Pre and Post Break Parameter Inference

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Introduction

- Much recent econometric and applied interest in models with time varying parameters. General question of what to do beyond testing null hypothesis of parameter stability.

- One strand of econometric and applied literature imposes additional structure of (few) discrete shifts at unknown date.

- Challenging econometric questions in general about date, magnitude and pre and post break values of parameters.

- We focus on inference on (pre and) post break parameter value in model with single break at unknown date.

Standard Approach

- With break date known, can simply rely on post break data for inference. Asymmetric effects of choosing wrong break date:
  - too late leads to loss in efficiency
  - too early leads to incorrect inference
- With break date unknown, these effects vanish only if break date can be estimated accurately.
- Bai (1994, 1997), Bai and Perron (1998) show that in linear regressions, least squares break date estimator is sufficiently accurate asymptotically as long as break magnitude is sufficiently large. Formally, requires that break magnitude is outside the local $T^{-1/2}$ neighborhood.
- Let δ_T be the size of the break measured in standard deviations of the full sample parameter estimator. Standard approach corresponds to $|\delta_T| \to \infty$, and local asymptotics to $\delta_T \to \delta \in \mathbb{R}$.

Standard Approach under Local Asymptotics

- For nominal 5% tests about the post break parameter value, and break date restricted to middle 70% of sample
  - Relying on least squares break date estimator results in size of $\simeq$ 30%.
  - Null rejection probability exceeds 10% for all $\delta \in [5, 11]$ for some break date.
  - Size distortions exacerbated under pretesting.
- Standard inference not uniformly valid.
- Hard to argue that parameter shifts of less than 11 standard deviations are not empirically relevant, as statistical evidence for parameter instability often less than overwhelming.

  $\Rightarrow$ consider one time shift in US quarterly productivity series, $T = 136$. $\delta = 11$ corresponds to 11.2% annualized growth shift.

This Paper

- Derivation of a uniformly valid test for the post break parameter value of a scalar parameter of interest with unknown break date for general time series GMM model.

- Test is approximately efficient in the sense that for a particular weighting function, its weighted average power is demonstrably within one percentage point of achievable weighted average power.

Limiting Testing Problem

- Consider simple Gaussian location problem

  $$y_t = T^{-1/2}\beta + \mathbf{1}[t \le \rho T]T^{-1/2}\delta + \varepsilon_t, \qquad \varepsilon_t \sim \text{iid}\mathcal{N}(0,1),\ t = 1, \cdots, T$$

  with break of magnitude $T^{-1/2}\delta$ after $100\rho\%$ of the sample. We want to test

  $$H_0 : \beta = 0 \quad \text{against} \quad H_1 : \beta \ne 0$$

- Then, for any $s \in \{t/T\}_{t=1}^{T}$,

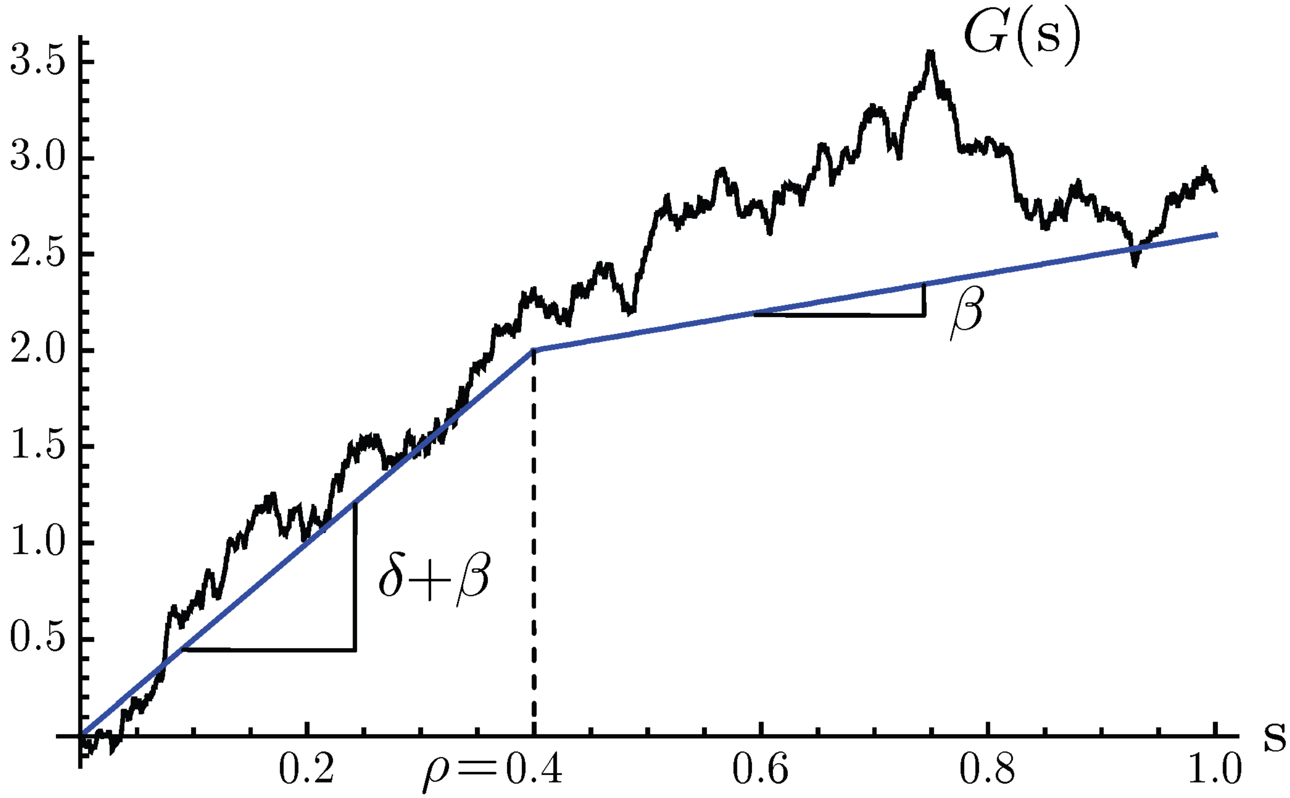
  $$T^{-1/2}\sum_{t=1}^{sT} y_t = \beta s + \delta \min(\rho, s) + T^{-1/2}\sum_{t=1}^{sT} \varepsilon_t \sim G(s)$$

