Appendix. Proof of relation (3) for $\alpha \leq 0.05$.

For the arguments, we will need the following result that follows from Lemma 1 Bakirov (1989) and its proof.

Lemma 1 *Let $g(s)$, $s \in (0,1]$ be a continuously differentiable function such that*

$$g(s) > 0, \quad g'(s) \geq 0 \ \ \textit{for all } s \in (0,1], \tag{A.1}$$

and let $f(s)$, $s \in (0,1]$ be a continuous function such that

$$f(s)(s - s_0) \geq 0 \ \ \textit{for some } s_0 \in [0,1), \tag{A.2}$$

$$\int_0^1 f(s)ds > 0. \tag{A.3}$$

Then $\displaystyle\int_0^1 f(s)g(s)ds > 0.$

For completeness, we provide the proof of Lemma 1 below.

Proof of Lemma 1. It suffices to show that $\displaystyle\int_\epsilon^1 f(s)g(s)ds > 0$ for all $\epsilon \in (0,1)$. It is easy to see that from conditions (A.2) and (A.3) it follows that $\displaystyle\int_\epsilon^1 f(s)ds > 0$ for all $\epsilon \in [0,1)$. This, together with (A.1), implies, using integration by parts, that

$$\int_\epsilon^1 f(s)g(s)ds = g(\epsilon)\int_\epsilon^1 f(s)ds + \int_\epsilon^1 g'(s)\int_x^1 f(s)ds\, dx > 0 \tag{A.4}$$

for all $\epsilon \in (0,1)$. The proof is complete.

Proof of relation (3) for $\alpha \leq 0.05$. Denote $R = \frac{q\,\mathrm{cv}^2}{\mathrm{cv}^2 + q - 1}$.

As in Bakirov (1989), we have

$$P(|t| > \mathrm{cv}) = \frac{1}{\pi}\int_0^1 \frac{\sqrt{R}s^{q/2}ds}{\sqrt{\Big(\sum_{k=1}^q \frac{x_k}{x_k+1} - R\Big)\prod_{k=1}^q (x_k + s)}},$$

where $x_k = R\sigma_k^2$ and, using multiplication of all σ_k, $k = 1, ..., q$, by a constant,

$$\sum_{k=1}^{q} \frac{x_k}{x_k + 1} = R. \tag{A.5}$$

Let $z = \max_{1 \le k \le q} x_k$. Fixing all x_k, except $x_i = z$ and $x_j = y$ and considering $x_j = y$ as a function of z, one obtains that the derivative of the probability $P(|t| > x)$ with respect to z at the point $x = (x_1, ..., x_q)$ is given by (see Bakirov (1989))

$$M = \frac{y - z}{2\pi} \int_0^1 \frac{L(z, y, P(x, s))}{\sqrt{1 - s}} U(x, s) ds, \tag{A.6}$$

where

$$L(z, y, v) = \frac{A + Bs}{(z + s)(y + s)} - \frac{2s(A + \frac{1+s}{2}B)}{(z + s)^2(y + s)^2 v}, \tag{A.7}$$

$$U(x, s) = \frac{s^{q/2-1}\sqrt{R}}{\sqrt{P(x, s) \prod_{k=1}^{q} (x_k + s)}}, \tag{A.8}$$

$$P(x, s) = \sum_{k=1}^{q} \frac{x_k}{(x_k + 1)(x_k + s)}, \tag{A.9}$$

$$A = yz - 1, \quad B = y + z + 2. \tag{A.10}$$

As in Bakirov (1989), from condition (A.5) it follows that $(z + s)(y + s)P(x, s) \ge sR$, and, therefore, the following estimate holds:

$$\frac{2\pi M}{y - z} \ge \int_0^1 \frac{U(x, s)}{(z + s)(y + s)\sqrt{1 - s}} \Big(A + Bs - \frac{2\big(A + \frac{1+s}{2}B\big)}{R} \Big). \tag{A.11}$$

Define the functions

$$f^*(s,y,z,R) = \frac{s^{-\alpha}}{\sqrt{1-s}}\Big(A+Bs-\frac{2(A+\frac{1+s}{2}B)}{R}\Big)\exp\Big[R\frac{\ln(s)-s}{2(1+z)}\Big]$$
$$g^*(s,y,z,R) = \frac{U(x,s)s^{\alpha}}{(z+s)(y+s)}\exp\Big[-R\frac{\ln(s)-s}{2(1+z)}\Big].$$

