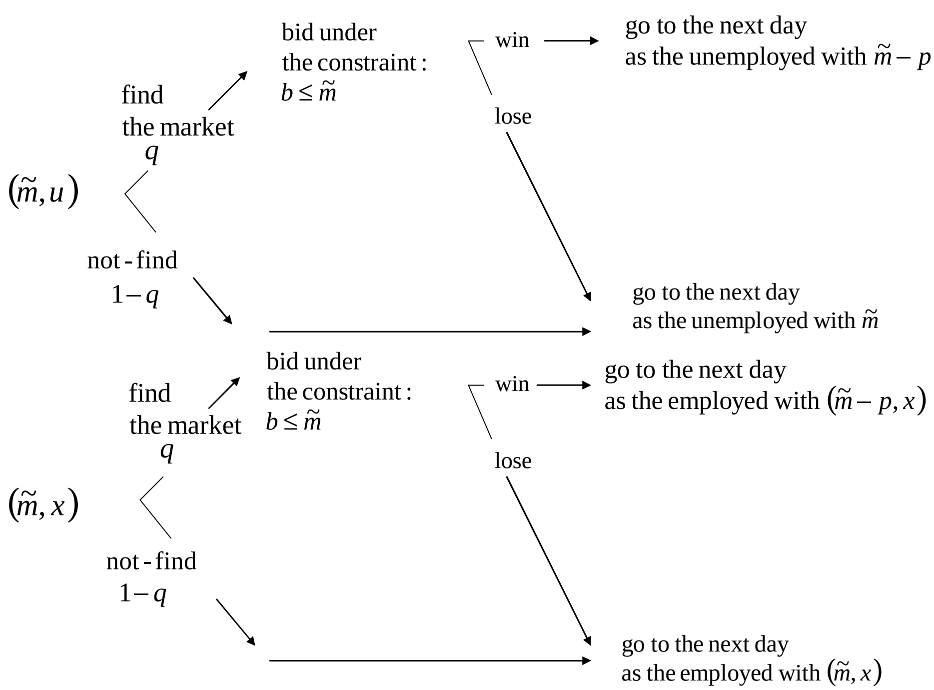


Figure 4: Transition at night

can omit p and let $\{0, 1, 2, 3, \cdots\}$ denote the monetary distribution and (m, i) be the state of the individual agents with $mp = \tilde{m}$.

To define stationarity, we need to define a measure on the agents' status. Let h be a probability measure on $\mathbf{Z}_+ \times \left(\{u\} \cup \left[\varepsilon, \frac{c}{1-\beta}\right]\right)$. For the unemployed agents, we define stationarity as $h_{m,u} \equiv h(m, u)$ being constant. As for the employed agents, let us define $h_{m,e} \equiv \int_{\left\{(m,x); x \in \left[\varepsilon, \frac{c}{1-\beta}\right]\right\}} dh$ and a corresponding state as $(m, e) \equiv \vee_{\left\{(m,x); x \in \left[\varepsilon, \frac{c}{1-\beta}\right]\right\}} (m, x)$. Then, from "law of large numbers", we can define stationarity as $h_{m,e}$ being constant.[4]

Moreover, we restrict our attention to stationary equilibria with Markov strategy that depends only on the matching probability $\theta = \theta\left(\sum_m h_{m,u}\right)$ and the degree of rationing $v = \Pr$ (getting one unit of the goods Thus, (θ, v) are only relevant states. Let us define $\Omega \equiv (v, \theta)$. Formally, we concentrate on equilibria with the Markov strategy defined as a pair $\sigma = (\omega, \rho)$[5] such that

- $\omega : \mathbf{Z}_+ \times \left(\{u\} \cup \left[\varepsilon, \frac{c}{1-\beta}\right]\right) \times [0, 1] \times [0, 1] \to \mathbf{R}_+$: Bidding strategy. $\omega(m, i; \Omega) = b$ means those with (m, i) bid $bp \leq m$ under Ω.

- $\rho : \mathbf{Z}_+ \times \left[\varepsilon, \frac{c}{1-\beta}\right] \times [0, 1] \times [0, 1] \to \{0, 1\}$: Working strategy, where $\rho(m, x; \Omega) = 0$ means those with m units of money at the beginning of the day quit or skip an opportunity with x disutility and $\rho = 1$ means application or reapplication.

After observing the disutility, the situation is the same regardless of the previous employment status. Thus, we do not distinguish between ρ for the previously unemployed and for the previously employed. The probability of winning when $b = 1$ is given by

$$v = \frac{\text{\# of supply} - \text{\# of bids more than } p}{\text{\# of bids equal to } p}. \tag{1}$$

In equilibria with p, $0 < v \leq 1$ must hold.

Since we assume the firm is competitive and that the labor market clears, the labor market-clearing condition is

$$w = p. \tag{2}$$

[4]There have been well-known difficulties in providing a probability-theoretic foundation to the idea of "law of large numbers" for a continuum of random variables. Here, following the convention of the labor search models, we assume "law of large numbers" holds. See Judd (1985) and Feldman and Gilles (1985) for "law of large numbers". The author is thankful to Satoru Takahashi for pointing out this problem.

[5]Since the agents are nonatomic, it suffices to show that there is no incentive to deviate from the equilibrium path. Thus, we only define the strategy on the equilibria.

We do not specify the labor market microstructure that induces (2). Rather, even with this restriction, we can show the real indeterminacy of the unemployment rate.

Using these definitions, we can now define stationary equilibria.

Definition 1 *A stationary equilibrium consists of a time-invariant profile* $\langle (\omega^*, \rho^*), h, v, \theta, p \rangle$ *such that*

1. $\{h_{m,u}, h_{m,e}\}$ *is stationary under the strategy* (ω^*, ρ^*),
2. $0 < v \leq 1$ *is derived by (1),*
3. $\theta = \theta \left(\sum_m h_{m,u} \right)$,
4. $p = \frac{M}{\sum_{m \geq 1} m(h_{m,e} + h_{m,u})}$,
5. (ω^*, ρ^*) *maximize the agents expected utility* $E \sum_{t=0}^{\infty} \beta^t \left(-I_t^e x_t + I_t^c c \right)$.

4 Optimal Strategy on the Equilibrium Path

Let us consider the optimal strategy of the agents next. Since the disutility shock is i.i.d., a Markov strategy is optimal and satisfies the reservation property.

Proposition 2 *For all* $q \in (0, 1]$, *there is a unique threshold* $R(m; \Omega) \in \left(-\frac{qc}{1-\beta}, \frac{qc}{1-\beta} \right)$ *for each* m *such that*

$$\rho^* (m, x; \Omega) = 1 \Leftrightarrow x \leq R(m; \Omega)$$

and $R(m; \Omega)$ *is continuous in* θ.[6] *In addition, for all* $q \in (0, 1]$ *and* Ω,

$$F(R(0; \Omega)) > 0.$$

Proof. See Appendix. ■

Next, we consider the optimal bid on the equilibrium path. The constant marginal utility of consumption, the constant price, and the discount of the future imply that the optimal bid is to maximize the probability of getting one unit of the goods for any Ω.

[6] We follow the convension in the labor search models that the agents work if they are indifferent.

Proposition 3 *For all* $q \in (0,1]$*,*

$$\begin{aligned} \omega^*(m,i;\Omega) &= 1 \text{ for } m=1 \\ \omega^*(m,i;\Omega) &\in (1,m] \text{ for } m \geq 2. \end{aligned} \quad (3)$$

Proof. See Appendix. ■

Finally, we show that those with sufficiently large amounts of money holdings do not work since there exists a positive lower bound of working disutility ε.

Proposition 4 *For all* $q \in (0,1]$*, there exists* N *such that for all* Ω*,* $R(m;\Omega) < \varepsilon$ *for all* $m > N$*.*

Proof. See Appendix. ■

5 Stationary Distribution

In this section, we find the stationary equilibrium. From **Proposition 3**, $v = \frac{\sum_{0 \leq m} h_{m,e} - q \sum_{2 \leq m} (h_{m,e}+h_{m,u})}{q(h_{1,e}+h_{1,u})}$. From **Proposition 4**, $h_{m,e} = h_{m,u} = 0$ for all $m > N$. Therefore, **Definition 1** is equivalent to;

1. $\rho^*(m,x;\Omega) = 1 \Leftrightarrow x \leq R(m;\Omega)$, and $\omega^*(m,i;\Omega) = 1$ for $m = 1$ and $\omega^*(m,i;\Omega) \in (1,m]$ for $m \geq 2$,
2. $\{h_{m,u}, h_{m,e}\}_{0 \leq m \leq N}$ is stationary under the strategy (ω^*, ρ^*),
3.

$$v = \frac{\sum_{0 \leq m \leq N} h_{m,e} - q \sum_{2 \leq m \leq N} (h_{m,e}+h_{m,u})}{q(h_{1,e}+h_{1,u})}, \quad (4)$$

$$\theta = \theta\left(\sum_{0 \leq m \leq N} h_{m,u}\right). \quad (5)$$

Given Ω, the transitions of h is summarized in Figure 5.