  where $G(s) \in D_{[0,1]}$ is the continuous time Gaussian process

  $$G(s) = \beta s + \delta \min(\rho, s) + W(s), \qquad s \in [0,1].$$

  Suggests analysis of 'limiting problem' where observation is G.

Example for G



Nuisance Parameters

- Let $\theta = (\beta, \delta, \rho)$, $\Theta_0 = \{\theta = (0, \delta, \rho) : \delta \in \mathbb{R}, \rho = [0.15, 0.85]\}$ and $\Theta_1 = \{\theta = (\beta, \delta, \rho) : \beta \neq 0, \delta \in \mathbb{R}, \rho = [0.15, 0.85]\}$, so that break date is restricted to middle 70% of the sample.

- Denote by f_θ the density of G relative to the measure ν of W, so that hypothesis problem becomes

$$H_0 : \text{The density of } G \text{ is } f_\theta,\ \theta \in \Theta_0$$
$$H_1 : \text{The density of } G \text{ is } f_\theta,\ \theta \in \Theta_1$$

- Tests are functions $\varphi : D_{[0,1]} \mapsto [0, 1]$, so that rejection probability under θ is $E_\theta[\varphi(G)] = \int \varphi f_\theta d\nu$.

- Size control requires $\sup_{\theta \in \Theta_0} E_\theta[\varphi(G)] \leq 5\%$.

- Good tests when null and alternative hypotheses are composite?

Alternative Hypothesis

- No uniformly most powerful test

- Focus on weighted average power criterion

$$\text{WAP}(\varphi) = \int E_\theta[\varphi(G)]dF(\theta)$$

  where under F, ρ is distributed uniform over $[0.15, 0.85]$ and (β, δ) is bivariate normal with $\beta \sim \mathcal{N}(0, \sigma^2_\beta)$, $\delta \sim \mathcal{N}(0, \sigma^2_\delta)$ and β independent of $\beta + \delta$, where $\sigma^2_\beta = 22$ and $\sigma^2_\delta = 400$.

- Because $\text{WAP}(\varphi) = \int\int \varphi f_\theta d\nu dF(\theta) = \int \varphi(\int f_\theta dF(\theta))d\nu$, maximizing WAP amounts to maximizing power against single alternative

$$H_F : \text{The density of } G \text{ is } \int f_\theta dF(\theta)$$

Null Hypothesis

- Key insight by Andrews, Moreira and Stock (2007) and Müller and Watson (2008):

  Lemma 1: Let φ be any level α test of H_0 against H_F. For any probability distribution Λ on Θ_0, let φ_Λ be the Neyman-Pearson level α test of

  $$H_\Lambda : \text{the density of } G \text{ is } \int f_\theta d\Lambda(\theta)$$

  against H_F. Then φ_Λ is at least as powerful as φ.