According to (A.11), thus,

$$\frac{2\pi M}{y-z} \geq \int_0^1 f^*(s,y,z,R)g^*(s,y,z,R)ds \tag{A.12}$$

(the above-defined functions f^* and g^* differ from the functions $f(s) = \frac{s^{-\alpha}}{\sqrt{1-s}}(A+Bs-2(A+\frac{1+s}{2}B)/R)$ and $g(s) = U(x,s)s^{\alpha}/((z+s)(y+s))$ in Bakirov (1989) by the additional factors $e^{R(\ln(s)-s)/(2(1+z))}$ and $e^{-R(\ln(s)-s)/(2(1+z))}$, respectively).

Let us show, using Lemma 1 that

$$\int_0^1 f^*(s,y,z,R)g^*(s,y,z,R)ds > 0. \tag{A.13}$$

Clearly, condition (A.2) is satisfied for f^* if $R \geq 2$.

We have

$$\frac{\partial \ln g^*(s,y,z,R)}{\partial s} = \frac{\frac{q}{2}-1+\alpha}{s} - \frac{1}{2}\sum_{k=1}^{q}\frac{1}{x_k+s} + \frac{1}{2}\frac{\sum_{k=1}^{q}\frac{x_k}{(x_k+1)(x_k+s)^2}}{P(x,s)} - \frac{1}{y+s} -$$
$$\frac{1}{z+s} - \frac{R(1-s)}{2s(1+z)} \geq \frac{1}{2s}\sum_{k=1}^{q}\frac{x_k}{x_k+s} - \frac{1-\alpha}{s} + \frac{1}{2(z+s)} - \frac{1}{y+s} - \frac{1}{z+s} - \frac{R(1-s)}{2s(1+z)}$$

since $\sum_{k=1}^{q}\frac{x_k(z+s)}{(x_k+1)(x_k+s)^2} \geq P(x,s)$ and $\sum_{k=1}^{q}(\frac{1}{s}-\frac{1}{x_k+s}) = \frac{1}{s}\sum_{k=1}^{q}\frac{x_k}{x_k+s}$. Now note that $\frac{x}{x+s}$ is a convex function of s, so that it is uniformly larger than its tangent at the point $s=1$, i.e.

$$\frac{x}{x+s} \geq \frac{x}{x+1} + (1-s)\frac{x}{(x+1)^2} \quad \text{for all } s \in [0,1]. \tag{A.14}$$

Therefore, $\sum_{k=1}^{q}\frac{x_k}{x_k+s}$ is a convex function of s, so that it is uniformly larger than its tangent at the point $s=1$, i.e.

$$\sum_{k=1}^{q}\frac{x_k}{x_k+s} \geq \sum_{k=1}^{q}\frac{x_k}{x_k+1} + (1-s)\sum_{k=1}^{q}\frac{x_k}{(x_k+1)^2} \quad \text{for all } s \in [0,1].$$

Also, $\sum_{k=1}^{q} \frac{x_k(z+1)}{(x_k+1)^2} \geq \sum_{k=1}^{q} \frac{x_k}{x_k+1} = R$, so that

$$\sum_{k=1}^{q} \frac{x_k}{x_k+s} \geq R + (1-s)\frac{R}{1+z}.$$

Using the last inequality, we have

$$\begin{aligned} \frac{\partial \ln g^*(s,y,z,R)}{\partial s} &\geq \frac{R}{2s} + \frac{R}{2(1+z)}\frac{1-s}{s} - \frac{1-\alpha}{s} - \frac{1}{y+s} - \frac{1}{2(s+z)} - \frac{R}{2(1+z)}\frac{1-s}{s} \\ &\geq \frac{R}{2s} - \frac{1-\alpha}{s} - \frac{1}{s} - \frac{1}{2(s+z)}. \end{aligned} \tag{A.15}$$