Using the modified definition, we can show the real indeterminacy of equilibria.

Theorem 5 *For any* $v \in (1,0]$*, there exists* $\{h_{m,e}, h_{m,u}\}_{0 \leq m \leq N}$ *that satisfies the stationary equilibrium conditions.*

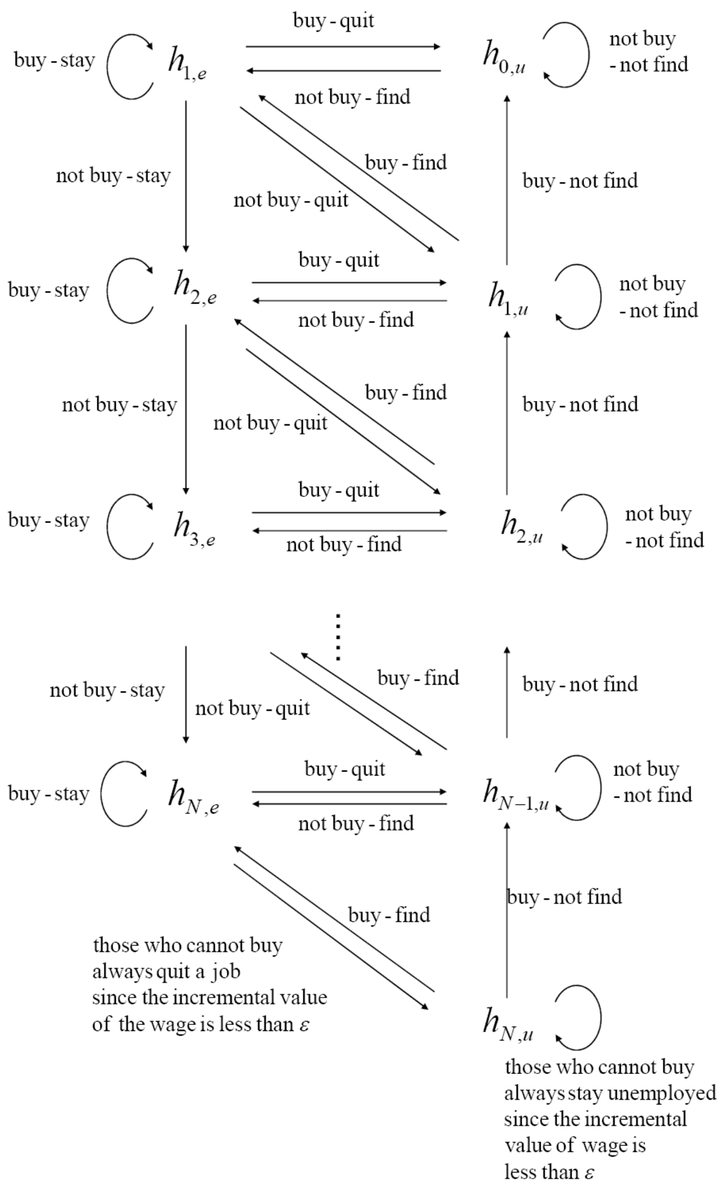


Figure 5: Transitions of h

Corollary 6 *Even if both the goods and labor markets are centralized, there exists* $\{h_{m,e}, h_{m,u}\}_{0\leq m\leq N}$ *that satisfies the equilibrium conditions for any* $v \in (1, 0]$.

A rigorous proof appears in the appendix. Here, we present an informal discussion. The technique to find a stationary distribution is the following. Firstly, arbitrarily fix $v \in (1, 0]$ and guess a stationary distribution $\{h_{m,e}, h_{m,u}\}_{0\leq m\leq N}$. Then, calculate the one-period forward distribution $\left\{h'_{m,e}, h'_{m,u}\right\}_{0\leq m\leq N}$ from $\{h_{m,e}, h_{m,u}\}_{0\leq m\leq N}$ using $(\rho^* (m, x; \Omega), \omega^* (m, i; \Omega))$, where $\Omega = (v, \theta)$ with $\theta = \theta\left(\sum_{0\leq m\leq N} h_{m,u}\right)$; that is, v is the initially fixed value and θ satisfies (5) under the guess. Since all the agents belonging to the outflow from some state must belong to the inflow into the other states, the total population is constant. Hence, $\sum_{0\leq m\leq N} \left(h'_{m,e} + h'_{m,u}\right) = 1$ always holds if $\sum_{0\leq m\leq N} (h_{m,e} + h_{m,u}) = 1$. Thus, since the mapping from $\{h_{m,e}, h_{m,u}\}_{0\leq m\leq N}$ to $\left\{h'_{m,e}, h'_{m,u}\right\}_{0\leq m\leq N}$ is a continuous function from the convex and compact set to itself, we can always find a stationary distribution $\left\{h^*_{m,e}, h^*_{m,u}\right\}_{0\leq m\leq N}$ satisfying (5). Thus, to verify $\left\{h^*_{m,e}, h^*_{m,u}\right\}_{0\leq m\leq N}$ is the equilibrium, it suffices to show that $\left\{h^*_{m,e}, h^*_{m,u}\right\}_{0\leq m\leq N}$ satisfies (4).

The key is that, regardless of the initially fixed value of v, $\left\{h^*_{m,e}, h^*_{m,u}\right\}_{0\leq m\leq N}$ satisfies (4); i.e., if some distribution is stationary under the transition, then it necessarily satisfies (4) automatically. The reason is that the amount of money someone pays is identical to that accepted by another; $\sum_m m (h_{m,e} + h_{m,u})$ is constant. On the other hand, the gross revenue of the firm is

$$\sum_m p h^*_{m,e}, \tag{6}$$

which is identical to the payment by the agents;

$$qp \left\{ v \left(h^*_{m,e} + h^*_{m,u}\right) + \sum_{2\leq m\leq N} \left(h^*_{m,e} + h^*_{m,u}\right) \right\}. \tag{7}$$

Since (6) and (7) are identically equal to each other, (4) is identity.

Therefore, the fundamental reason for the real indeterminacy is the following. Since the amount of money the buyer pays is equal to that the seller accepts even with the existence of excess demand, we can find a stationary equilibrium corresponding to any degree of excess demand.

In the Vickrey auction with equilibrium price p, while the probability of winning is 1 for those with bids more than p, the probability is less than 1 for those with bids equal to p if there exists excess demand. In such a situation, even if one has p units of money and strictly prefers consuming, the cash-in-advance constraint prevents her from bidding more than p. Then, the degree of quota, v, changes the incremental value of money, which also changes the unemployment rate through the agents' incentives to work. If we neglect the continuation value for simplicity, getting p more units of money for those with 0 current money holdings is qvc. On the other hand, for those with p current money holdings, this incremental value becomes $q(1-v)c$. Therefore, the marginal value of money changes as v changes. Since this makes $R(0)$ and $R(1)$ change with v, so does the unemployment rate.

The above logic does not depend on the value of q and the form of θ. Thus, even if the goods and labor markets are centralized, the real indeterminacy persists.

For comparison, we cite the result when we impose the restriction that the goods markets clear; that is to say, a Walrasian equilibrium is realized. See Sugaya (2007) for more details.

Theorem 7 *(i) There always exists a Walrasian equilibrium*[7]*.*

(ii) Under some regularity conditions, the Walrasian equilibrium is unique.

(iii) The continuum of stationary equilibria in ***Theorem 5*** *includes the Walrasian equilibrium as a point with* $v=1$*.*

As we have mentioned, everyone with p money holdings strictly prefers consumption in stationary equilibria, which means they must consume in Walrasian equilibria with price p. Thus, $v=1$. Therefore, in spite of the existence of the real indeterminacy above, we can show that the Walrasian is unique. Therefore, contrary to the conventional wisdom, the large auction market cannot induce the Walrasian.

