The Impossibility of Consistent Discrimination between I(0) and I(1) Processes

Ulrich K. Müller
Princeton University

August 19, 2005

Motivation

- Huge literature on discriminating between I(0) and I(1) processes

- Two approaches:
  - Unit root tests: H_0: "process is I(1)" against H_1: "process is I(0)"
  - Stationarity tests H_0: "process is I(0)" against H_1: "process is I(1)"

- Why discriminate between the two?
  - Subsequent inference depends on right choice (e.g. cointegration)
  - Forecasting
  - Economic Theory

Definition of I(0) and I(1) processes

- $\{y_{T,t}\}_{t=1}^{T}$ is I(0): $\quad \exists\ \sigma > 0 : T^{-1/2}\sum_{t=1}^{[\cdot T]} y_{T,t} \Rightarrow \sigma W(\cdot)$

- $\{y_{T,t}\}_{t=1}^{T}$ is I(1): $\quad \exists\ \sigma > 0 : T^{-1/2} y_{T,[\cdot T]} \Rightarrow \sigma W(\cdot)$

- Comments:

  - Number of authors use this definition of I(0)/I(1) (e.g. Stock (1994) or Davidson (1999))

  - I(0) equivalent to Functional Central Limit Theorem (FCLT) for $\{y_{T,t}\}_{t=1}^{T}$

  - Deterministics are known

Related Impossibility Literature

Blough (1992), Faust (1996), Pötscher (2002)

- Definition of I(0) time series: stationary with positive spectral density at frequency zero. Interest in *uniform* properties of discrimination procedures over parameter space.

- Example:

$$y_t = \rho y_{t-1} + \varepsilon_t \qquad t = 1, \cdots, T$$

  with $\varepsilon_t \sim iid\mathcal{N}(0,1)$.
  For any T one can choose $|\rho| < 1$ such that distribution of any statistic of $\{y_t\}_{t=1}^T$ becomes arbitrarily close to the distribution when $\rho = 1$.
  $\Rightarrow$ Stationarity tests cannot control size uniformly, and unit root tests have no power against some alternatives.

Related Impossibility Literature

- This result has no implication for whether or not it is possible to discriminate between I(0) and I(1) processes as defined earlier: For each T, let ρ_T be such that a given 5% level stationarity test has rejection probability 90%. Then it is not clear whether $y_{T,t} = \rho_T y_{T,t-1} + \varepsilon_t$ is I(0) in the sense of $T^{-1/2} \sum_{t=1}^{[\cdot T]} y_{T,t} \Rightarrow \sigma W(\cdot)$.

- For subsequent Wiener process approximations or forecasting, definition given earlier is relevant one. Can one construct valid and consistent unit root and stationarity tests under that definition?

Tests, Consistency and Size Control

- A (possibly randomized) test φ_T is a sequence of measurable functions of $\mathbb{R}^T \mapsto [0,1]$, where $\varphi_T(\{x_{T,t}\}_{t=1}^T)$ indicates the rejection probability conditional on observing $\{y_{T,t}\}_{t=1}^T = \{x_{T,t}\}_{t=1}^T$. Unconditional rejection probability is $E\varphi_T(\{y_{T,t}\}_{t=1}^T)$.

- Consistent unit root test φ_T: $E\varphi_T(\{y_{T,t}\}_{t=1}^T) \to 1$ whenever $\{y_{T,t}\}_{t=1}^T$ is I(0).

- Asymptotic size control of unit root test φ_T of level α: $E\varphi_T(\{y_{T,t}\}_{t=1}^T) \to \alpha$ whenever $\{y_{T,t}\}_{t=1}^T$ is I(1).

- Scale invariant tests: $\varphi_T(\{x_{T,t}\}_{t=1}^T) = \varphi_T(\{cx_{T,t}\}_{t=1}^T)$ for all $\{x_{T,t}\}_{t=1}^T$ and $c \neq 0$.

Main results

1. There exists a consistent unit root test that controls asymptotic size.

2. There does not exist a consistent stationarity test that controls asymptotic size, although there are inconsistent stationarity tests that do.

Consistent Unit Root test

Constructive proof using Breitung's (2002) statistic

$$G_T = T^{-2} \frac{\sum_{t=1}^{T} \left[\sum_{l=1}^{t} y_{T,l} \right]^2}{\sum_{t=1}^{T} y_{T,t}^2}$$

that rejects for small values.

$\Rightarrow$ Under null hypothesis of I(1), $G_T \Rightarrow \int_0^1 \left[\int_0^s W(u) du \right]^2 ds / \int W(s)^2 ds$.

$\Rightarrow$ Breitung (2002) shows consistency against stationary and ergodic alternative.

$\Rightarrow$ This paper shows consistency against general I(0) alternative.

No Consistent Stationarity test

Theorem:

Let $y_{T,t} = \sum_{l=1}^{t} \varepsilon_l$, $\varepsilon_t \sim iid\mathcal{N}(0,1)$.

For any scale invariant stationarity test φ_T that satisfies $\varphi_T(\{y_{T,t}\}_{t=1}^{T}) \xrightarrow{p} 1$, there exists a sequence of $T \times 1$ Gaussian random vectors $Z_T = (z_{T,1}, \cdots, z_{T,T})'$ satisfying $\varphi_T(\{z_{T,t}\}_{t=1}^{T}) \xrightarrow{p} 1$ and $\sup_s |T^{-1/2} \sum_{t=1}^{[sT]} z_{T,t} - W(s)| \xrightarrow{a.s.} 0$, such that $\{z_{T,t}\}_{t=1}^{T}$ is I(0).