- Proof: Since φ is of size α under H_0, it is also a valid level α test of H_Λ against H_1. But by assumption, φ_Λ is the best level α test in this problem, so its power is at least as high.

- *If* Λ^{**} is such that $\varphi_{\Lambda^{**}}$ is of level α under H_0, then Λ^{**} is the *least favorable distribution*, and $\varphi_{\Lambda^{**}}$ is the best test of H_0 against H_F. Unrestricted statement of Lemma 1 useful, because Λ^{**} often hard to identify.

Approximately Least Favorable Distribution

- Suppose we could identify a distribution Λ^* such that

  1. the 5% level test φ_{Λ^*} of H_{Λ^*} against H_F has power of 50.0%
  2. the 4.7% level test $\tilde{\varphi}_{\Lambda^*}$ of H_{Λ^*} against H_F has power of 49.0%, and $\tilde{\varphi}_{\Lambda^*}$ is a 5% level test of H_0

  Then, by Lemma 1, $\tilde{\varphi}_{\Lambda^*}$ is close to maximizing weighted average power among all 5% level test, and Λ^* might be called an *approximately least favorable distribution.*

- We extend algorithm of Müller and Watson (2008) to numerically identify a suitable Λ^*.

Main Result for Limiting Problem

Proposition 1:

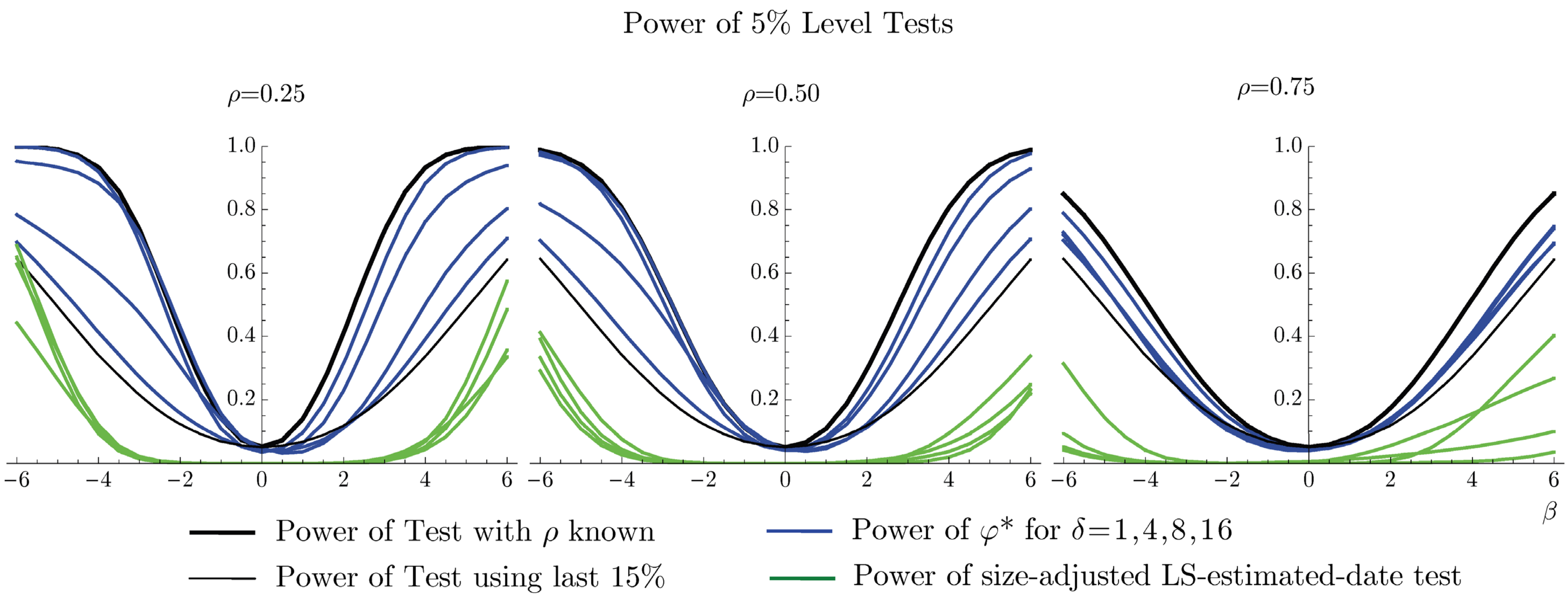
(i) any 5% level test φ of $H_0 : \theta \in \Theta_0$ has WAP(φ) $\leq \bar{\pi} \simeq 50.0\%$;

(ii) the (non-randomized) test φ^* defined in the paper is of level $\alpha^* \simeq 5\%$, and has WAP(φ^*) $= \pi^* \simeq 49.0\%$.