6 Welfare Comparison

Since the discussion about unemployment is mainly related to social welfare, we will consider the welfare comparison among different degrees of excess demand. Let us consider a benevolent social

[7] We define a Walrasian equilibrium with p as a stationary equilibrium that satisfies all the conditions of **Definition 1** plus the requirement that all the agents with no less than p money holdings who strictly prefer consuming can consume for sure.

planner who wants to maximize the social welfare;

$$\sum_{m\geq 0}\left\{\int_{\left\{(m,x');x'\in\left[\varepsilon,\frac{c}{1-\beta}\right]\right\}} V\left(m,x'\right)dh + V\left(m,u\right)h\left(m,u\right)\right\}.$$

Assume that the social planner is subject to the same matching constraints as the firm and the agents; that is, he takes q, $\theta(\cdot)$, $F(\cdot)$, and $G(\cdot)$ as given. Further, the planner only uses stationary policies.[8] Then, the maximization problem is

$$\begin{aligned}
&\max_{R,q}\sum_{t=0}^{\infty}\beta^{t}\left(\min\left\{1-U_t,q\right\}c-y_t\right)\\
&s.t.\begin{cases} U_t\geq U_{t-1}+\lambda\left(1-G\left(R\right)\right)\left(1-U_{t-1}\right)-\theta\left(U_{t-1}\right)U_{t-1}F\left(R\right),\\ y_t\leq y_{t-1}-\lambda y_{t-1}+\lambda\left(1-U_{t-1}\right)\int_{\varepsilon}^{R}xdG\left(x\right)+\theta\left(U_{t-1}\right)U_{t-1}\int_{\varepsilon}^{R}xdF\left(x\right),\end{cases}
\end{aligned}$$

where y_t denotes the social disutility in the labor force at date t, R denotes the reservation disutility.

The logic is as follows. Socially accumulated utility per day is the amount of consumed goods multiplied by c minus current disutility. Today's unemployment rate is yesterday's unemployment rate plus those newly unemployed minus those newly employed. Today's social disutility is yesterday's disutility minus disutility of those hit by the shocks plus the expected amount of disutility for those who are hit by the shocks and stay and those newly employed.

Solving this and evaluating the Euler conditions yield

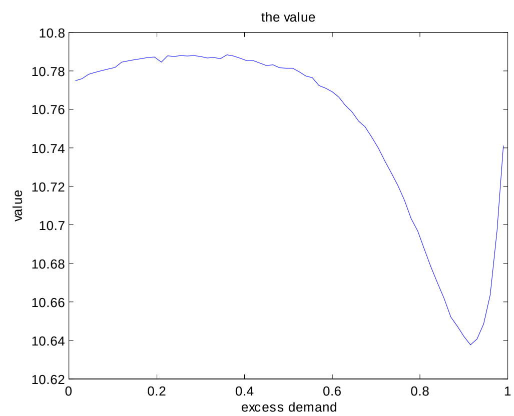
$$\begin{aligned}
\text{if } U &\geq 1-q,\ \begin{cases}\beta\left\{\theta\left(U\right)+\theta'\left(U\right)U\right\}\int_0^R F\left(x\right)dx=\left(1-\beta\left(1-\lambda\right)\right)\left(c-R\right)+\beta\lambda\int_0^R G\left(x\right)dx,\\ U=\frac{\lambda\left(1-G\left(R\right)\right)}{\lambda\left(1-G\left(R\right)\right)+\theta\left(U\right)F\left(R\right)},\end{cases}\\
\text{if } U &\leq 1-q,\ U=1-q.
\end{aligned}$$

Sugaya (2007) shows that if and only if $\theta'(U)=0$ and $q=1$, this condition is equivalent to the equilibrium conditions with $v=1$. If $\theta'(U)<0$, employment has positive externalities which make it easier for others to find a job. In competitive equilibria, however, the agents are nonatomic and do not consider this effect. Thus, the employment rate with cleared markets is lower than the social optimum. Similarly, if $\theta'(U)>0$, the employment rate with cleared markets is higher than the social optimum.

[8] See section 8 of Pissarides (2000) for more details.

Therefore, cleared markets do not necessarily entail higher welfare than excess demand. Usually, employment has positive externalities. In such a situation, while the decline in v is inefficient for the distribution of goods, it may be efficient for job creation since it makes the agents more patient and decreases the unemployment rate by enlarging the difference between $V(1,u)$ and $V(2,e)$. As long as this positive effect in the labor market outweighs the negative effect in the goods markets, the decrease in v is desirable.

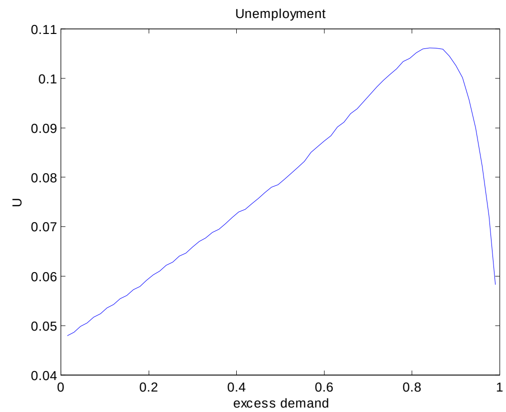
The following result of the numerical analysis verifies the claim. We use the following parameter values; $c = 3$, $q = 1$, $\theta(U) = \frac{2}{3} - \frac{2}{3}U^2$, $f(x) = g(x) = \frac{9-x^2}{18}$.[9]

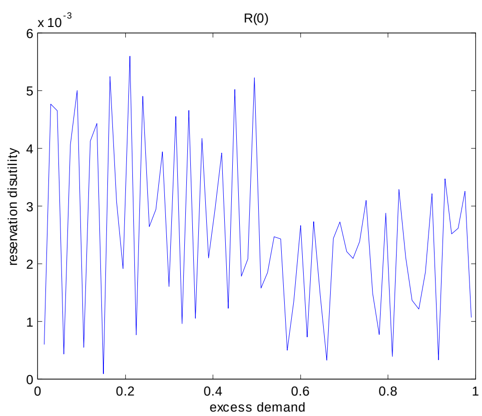


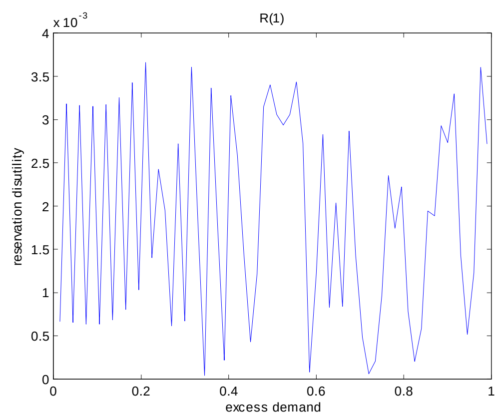
The maximum is attained with $v = 0.36$. The reason is that the unemployment rate is low for low v, which is because the reservation for low v is high since having less money when v is low entails the risk of being rationed. Since low U has a positive externality for θ, the maximum value

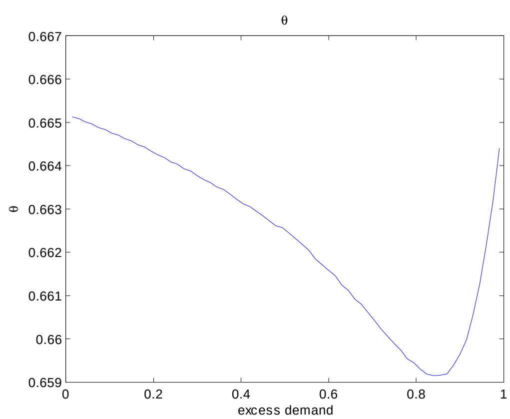
[9]Note that without the assumption that $F(\varepsilon) = G(\varepsilon) = 0$ for some $\varepsilon > 0$, the monetary holdings have an upper bound when $q = 1$. Thus, we use these functions for the simplicity of calculation.

is attained with low v.









Therefore, in summary,

Theorem 8 *A competitive equilibrium with $v = 1$, which is a Walrasian equilibrium, is not efficient and may be dominated by some stationary equilibrium with excess demand if and only if $\theta' \neq 0$.*

7 Conclusion

This paper constructs a dynamic general equilibrium model with goods and labor markets, where money is a medium of exchange. What is new compared to the existing literature on the unemployment problem is that we consider the interaction between the goods and labor markets in the presence of money as a medium of exchange.

Contrary to the conventional wisdom, we have shown that the competition of many agents in the large auction market does not realize a equilibrium with cleared markets (that is, a Walrasian equilibrium). There exists a continuum of equilibria differing in the unemployment rate.

The underlying logic is as follows. Since the amount of money the buyer pays is equal to that the seller accepts even in the presence of excess demand in monetary economies, the traded volume is always balanced. Thus, we can pin down a stationary equilibrium only after arbitrarily fixing the degree of excess demand. In the Vickrey auction with equilibrium price p, those with p money holdings are rationed with some probability if there exists excess demand since the cash-in-advance constraint prevents her from bidding more than p. This rationing in the goods markets changes the incremental value of currency; even if the monetary holdings are constant at p, the probability of consuming varies with excess demand. This alters the unemployment rate through agents' incentives to work, which induces the real indeterminacy of the unemployment rate. Since the production level determines consumption, social welfare is also indeterminate.

Though our work has shed some light on this subject, the result here does not mean the monetary economies per se are instable. There are many other issues that are worth investigating. In particular, whether our result implies that an equilibrium in the real world is intrinsically fragile or some aspects could lead the economy to some specific state is an important yet unsolved question. Whether there exists the real indeterminacy in monetary economies has crucial implication for business cycles and the unemployment problem. Therefore, it is important to explore the factors in monetary economies that we have not considered fully but which may stabilize the economy. The transition paths, learning effect or behavioral tendency of agents, the evolutionary stability, the role of credit, and the action of central banks such as inflation targets or control of the money supply are the main candidates and the important topics for further research.