Idea of Proof

- Itô and Nisio (1968): With $\xi_l \sim iid\mathcal{N}(0,1)$,
$$W(s) = \sum_{l=1}^{\infty} \frac{\sqrt{2}\sin((l-1/2)\pi s)}{(l-1/2)\pi}\xi_l \equiv \sum_{l=1}^{\infty} \phi_l(s)\xi_l$$
since r.h.s. converges almost surely and uniformly in s.

- For fixed n, define
$$W_n(s) = \sum_{l=1}^{n}(l-1/2)\phi_l(s)\xi_l + \sum_{l=n+1}^{\infty} \phi_l(s)\xi_l.$$
Since $\varphi_T(\{W(t/T)\}_{t=1}^T) \xrightarrow{p} 1$, must be true that also $\varphi_T(\{W_n(t/T)\}_{t=1}^T) \xrightarrow{p} 1$, as measures of W_n and W are equivalent.

- Carefully pick sequence $n_T \to \infty$ so that also $\varphi_T(\{W_{n_T}(t/T)\}_{t=1}^T) \xrightarrow{p} 1$. But with $n_T \to \infty$, $W_{n_T}(t/T)$ becomes I(0).

Implications of Impossibility

- Rejections of stationarity tests should be interpreted with care. Problems cannot be solved by considering alternative tests.

- Cannot consistently estimate the fractional parameter $d \subset A \supset \{0, 1\}$ when the fractional process I(d) is defined in terms of weak convergence to a Fractional Brownian Motion.

- Scale invariance not necessary if definition of I(0) and I(1) is altered to

$$\text{I(0):} \qquad \exists\ \{\sigma_T > \ 0\}_{T=1}^{\infty} : T^{-1/2}\sigma_T^{-1} \sum_{t=1}^{[\cdot T]} y_{T,t} \Rightarrow W(\cdot)$$

$$\text{I(1):} \qquad \exists\ \{\sigma_T > \ 0\}_{T=1}^{\infty} : T^{-1/2}\sigma_T^{-1} y_{T,[\cdot T]} \Rightarrow W(\cdot)$$

Uniform Properties?

Let $\mathcal{G}_0$ be the set of all I(0) processes $\{\{y_{T,t}\}_{t=1}^{T}\}_{T=1}^{\infty}$, i.e. for each $\{\{y_{T,t}\}_{t=1}^{T}\}_{T=1}^{\infty} \in \mathcal{G}_0$, $\exists\ \sigma > 0 : T^{-1/2}\sigma^{-1}\sum_{t=1}^{[\cdot T]} y_{T,t} \Rightarrow W(\cdot)$. Then clearly, for any stationarity test φ_T and given T,

$$\sup_{\{\{y_{T,t}\}_{t=1}^{T}\}_{T=1}^{\infty} \in \mathcal{G}_0} E\varphi_T(\{y_{T,t}\}_{t=1}^{T}) \geq E\varphi_T(\{T^{1/2}W(t/T)\}_{t=1}^{T})$$

$\Rightarrow$ no uniform size control without lower bound of speed of convergence

Uniform Properties?

- Let $\mathcal{H}_0(\{\delta_T\}_{T=1}^{\infty})$ be the set of all processes $\{\{y_{T,t}\}_{t=1}^{T}\}_{T=1}^{\infty}$ satisfying

$$\exists\ \sigma > 0 : \sup_t |T^{-1/2} \sum_{t=1}^{[\cdot T]} y_{T,t} - \sigma W(t/T)| < \delta_T \text{ a.s. for all } T.$$

- Let $\mathcal{H}_1(\{\delta_T\}_{T=1}^{\infty})$ be the set of I(1) processes satisfying

$$\exists\ \sigma > 0 : \sup_t |T^{-1/2} y_{T,[\cdot T]} - \sigma W(t/T)| < \delta_T \text{ a.s. for all } T.$$

- Sequence $\delta_T \to 0$ describes speed of convergence towards Wiener process.

Uniform Properties!

- For any choice of sequence $\delta_T \to 0$, Breitung's (2002) test statistic G_T yields a uniformly consistent unit root test with uniform size control:

$$\begin{aligned} \inf_{\{\{y_{T,t}\}_{t=1}^T\}_{T=1}^\infty \in \mathcal{H}_0(\{\delta_T\}_{T=1}^\infty)} E\varphi_T(\{y_{T,t}\}_{t=1}^T) &\to 1 \\ \sup_{\{\{y_{T,t}\}_{t=1}^T\}_{T=1}^\infty \in \mathcal{H}_1(\{\delta_T\}_{T=1}^\infty)} E\varphi_T(\{y_{T,t}\}_{t=1}^T) &\to \alpha. \end{aligned}$$

- For any scale invariant stationarity test that is consistent for a Gaussian Random walk, there exists $\{\{z_{T,t}\}_{t=1}^T\}_{T=1}^\infty \in \mathcal{H}_0(\{\delta_T\}_{T=1}^\infty)$ with $\delta_T \to 0$ such that $E\varphi_T(\{z_{T,t}\}_{t=1}^T) \to 1$.

- $\Rightarrow$ Need to know speed of convergence for construction of consistent stationarity tests, but not for consistent unit root tests.

Constructive Note

- Existence of *inconsistent* stationarity test that controls size:

$$J_T = \frac{\sum_{t=[T/2]+1}^{T} y_{T,t}}{\sum_{t=1}^{[T/2]} y_{T,t}}$$

converges to Cauchy under I(0) and to $J_T \Rightarrow \int_{1/2}^{1} W(s)ds / \int_0^{1/2} W(s)ds$ under I(1), which is not Cauchy.

- Interesting research question of how to construct in some sense efficient stationarity and unit root tests that behave well under the general I(0) and I(1) definition.