Power bound and level claim in Proposition 1 are *only* subject to Monte Carlo error (using 50,000 replications). No additional qualifications such as

- size was only established at a finite set of grid points
- Wiener processes were approximated by a discrete Random Walk

Power Profile of Suggested Test



Multivariate Limiting Problem

- Suppose instead of observing G, we observe the $(k+1)\times 1$ vector Gaussian process

$$\begin{pmatrix} G(s) \\ \tilde{G}(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} W(s) \\ \tilde{W}_k(s) \end{pmatrix} + \int_0^s \begin{pmatrix} \beta + \delta \mathbf{1}[\lambda \le \rho] \\ \tilde{\beta} + \tilde{\delta}\mathbf{1}[\lambda \le \rho] \end{pmatrix} d\lambda$$

  where $\tilde{\delta}, \tilde{\beta} \in \mathbb{R}^k$. Denote by φ^* the test that ignores $\tilde{G}$, so that φ^* is robust to properties of $\tilde{G}$.

- **Proposition**:
  - With $\tilde{\delta}, \tilde{\beta}$ unknown, φ^* remains an approximately admissible test.
  - Upper bound on power gains if $\tilde{\delta} = 0$ is known under invariance

$$\tilde{G}(s) \to \tilde{G}(s) + bs$$

    as function of $R^2 = A_{21}'(A_{21}A_{21}' + A_{22}A_{22}')^{-1}A_{21}$.

Inference in Parametric and GMM Models

- **Proposition** (via Limits of Experiments): Relevant limiting problem in correctly specified, sufficiently regular parametric models is multivariate limiting problem.

- Small sample version $\hat{\varphi}^*$ of φ^* implemented as function of *partial sample GMM estimators* (cf. Andrews 1993). Because φ^* is function of $\{G(15/100), G(16/100), \cdots, G(85/100); G(1) - G(15/100), \cdots, G(1) - G(85/100)\}$, requires 142 GMM estimations.

- **Proposition**: Under suitable conditions: (i) $\hat{\varphi}^*$ controls size uniformly over δ; (ii) $\hat{\varphi}^*$ is consistent for non-local alternatives; (iii) $\hat{\varphi}^*$ behaves close to optimal under nonlocal breaks; (iv) $\hat{\varphi}^*$ has local asymptotic power equal to φ^* under local breaks and alternatives.

Monte Carlo

Gaussian location problem $y_t = \beta + \delta T^{-1/2}\mathbf{1}[t \le \rho T] + u_t$, $T = 180$

ρ	Size ($\beta = 0$) $\hat{\varphi}^*$, $\delta =$ 1	4	8	16	$\hat{\varphi}_\rho$	Power ($\beta = 4$) $\hat{\varphi}^*$, $\delta =$ 1	4	8	16	$\hat{\varphi}_\rho$
					i.i.d. disturbances					
0.25	4.9	5.1	4.8	4.2	5.3	41.6	53.2	78.2	89.8	93.2
0.50	4.8	5.1	5.2	4.8	5.7	41.3	50.7	68.0	74.6	80.7
0.75	4.7	4.5	5.6	5.9	6.0	41.6	42.0	44.6	44.2	52.7
					AR(1) disturbances					
0.25	8.9	9.0	8.1	7.4	8.2	52.2	63.8	83.2	92.6	94.6
0.50	9.1	8.7	8.4	7.9	9.0	51.5	60.5	74.4	80.1	85.0
0.75	9.0	7.9	8.6	10.3	11.6	51.1	49.6	52.5	55.3	63.1
					MA(1) disturbances					
0.25	7.0	7.3	6.4	5.8	6.9	48.0	59.5	80.9	91.2	93.6
0.50	7.1	7.2	7.0	6.3	7.3	47.5	56.3	71.4	77.8	82.5
0.75	6.8	7.0	7.7	8.3	8.9	47.5	46.6	48.7	50.0	58.1

Conclusions

- Approximately weighted average power maximizing, uniformly valid test for a standard problem in time series econometrics

- Hard due to presence of nuisance parameter under null hypothesis, and apparently no special structure to statistical model that could be exploited

- Suggested test not pretty, but algorithm of its determination is generic. Given the size control approach, suggested test as valid as any whose critical value is determined by Monte Carlo, and demonstrably close to efficient in some sense

  $\Rightarrow$ 'proof of concept' for other problems involving nuisance parameters under the null hypothesis that cannot be estimated consistently