References

[1] A. Rao Aiyagari and Neil Wallace. Existence of steady states with positive consumption in the Kiyotaki-Wright model. *Review of Economic Studies*, 58:901-916, 1991.

[2] Robert J. Auman. Markets with a continuum of traders. *Econometrica*, 32:39-50, 1964.

[3] Russell Cooper and Andrew John. Coordinating Coordination Failures in Keynesian Models. *Quarterly Journal of Economics*, 103:441-463, 1988

[4] Peter. A. Diamond. Aggregate demand management in search equilibrium. *Journal of Political Economy*, 90:881-94, 1982.

[5] Darrell Duffie and Yeneng Sun. The exact law of large numbers for independent random matching. mimeo, 2004

[6] Darrell Duffie and Yeneng Sun. Existence of independent random matching. Forthcoming in *the Annals of Applied Probability*, 2006

[7] Mark Feldman and Christian Gilles. An expository note on individual risk without aggregate uncertainty. *Journal of Economic Theory*, 102:26-32

[8] Douglas Gale. Bargaining and competition part i: Characterization. *Econometrica*, 54:785-806, 1986(a).

[9] Douglas Gale. Bargaining and competition part ii: Existence. *Econometrica*, 54:807-818, 1986(b).

[10] Douglas Gale. Limit theorems for markets with sequential bargaining. *Journal of Economic Theory*, 43:20-54, 1987

[11] Douglas Gale. *Strategic Foundations of General Equilibrium: Dynamic Matching and Bargaining Games (Churchill Lectures in Economic Theory)*. Cambridge University Press, 2000.

[12] Edward J. Green and Ruilin Zhou. A rudimentary random-matching model with divisible money and prices. *Journal of Economic Theory*, 81:252-271, 1998.

[13] Edward J. Green and Ruilin Zhou. Dynamic monetary equilibrium in a random matching economy. *Econometrica*, 70:929-969, 2002.

[14] Kenneth L. Judd. The law of large numbers with a continuum of IID random variables. *Journal of Economic Theory*, 35:19-25, 1985

[15] Kazuya Kamiya and Takashi Shimizu. Real indeterminacy of stationary equilibria in matching models with divisible money. Forthcoming in *Journal of Mathematical Economics*.

[16] Kazuya Kamiya and Takashi Shimizu. A dynamic general equilibrium model with centralized auction markets. *CIRJE Discussion Paper*, 2006.

[17] Nobuhiro Kiyotaki and Randall Wright. On money as a media of exchange. *Journal of Political Economy*, 97:927-954, 1989.

[18] Ricard Lagos and Randall Wright. A unified framework for monetary theory and policy analysis. *Journal of Political Economy*, 113:463-484, 2005.

[19] Lars Ljungqvist and Thomas J. Sargent. *Recursive Macroeconomic Theory*. 2nd ed. MIT press, 2004.

[20] Robert E. Lucas, Jr., Two illustrations of the quantity theory of money. *American Economic Review*, 70:1005-1014, 1980.

[21] Akihiko Matsui and Takashi Shimizu. A theory of money and marketplaces. *International Economic Review*, 46:35-59, 2005.

[22] Paul Milgrom. *Putting Auction Theory to Work (Churchill Lectures in Economic Theory)*. Cambridge University Press, 2004.

[23] Dale T. Mortensen and Christpher A. Pissarides. Job creation and job destruction in the theory of unemployment. *Review of Economic Studies*, 61:397-415.

[24] Christopher A. Pissarides. *Equilibrium unemployment theory*. 2nd ed. MIT Press, 2000.

[25] Ariel Rubinstein and Asher Wolinsky. Equilibrium in a market with sequential bargaining. *Econometrica*. 53:1133-1150, 1985.

[26] Ariel Rubinstein and Asher Wolinsky. Decentralized trading, strategic behavior and the walrasian outcome. *Review of Economic Studies*, 57:63-78, 1990.

[27] Nancy L. Stokey, Robert E. Lucas, Jr., and with Edward C. Prescott. *Recursive Methods in Economic Dynamics*. Harvard University Press, 1989.

[28] Takuo Sugaya. The real indeterminacy of the unemployment rate in a dynamic general equilibrium model. *Master thesis at the University of Tokyo*

[29] Yeneng Sun. The exact law of large numbers via Fubini extension and characterization of insurable risks. *Journal of Economic theory*, 126:31-69

[30] Ruilin Zhou. Individual and aggregate real balances in a random-matching model. *International Economic Review*, 40:1009-1038, 1999.

8 Appendix

8.1 The Optimal Strategy of the Agents

8.1.1 Value Function

We take the value function approach. In a stationary equilibrium, the relevant information is

$$\Omega \equiv (v, \theta) = \left(\frac{\text{\# of supply} - \text{\# of bids more than } p}{\text{\# of bids equal to } p}, \theta \left(\sum_m h_{m,u} \right) \right).$$

Therefore, we study the value function in terms of the Bellman equation corresponding to some arbitrarily fixed Ω. The value for the agents with (m, u) can be written as

$$\begin{aligned} V(m,u;\Omega) &= T(V(m,u;\Omega)) \\ &\equiv \max_{0 \le b \le m} \tilde{v}(b) \left\{ \begin{array}{c} c + \beta\theta \int_{\varepsilon}^{\frac{c}{1-\beta}} \max\{V(m,x';\Omega) - x', V(m-1,u;\Omega)\} dF(x') \\ +\beta(1-\theta) V(m-1,u;\Omega) \end{array} \right\} \\ &\quad + (1-\tilde{v}(b)) \left\{ \begin{array}{c} \beta\theta \int_{\varepsilon}^{\frac{c}{1-\beta}} \max\{V(m+1,x';\Omega) - x', V(m,u;\Omega)\} dF(x') \\ +\beta(1-\theta) V(m,u;\Omega) \end{array} \right\} \end{aligned} \quad (8)$$

for $m \ge 0$,

where $V(m,x)$ is the value for those with (m,x) at the beginning of night and

$$\tilde{v}(b) \equiv \begin{cases} 0 & \text{if } b < 1, \\ qv & \text{if } b = 1, \\ q & \text{if } b > 1; \end{cases}$$

that is, the probability of getting one unit of the goods when a bid is bp. Note that although $m-1<0$ if $m=0$, we can identify the support of m as $\mathbf{Z}_+$ since $\tilde{v}(0)=0$.

The logic is the following. If she bids bp if she finds the goods markets, the probability of getting one unit of the goods is $\tilde{v}(b)$. If she wins, she can consume one unit of the goods and her money holdings become $m-1$. On the next day, with probability θ, she can find a working opportunity. The disutility resulting from working is drawn from F. When she applies for the job with disutility x', her state will become (m,x') since there is no excess supply in the labor market. If she skips, the state will become $(m-1,u)$. Therefore, she will work as long as $V(m,x')-x' \geq V(m-1,u)$.[10] When she cannot find a working opportunity, she continues to be unemployed. Thus, her state will stay at $(m-1,u)$. If she loses the auction or cannot find the goods markets, she cannot consume but the money holdings are still m at the beginning of the next day. The decision at the next day market can be analogously studied with $m-1$ replaced by m.

Next, we consider the value for the agents with (m,x), which can be written as

$$\begin{aligned} V(m,x;\Omega) &= T(V(m,x;\Omega)) \\ &\equiv \max_{0\leq b\leq m} \tilde{v}(b) \left\{ \begin{array}{c} c+\beta\lambda\int_{\varepsilon}^{\frac{c}{1-\beta}} \max\{V(m,x';\Omega)-x', V(m-1,u;\Omega)\}\, dG(x') \\ +\beta(1-\lambda)\max\{V(m,x;\Omega)-x, V(m-1,u;\Omega)\} \end{array} \right\} \\ &\quad +(1-\tilde{v}(b)) \left\{ \begin{array}{c} \beta\lambda\int_{\varepsilon}^{\frac{c}{1-\beta}} \max\{V(m+1,x';\Omega)-x', V(m,u;\Omega)\}\, dG(x') \\ +\beta(1-\lambda)\max\{V(m+1,x;\Omega)-x, V(m,u;\Omega)\} \end{array} \right\} \end{aligned} \tag{9}$$

$$\text{for } m\geq 0, x\in\left[-\frac{c}{1-\beta}, \frac{c}{1-\beta}\right].$$

Though we restrict the support of F and G to $\left[\varepsilon, \frac{c}{1-\beta}\right]$, we define the value function for $x \in \left[-\frac{c}{1-\beta}, \frac{c}{1-\beta}\right]$ for the convenience of the following proofs.

[10] We follow the convension in the labor search models that the agents work if they are indifferent.

The logic is as follows. With probability $\tilde{v}(b)$, the agent can consume one unit of the goods and her money holdings become $m-1$. On the next day, the shock arrives with probability λ. She decides whether to quit her job or not. If she quits, her state will be $(m-1, u)$ while, if she continues, her state will be (m, x').[11] Thus, the agent will be employed if and only if $V(m, x') - x' \geq V(m-1, u)$. If she loses the auction or cannot find the goods markets, she cannot consume but the money holdings are still m. The decision at next day market can be analogously studied with $m-1$ replaced by m.

The solution to the Bellman equation is the solution to the functional equation

$$TV = V.$$

We will show that the functional equation above has the following property.

Lemma 9 *The Bellman equation above satisfies Blackwell's sufficient conditions for all $q \in (0,1]$; i.e., given v, there is a unique solution in the functional space $C[(m, i; \theta)]$, where $C[(m, i; \theta)]$ denotes the space of bounded functions continuous in m, $i \in \{u\} \cup \left[-\frac{c}{1-\beta}, \frac{c}{1-\beta}\right]$, and θ with* sup *norm.*

Proof.

1. Boundedness

   For all $m \geq 0$, $\tilde{v}(b)$, β, λ, θ, $\frac{c}{1-\beta}$ and c are bounded. Furthermore, since the domain of integral is compact, the integral of bounded functions is bounded. Thus, TV is bounded if V is bounded.

2. Continuity

   Since m and $i = u$ are isolated, it suffices to show that TV is continuous in θ and $i \in \left[-\frac{c}{1-\beta}, \frac{c}{1-\beta}\right]$.

   Firstly, $\tilde{v}(1) \in \{0, qv\}$ and $\tilde{v}(b) \in \{0, qv, q\}$ for $m \geq 2$. Thus, $\tilde{v}(b)$ is independent of (θ, i) and compact. Secondly, since (θ, i) are in a compact subset of $\mathbf{R}^2$, continuity of V is equivalent to uniform continuity and integral preserves continuity. Therefore, the Theorem of Maximum guarantees TV is continuous if V is continuous.

[11] $x = x'$ if no shock occurs.

3. Discounting

   By substitution, we get

$$(TV + a)(m, i; \Omega) = TV(m, i; \Omega) + \beta a.$$

4. Monotonicity

   Monotonicity is obvious since all the coefficients in our expression for V is positive.

■

8.1.2 Proof of Proposition 2

Proof. From **Lemma 9**, we can identify the solution to the Bellman equation as the optimal value in the sequential problem. The agents with (m, x) can get at least as high value as the agents with (m, x') $(x' > x)$ by the same strategy[12] as (m, x') since x is i.i.d. Therefore, $V(m, x; \Omega)$ is nonincreasing in x. Similarly, $V(m, i; \Omega)$ is nondecreasing in m and bounded by $\left[0, \frac{qc}{1-\beta}\right]$.

Noting that $V(m+1, x; \Omega) - x \in \left[-x, \frac{qc}{1-\beta} - x\right]$ is strictly decreasing and continuous in x while $V(m, u; \Omega) \in \left[0, \frac{qc}{1-\beta}\right]$ is constant in x, there is a unique threshold $R(m; \Omega) \in \left[-\frac{qc}{1-\beta}, \frac{qc}{1-\beta}\right]$ such that $x \leq R(m; \Omega) \Leftrightarrow V(m+1, x; \Omega) - x \geq V(m, u; \Omega)$. Since $V(m+1, x; \Omega)$ and $V(m, u; \Omega)$ are bounded and continuous in θ, so is $R(m; \Omega)$.

Suppose $F(R(0; \Omega)) = 0$ and those with $(0, u)$ do not work with probability 1. Then, she cannot make a positive bid. Thus, life-time utility is 0. On the other hand, consider the following strategy;

work only if she finds an opportunity with $x' \leq \delta$ at $t = 0$

$b = 0$ at $t = 0$

work only if she worked at $t = 0$ and no shock occurs at $t = 1$

$b = m$ at $t = 1$

obey the strategy ρ^*, ω^* for all $t \geq 2$

[12] This is not restricted to the case with Markov strategy.

Then, the expected disutility is $\{1+\beta(1-\lambda)\}\int_{\varepsilon}^{\delta} x' dF(x')$. On the other hand, at least with probability $F(\delta)(1-\lambda)$, she can bid $b=2$ at $t=1$. Thus, the expected utility is at least $F(\delta)(1-\lambda)\beta c$. From the assumption, $F(\delta)(1-\lambda)\beta c-\{1+\beta(1-\lambda)\}\int_{\varepsilon}^{\delta} x' dF(x')>0$, which is a contradiction. ■

8.1.3 Proof of Proposition 3

Since we take Ω as given for **Proposition 3**, we omit Ω. Firstly, we prove the following claim.

Claim 10 *For all q and for all $\epsilon>0$, there exists N_ϵ such that for all Ω, for all $m \geq N_\epsilon$,*

$$\begin{cases} V(m,x) \in \left(q\frac{c}{1-\beta}-\epsilon, q\frac{c}{1-\beta}\right] \\ V(m,u) \in \left(q\frac{c}{1-\beta}-\epsilon, q\frac{c}{1-\beta}\right] \\ R(m)<\epsilon \end{cases}. \tag{10}$$

Proof. For all $i \in \left[\varepsilon, \frac{c}{1-\beta}\right] \cup \{u\}$, $V(m,i) \leq q\frac{c}{1-\beta}$. On the other hand, $q\frac{c-(\beta)^{m-1}c}{1-\beta} \leq V(m,i)$ since the agents with (m,i) can get $q\frac{c-(\beta)^{m-1}c}{1-\beta}$ by always quitting their jobs and bidding $2p$ for at least $m-2$ times. Hence, if $m \geq N_\epsilon \equiv \left\lfloor \frac{\log\epsilon-\log qc+\log(1-\beta)}{\log\beta} \right\rfloor+2$, (10) holds. ■

Remark 11 ***Proposition 4*** *directly follows from* ***Claim 10*** *if we take $N=m_\varepsilon$.*

Using **claim 10**, we can prove **Proposition 3**.

Proof. We will show that it is strictly optimal to make $\tilde{v}(b)$ highest for all $m \geq 1$ by mathematical induction.

1. For $m=N$, the argument holds from **Claim 10**.
2. When it is strictly optimal to make $\tilde{v}(b)$ highest for all $m \geq n+1$, then it is strictly optimal to make $\tilde{v}(b)$ highest for $m=n$.
    (a) The case for the unemployed

Suppose, in the contrary, $b < \arg\max_{b \leq m} \tilde{v}(b)$ is one of the optimal strategies; that is,

$$\begin{aligned} & V(n,u) \\ = & \int_{\varepsilon}^{\frac{c}{1-\beta}} \beta\theta \max\left\{V\left(n+1,x'\right)-x', V(n,u)\right\} dF\left(x'\right) + \beta(1-\theta)V(n,u) \\ \geq & c + \beta\theta \int_{\varepsilon}^{\frac{c}{1-\beta}} \max\left\{V\left(n,x'\right)-x, V(n-1,u)\right\} dF\left(x'\right) + \beta(1-\theta)V(n-1,u). \end{aligned} \tag{11}$$

In addition, since $b = 0$ is always plausible,

$$V(n-1,u) \geq \int_{\varepsilon}^{\frac{c}{1-\beta}} \beta\theta \max\left\{V\left(n,x'\right)-x', V(n-1,u)\right\} dF\left(x'\right) + \beta(1-\theta)V(n-1,u). \tag{12}$$

and

$$\begin{aligned} V(n,x) \geq & \beta\lambda \int_{\varepsilon}^{\frac{c}{1-\beta}} \max\left\{V\left(n+1,x'\right)-x', V(n,u)\right\} dG\left(x'\right) \\ & +\beta(1-\lambda)\max\left\{V(n+1,x)-x, V(n,u)\right\}. \end{aligned} \tag{13}$$

From the assumption of mathematical induction,

$$\begin{aligned} V(n+1,x) \leq & c + \beta\lambda \int_{\varepsilon}^{\frac{c}{1-\beta}} \max\left\{V\left(n+1,x'\right)-x', V(n,u)\right\} dG\left(x'\right) \\ & +\beta(1-\lambda)\max\left\{V(n+1,x)-x, V(n,u)\right\}. \end{aligned} \tag{14}$$

From (13) and (14),

$$c + V\left(n,x'\right) - V\left(n+1,x'\right) \geq 0. \tag{15}$$

From (15),

$$c + V(n-1,u) - V(n,u) \leq 0 \tag{16}$$

is necessary for (11) to hold.

However, from (11) and (12),

$$
\begin{aligned}
& c+V(n-1,u)-V(n,u) \\
\geq\; & c+\int_{\varepsilon}^{\frac{c}{1-\beta}} \beta\theta \max\left\{V\left(n,x'\right)-x', V(n-1,u)\right\} dF\left(x'\right)+\beta(1-\theta) V(n-1,u) \\
& -\int_{\varepsilon}^{\frac{c}{1-\beta}} \beta\theta \max\left\{V\left(n+1,x'\right)-x', V(n,u)\right\} dF\left(x'\right)-\beta(1-\theta) V(n,u) \\
=\; & (1-\beta) c+\int_{\varepsilon}^{\frac{c}{1-\beta}} \beta\theta \max\left\{V\left(n,x'\right)-x', V(n-1,u)\right\} dF\left(x'\right)+\beta(1-\theta) V(n-1,u) \\
& -\int_{\varepsilon}^{\frac{c}{1-\beta}} \beta\theta \max\left\{-c+V\left(n+1,x'\right)-x', -c+V(n,u)\right\} dF\left(x'\right)-\beta(1-\theta)(-c+V(n,u)) \\
\geq\; & (1-\beta) c+\int_{\varepsilon}^{\frac{c}{1-\beta}} \beta\theta \max\left\{V\left(n,x'\right)-x', V(n-1,u)\right\} dF\left(x'\right)+\beta(1-\theta) V(n-1,u) \\
& -\int_{\varepsilon}^{\frac{c}{1-\beta}} \beta\theta \max\left\{V\left(n,x'\right)-x', -c+V(n,u)\right\} dF\left(x'\right)-\beta(1-\theta)(-c+V(n,u)) \\
\geq\; & (1-\beta) c-\beta\left|c+V(n-1,u)-V(n,u)\right| \\
=\; & (1-\beta) c+\beta(c+V(n-1,u)-V(n,u)) \\
\geq\; & c>0,
\end{aligned}
\tag{17}
$$

which is a contradiction. The second inequality uses (15) and the second equality uses (16). Therefore, the value when she wins is strictly larger than the value when she loses.

(b) The case for the employed

From the result for the unemployment case,

$$
V(n,u)<c+V(n-1,u). \tag{18}
$$

Thus,

$$
\begin{aligned}
& c+\beta\lambda\int_{\varepsilon}^{\frac{c}{1-\beta}}\max\left\{V\left(n,x'\right)-x',V\left(n-1,u\right)\right\}dG\left(x'\right) \\
& +\beta\left(1-\lambda\right)\max\left\{V\left(n,x\right)-x,V\left(n-1,u\right)\right\} \\
& -\beta\lambda\int_{\varepsilon}^{\frac{c}{1-\beta}}\max\left\{V\left(n+1,x'\right)-x',V\left(n,u\right)\right\}dG\left(x'\right) \\
& -\beta\left(1-\lambda\right)\max\left\{V\left(n+1,x\right)-x,V\left(n,u\right)\right\} \\
\geq\ & c+\beta\lambda\int_{\varepsilon}^{\frac{c}{1-\beta}}\max\left\{V\left(n,x'\right)-x',V\left(n-1,u\right)\right\}dG\left(x'\right) \\
& +\beta\left(1-\lambda\right)\max\left\{V\left(n,x\right)-x,V\left(n-1,u\right)\right\} \\
& -\beta\lambda\int_{\varepsilon}^{\frac{c}{1-\beta}}\max\left\{c+V\left(n,x'\right)-x',c+V\left(n-1,u\right)\right\}dG\left(x'\right) \\
& -\beta\left(1-\lambda\right)\max\left\{c+V\left(n,x\right)-x,c+V\left(n-1,u\right)\right\} \\
\geq\ & \left(1-\beta\right)c>0.
\end{aligned}
$$

The first inequality follows from (15) and (18). Therefore, the value when she wins is strictly larger than the value when she loses.

■

Remark 12 *If we admit mixed strategies, the agents with mp can choose the probability of getting one unit of the goods in* $[0,\max_{b\leq m}\tilde{v}(b)]$ *by combination of bidding* 0 *and* m *properly. This makes no change in the proof above. Since the optimal strategy is unique, we concentrate on pure strategies without loss of generality.*

8.2 Proof of the Real Indeterminacy

Here, we provide the rigorous proof of **Theorem 5**. As have mentioned, **Definition 1** is equivalent to the following.

1. Fix $v\in(1,0]$
2. Guess a stationary distribution $\{h_{m,e},h_{m,u}\}_{0\leq m\leq N}$
3. Given $\Omega=(v,\theta)$ with $\theta=\theta\left(\sum_{0\leq m\leq N}h_{m,u}\right)$ and $(\rho^{*}(m,x;\Omega),\omega^{*}(m,i;\Omega))$, calculate the one period forward distribution denoted by $\left\{h'_{m,e},h'_{m,u}\right\}_{0\leq m\leq N}$ taking $\{h_{m,e},h_{m,u}\}_{0\leq m\leq N}$ as

the initial distribution.

4. Ω is consistent with $\{h_{m,e}, h_{m,u}\}_{0\le m\le N}$ and (ω^*, ρ^*);

$$\{h_{m,e}, h_{m,u}\}_{0\le m\le N} = \left\{h'_{m,e}, h'_{m,u}\right\}_{0\le m\le N}, \tag{19}$$

$$v = \frac{\sum_{0\le m\le N} h_{m,e} - q\sum_{2\le m\le N}(h_{m,e}+h_{m,u})}{q(h_{1,e}+h_{1,u})}. \tag{20}$$

Firstly, we formally define the function from $\{h_{m,e}, h_{m,u}\}_{0\le m\le N}$ to $\left\{h'_{m,e}, h'_{m,u}\right\}_{0\le m\le N}$ as

$$h(0,e) = 0, \tag{21}$$

$$\begin{aligned} h'_{m,e} \equiv\ & v_m (1-\lambda+\lambda G(R(m-1;\Omega)))\, h_{m,e} \\ & + (1-v_{m-1})((1-\lambda) H(R(m-1;\Omega);m-1) + \lambda G(R(m-1;\Omega)))\, h_{m-1,e} \\ & + (1-v_{m-1})\,\theta F(R(m-1;\Omega))\, h_{m-1,u} \\ & + v_m \theta F(R(m-1;\Omega))\, h_{m,u} \text{ for } 1\le m\le N, \end{aligned} \tag{22}$$

$$\begin{aligned} h'_{m-1,u} \equiv\ & (1-v_{m-1})((1-\lambda)(1-H(R(m-1;\Omega);m-1)) + \lambda(1-G(R(m-1;\Omega))))\, h_{m-1,e} \\ & + v_m \lambda G(R(m-1;\Omega))\, h_{m,e} \\ & + (1-v_{m-1})(1-\theta+\theta(1-F(R(m-1;\Omega))))\, h_{m-1,u} \\ & + v_m (1-\theta+\theta(1-F(R(m-1;\Omega))))\, h_{m,u} \text{ for } 1\le m\le N, \end{aligned} \tag{23}$$

$$\begin{aligned} h(N,u) \equiv\ & (1-v_N)((1-\lambda)(1-H(R(N:\Omega);N)) + \lambda(1-G(R(N;\Omega))))\, h_{N,e} \\ & + (1-v_N)(1-\theta+\theta(1-F(R(N;\Omega))))\, h_{N,u}, \end{aligned} \tag{24}$$

where $v_m \equiv \max_{b\le m} qv(b)$, and

$$H(x;m) \equiv \begin{cases} \dfrac{h(\{(m,x');x'\le x\})}{h_{m,e}} \text{ if } h_{m,e} > 0, \\ 1 \text{ if } h_{m,e} = 0. \end{cases}$$

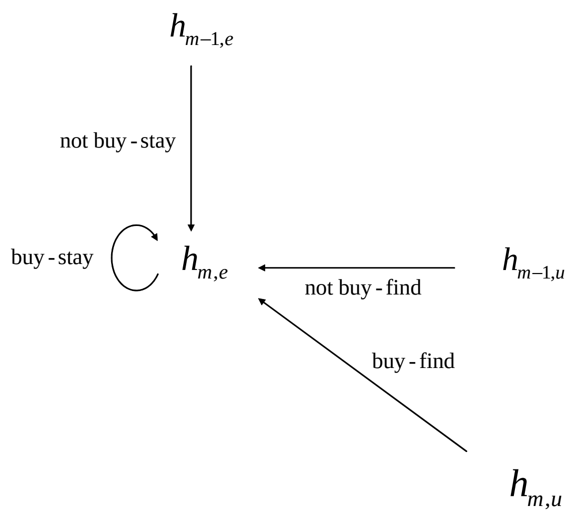


Figure 6: The inflow into (m, e)

Note that v_m is the probability of getting goods for agents with m money holdings and $H(x, m)$ is the probability that the disutility is no more than x given that the agents are previously employed and have m money holdings at the beginning of the night. Here, although we use measure h, we will show that we can express $H(x; m)$ solely by $\{h_{m,e}, h_{m,u}\}_{0 \le m \le N}$ and $\{R(m-1; \Omega)\}_{0 \le m \le N}$.

First, noting that all the agents with $(0, e)$ cannot buy at night, their next states will be $(0, u)$ or $(1, e)$. Since any agent who worked during the previous day has some money, there are no inflows into $(0, e)$. Therefore, $h_{0,e} = 0$.[13]

Second, we consider the stationary condition for (m, e). From **Proposition 3**, the agents with m money holdings work if and only if the disutility is no more than $R(m-1)$. From Figure 5, the inflow into (m, e) consists of those with (m, e) who can buy one unit of the goods at night and stay at the job on the next day, those with $(m-1)$ who cannot buy and stay, those with (m, u) who can buy and find a job, and those with $(m-u)$ who cannot buy and find. Figure 6 collects the arrows into $h_{m,e}$ from Figure 5.

It is verified from Figure 7 that those with (m, e) who can buy stay at (m, e) if and only if the shock does not occur or the shock not more than $R(m-1; \Omega)$ arrives, which yields the first component;

$$v_m (1 - \lambda + \lambda G(R(m-1; \Omega))) h_{m,e} \tag{25}$$

[13] Note that $H(R(m'), -1)$ is irrelevant in (22) with $m = 1$ since $h(0, e) = 0$.

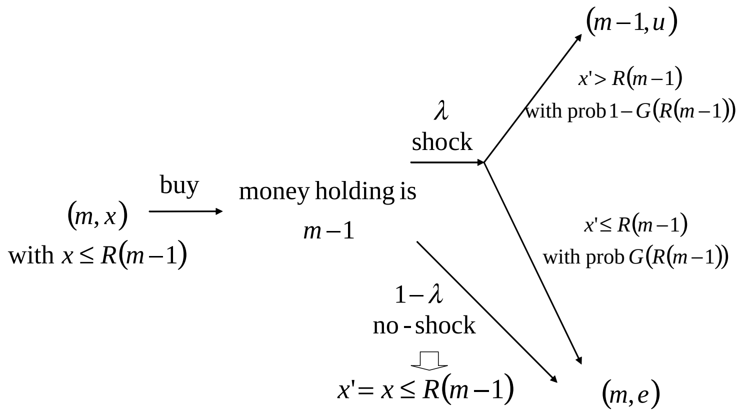


Figure 7: Those with (m, e) who can buy and stay at the job

The situation for $(m-1, e)$ who cannot buy is more complicated. They have $m-1$ money at the beginning of the next day. Thus, they stay as long as $x \leq R(m-1; \Omega)$. Note that, at the beginning of the previous day, their money holdings were only $m-2$. Hence, they decided to work if and only if $x \leq R(m-2; \Omega)$. Thus, those with initial disutility between $R(m-2; \Omega)$ and $R(m-1; \Omega)$ quit even without a shock. Therefore, $H(R(m-1; \Omega); m-1)$ is the probability of staying without a shock, which can be verified from Figure 8. Thus, the second component is

$$(1-v_{m-1})\left((1-\lambda) H(R(m-1; \Omega); m-1) + \lambda G(R(m-1; \Omega))\right) h_{m-1,e} \tag{26}$$

As for those with $(m-1, u)$ who cannot buy and those with (m, u) who can buy, since they have $m-1$ units of money at the beginning of the next day, they work with probability $\theta F(R(m-1; \Omega))$, which is verified from Figure 9. Thus, the third and fourth components are

$$(1-v_{m-1})\, \theta F(R(m-1; \Omega))\, h_{m-1,u} + v_m \theta F(R(m-1; \Omega))\, h_{m,u} \tag{27}$$

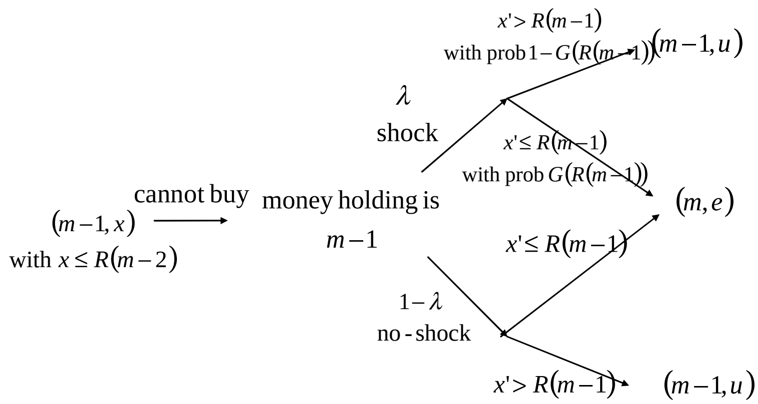


Figure 8: Those with $(m-1, e)$ who cannot buy and stay

Third, we consider the stationary condition for $(m-1, u)$. The inflow into those with $(m-1, u)$ consists of those with $(m-1, e)$ who cannot buy and quit, those with (m, e) who can buy and quit, those with $(m-1, u)$ who cannot buy and cannot find a job and those with (m, u) who can buy and cannot find. Figure 10 collects the arrows into $h_{m-1,u}$ from Figure 5.

The first component, those with $(m-1, e)$ who cannot buy and quit, is equal to those with $(m-1, e)$ who cannot buy minus (26). The second component, those with (m, e) who can buy quit, is equal to those with (m, e) who can buy minus (25). The third and fourth components, those with $(m-1, u)$ who cannot buy and do not work and those with (m, u) who can buy and do not work, are equal to those with $(m-1, u)$ who cannot buy plus those with (m, u) who can buy minus (27). Adding them up yields (23).

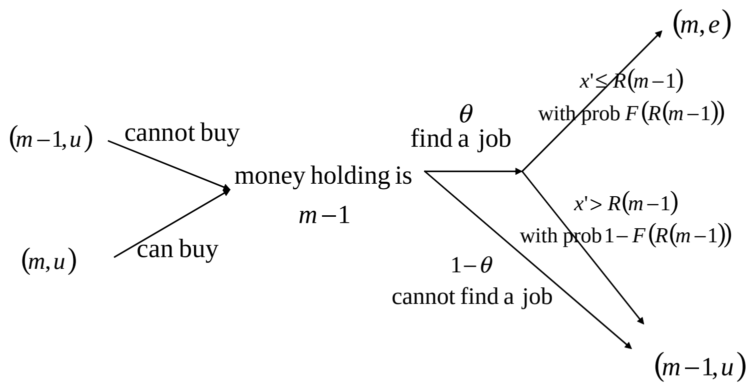


Figure 9: Those with $(m-1,u)$ who cannot buy and apply
Those with (m,u) who can buy and apply

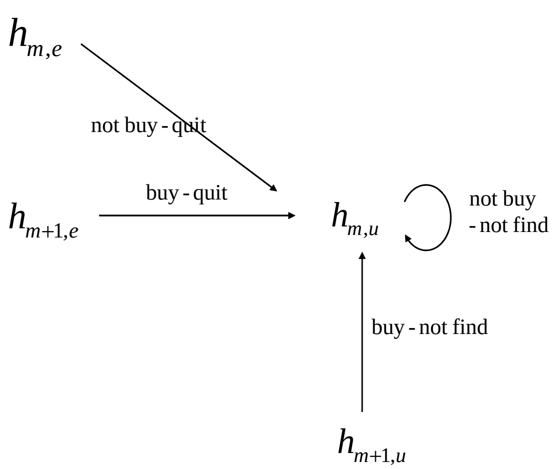


Figure 10: The inflow into $(m-1,u)$

Finally, we explain the expression of $H\left(R\left(m';\Omega\right),m\right)=\begin{cases}\dfrac{h\left(\left\{(m,x');x'\leq R\left(m';\Omega\right)\right\}\right)}{h_{m,e}} \text{ if } h_{m,e}>0,\\ 1 \text{ if } h_{m,e}=0.\end{cases}$

From (24),

$$\begin{aligned}
& h\left(\left\{\left(m,x'\right);x'\leq R\left(m';\Omega\right)\right\}\right)\\
= & \left(1-v_{m-1}\right)\begin{pmatrix}\left(1-\lambda\right)H\left(R\left(m';\Omega\right);m-1\right)\\ +\lambda G\left(\min\left\{R\left(m';\Omega\right),R\left(m-1;\Omega\right)\right\}\right)\end{pmatrix}h_{m-1,e}\\
& +v_m\left(\left(1-\lambda\right)H\left(R\left(m';\Omega\right);m\right)+\lambda G\left(\min\left\{R\left(m';\Omega\right),R\left(m-1;\Omega\right)\right\}\right)\right)h_{m,e}\\
& +\left(1-v_{m-1}\right)\theta F\left(\min\left\{R\left(m';\Omega\right),R\left(m-1;\Omega\right)\right\}\right)h_{m-1,u}\\
& +v_m\theta F\left(\min\left\{R\left(m';\Omega\right),R\left(m-1;\Omega\right)\right\}\right)h_{m,u}
\end{aligned}$$

Hence, dividing this by $h_{m,e}$ yields

$$\begin{aligned}
& H\left(R\left(m';\Omega\right);m\right)\\
= & v_m\left(\left(1-\lambda\right)H\left(R\left(m';\Omega\right);m\right)+\lambda G\left(\min\left\{R\left(m';\Omega\right),R\left(m-1;\Omega\right)\right\}\right)\right)\\
& +\left(1-v_{m-1}\right)\left(\left(1-\lambda\right)H\left(R\left(m';\Omega\right);m-1\right)+\lambda G\left(\min\left\{R\left(m';\Omega\right),R\left(m-1;\Omega\right)\right\}\right)\right)\frac{h_{m-1,e}}{h_{m,e}}\\
& +\theta F\left(\min\left\{R\left(m';\Omega\right),R\left(m-1;\Omega\right)\right\}\right)\left\{\left(1-v_{m-1}\right)\frac{h_{m-1,u}}{h_{m,e}}+v_m\frac{h_{m,u}}{h_{m,e}}\right\}
\end{aligned}$$

Therefore, given $\left\{R\left(m;\Omega\right)\right\}_{0\leq m\leq N}$ and $\left\{h_{m,e},h_{m,u}\right\}_{0\leq m\leq N}$, we can calculate $\left\{\left\{H\left(R\left(m';\Omega\right);m\right)\right\}_{0\leq m'\leq N}\right\}_{0\leq m\leq}$ as follows. If $h_{1,e}>0$

$$\begin{aligned}
& H\left(R\left(m';\Omega\right);1\right)\\
= & \min\left\{\tfrac{1}{1-v_1(1-\lambda)}\begin{pmatrix}qv\lambda G\left(\min\left\{R\left(m';\Omega\right),R\left(0;\Omega\right)\right\}\right)\\ +\theta F\left(\min\left\{R\left(m';\Omega\right),R\left(0;\Omega\right)\right\}\right)\left\{\frac{h_{0,u}}{h_{1,e}}+qv\frac{h_{1.u}}{h_{1,e}}\right\}\end{pmatrix},1\right\}
\end{aligned}\tag{28}$$

for all $0 \leq m' \leq N$. If $h_{1,e} = 0$, $H(R(m';\Omega);1) = 1$ for all $0 \leq m' \leq N$. If $h_{m,e} > 0$,

$$H\left(R\left(m';\Omega\right);m\right) = \min\left\{\frac{1}{1-v_m(1-\lambda)}\left(\begin{array}{c} v_m\lambda G\left(\min\left\{R\left(m';\Omega\right),R\left(m-1;\Omega\right)\right\}\right) \\ +\left(\begin{array}{c}\left(1-v_{m-1}\right)\left(1-\lambda\right)H\left(R\left(m';\Omega\right);m-1\right) \\ +\lambda G\left(\min\left\{R\left(m';\Omega\right),R\left(m-1;\Omega\right)\right\}\right)\end{array}\right)\frac{h_{m-1,e}}{h_{m,e}} \\ +\theta F\left(\min\left\{R\left(m';\Omega\right),R\left(m-1;\Omega\right)\right\}\right)\left\{\left(1-v_{m-1}\right)\frac{h_{m-1,u}}{h_{m,e}}+v_m\frac{h_{m,u}}{h_{m,e}}\right\}\end{array}\right),1\right\}, \tag{29}$$

for all $0 \leq m' \leq N$. If $h_{m,e} = 0$, $H(R(m';\Omega);1) = 1$ for all $0 \leq m' \leq N$.

Thus, let us define a function f_v from $\{h_{m,e}, h_{m,u}\}_{0\leq m\leq N}$ to $\left\{h'_{m,e}, h'_{m,u}\right\}_{0\leq m\leq N}$ as follows. Given v. Calculate $\theta = \theta\left(\sum_{0\leq m\leq N} h_{m,u}\right)$. From $\Omega = (v,\theta)$, calculate $R(m;\Omega)$. Using these, recursively calculate $\left\{\{H(R(m');m)\}_{0\leq m'\leq N}\right\}_{0\leq m\leq N}$ from (28) and (29). Using these, calculate $\left\{h'_{m,e}, h'_{m,u}\right\}_{0\leq m\leq N}$ from (21), (22), (23), and (24).

Then, **Definition 1** is equivalent to the condition that $\left\{h^*_{m,e}, h^*_{m,u}\right\}_{0\leq m\leq N}$ is a fixed point of f_v and

$$\textstyle\sum_{0\leq m\leq N}\left(h^*_{m,e}+h^*_{m,u}\right) = 1, \tag{30}$$

$$v = \frac{\sum_{1\leq m\leq N} h^*_{m,e} - q\sum_{2\leq m\leq N}\left(h^*_{m,e}+h^*_{m,u}\right)}{q\left(h^*_{1,e}+h^*_{1,u}\right)}. \tag{31}$$

Therefore, if and only if the following proposition holds, **Theorem 5** holds.

Proposition 13 *For any $v \in (1,0]$, if G and F are continuous, there exists $\left\{h^*_{m,e}, h^*_{m,u}\right\}_{0\leq m\leq N}$ that is a fixed point of f_v satisfying (30) and (31).*

Proof.

1. For any v, there exists a fixed point of f_v that satisfies (30)

   Let the domain of f_v be

$$\left\{\{h_{m,e},h_{m,u}\}_{0\leq m\leq N};\begin{array}{c}\sum_{0\leq m\leq N}\left(h_{m,e}+h_{m,u}\right)=1 \\ h_{m,i}\geq 0 \text{ for all } 0\leq m\leq N \text{ and } i\in\{e,u\}\end{array}\right\}.$$

From (21), (22), (23), and (24),

$$\sum_{0\leq m\leq N}\left(h'_{m,e}+h'_{m,u}\right)=\sum_{0\leq m\leq N}\left(h_{m,e}+h_{m,u}\right)=1$$
$$h'_{m,i}\geq 0 \text{ for all } 0\leq m\leq N \text{ and } i\in\{e,u\}.$$

Thus, f_v is a function from a convex and compact set to itself.

From the following reasons, f_v is continuous. Firstly, θ is continuous in $\{h_{m,e},h_{m,u}\}_{0\leq m\leq N}$. Secondly, from **Proposition 2**, $\{R(m;\Omega)\}_{0\leq m\leq N}$ is also continuous in θ. Thirdly, if $h_{m,e}>0$, $\left\{\{H(R(m';\Omega);m)\}_{0\leq m'\leq N}\right\}_{0\leq m\leq N}$ are continuous in θ and $\{h_{m,e},h_{m,u}\}_{0\leq m\leq N}$ since max operator preserves continuity and $(N+1)^2$ is finite. If $h_{m,e}=0$, all the relevant terms take the form of $H(R(m';\Omega);m)\,h_{m,e}$. For $h_{m,e}>0$, $H(R(m';\Omega);m)\in[0,1]$. Thus, $\lim_{h_{m,e}\to 0}H(R(m';\Omega);m)\,h_{m,e}=0$. Thus, $H(R(m';\Omega);m)\,h_{m,e}$ are continuous. Finally, (21), (22), (23), and (24) are continuous in Ω, $\left\{\{H(R(m');m)\,h_{m,e}\}_{0\leq m'\leq N}\right\}_{0\leq m\leq N}$, and $\{h_{m,e},h_{m,u}\}_{0\leq m\leq N}$.

Therefore, Brouwer's fixed point theorem guarantees the existence for all $v\in(1,0]$.

2. Redundancy of v

What we want to show is that if $\left\{h^*_{m,e},h^*_{m,u}\right\}_{0\leq m\leq N}$ is a fixed point of f_v,

$$v=\frac{\sum_{1\leq m\leq N}h^*_{m,e}-q\sum_{2\leq m\leq N}\left(h^*_{m,e}+h^*_{m,u}\right)}{q\left(h^*_{1,e}+h^*_{1,u}\right)};$$

that is, the condition for v is redundant. Note that, from **Proposition 3**, $h^*_{1,e}>0$. Thus, v is always well defined.

Adding up the stationary condition for $h_{m,e}$ and $h_{m-1,u}$, we get

$$\begin{aligned}(1-qv)\,h^*_{1,e}&=qvh^*_{1,u}\text{ for }m=1,\\(1-q)\,h^*_{m,e}&=qh_{m,u}\text{ for }m\geq 2.\end{aligned}$$

Combining these yields

$$v = \frac{\sum_{1 \leq m \leq N} h^*_{m,e} - q \sum_{2 \leq m \leq N} \left(h^*_{m,e} + h^*_{m,u}\right)}{q\left(h^*_{1,e} + h^*_{1,u}\right)}.$$

■