From (A.15) it follows that conditions (A.1) are satisfied for g^* if

$$R \geq 4 - 2\alpha + \frac{1}{1+z}. \tag{A.16}$$

We have

$$\frac{\partial f^*(s,y,z,R)}{\partial y} = \int_0^1 \tilde{f}(s,z,R)\tilde{g}(s,z,R)ds,$$

where

$$\tilde{f}(s,z,R) = \frac{s^{-\alpha}}{\sqrt{1-s}}\Big(z - \frac{1+2z}{R} + s\big(1 - \frac{1}{R}\big)\Big),$$

$$\tilde{g}(s,z,R) = \exp\Big[R\frac{\ln(s)-s}{2(1+z)}\Big].$$

Since

$$\begin{aligned} \int_0^1 \tilde{f}(s,z,R)ds = \Big(z - \frac{1+2z}{R}\Big)\frac{\Gamma(0.5)\Gamma(1-\alpha)}{\Gamma(1.5-\alpha)} + \Big(1-\frac{1}{R}\Big)\frac{\Gamma(0.5)\Gamma(2-\alpha)}{\Gamma(2.5-\alpha)} = \\ \frac{\Gamma(0.5)\Gamma(2-\alpha)}{\Gamma(2.5-\alpha)}\Big[\Big(z - \frac{1+2z}{R}\Big)(1.5-\alpha) + \Big(1-\frac{1}{R}\Big)(1-\alpha)\Big], \end{aligned} \tag{A.17}$$

the functions $\tilde{f}(s,z,R)$ and $\tilde{g}(s,z,R)$ satisfy conditions of Lemma 1 and, thus, $\partial f^*(s,y,z,R)/\partial y > 0$ for

$$R > \frac{5-4\alpha+2(3-2\alpha)z}{2-2\alpha+(3-2\alpha)z}. \tag{A.18}$$

Obviously, (A.18) is satisfied if

$$R > \frac{5 - 4\alpha}{2 - 2\alpha}. \tag{A.19}$$

From the above we conclude that, under (A.19), inequality (A.3) holds if

$$\int_0^1 f^*(s, 0, z, R)ds = \int_0^1 \tilde{f}_1(s, z, R)\tilde{g}(s, z, R)ds > 0, \tag{A.20}$$

where $\tilde{f}_1(s, z, R) = \frac{s^{-\alpha}}{\sqrt{1-s}} \frac{-(R+z)+(R-1)(2+z)s}{R}$. We have that the functions $\tilde{f}_1$ and $\tilde{g}$ satisfy the conditions of Lemma 1 if

$$\int_0^1 \tilde{f}_1(s, z, R)ds > 0, \tag{A.21}$$

that is, for

$$\frac{(-2 - z + R(2 + z))\Gamma(2 - \alpha)}{\Gamma(2.5 - \alpha)} - \frac{(R + z)\Gamma(1 - \alpha)}{\Gamma(1.5 - \alpha)} > 0 \tag{A.22}$$

or, equivalently,

$$(-2 - z + R(2 + z))(1 - \alpha) - (R + z)(1.5 - \alpha) > 0. \tag{A.23}$$

Condition (A.23) is satisfied if

$$R > \frac{4 - 4\alpha + (5 - 4\alpha)z}{1 - 2\alpha + (2 - 2\alpha)z}. \tag{A.24}$$

Clearly, (A.24) implies (A.19).

First, let $R \geq \tilde{R} = (5 + \sqrt{5})/2 \approx 3.618$ and let $\alpha = (4 - R_0)/2 + 1/(2(1 + z)) = 3/4 - \sqrt{5}/4 + 1/(2(1 + z)) \approx 0.191 + 1/(2(1 + z))$. Condition (A.16) is trivially satisfied. It is easy to check that inequality (A.22) and, thus, (A.19), are satisfied for $z > \tilde{z} = \frac{-\tilde{R}^2+6\tilde{R}-6}{\tilde{R}^2-4\tilde{R}+3} = (3 + \sqrt{5})/(1 + \sqrt{5}) \approx 1.618$. We conclude that (A.13) holds for $R \geq \tilde{R}$ and $z > \tilde{z}$.

Let us now show that (A.13) is also satisfied for $R \geq \tilde{R}$ and $0 \leq z \leq \tilde{z}$.

We first make the following observation. Let z_0 and z_1 be such that $0 \leq z_0 \leq z \leq z_1$ and let $R_0 \leq R$. Then

$$\int_0^1 \tilde{f}_1(s,z,R)\tilde{g}(s,z,R)ds = \int_0^1 \tilde{f}_1(s,z,R)\tilde{g}(s,z_1,R_0)\frac{\tilde{g}(s,z,R)}{\tilde{g}(s,z_1,R_0)}ds.$$

Since

$$\frac{\partial \ln \frac{\tilde{g}(s,z,R)}{\tilde{g}(s,z_1,R_0)}}{\partial s} = \frac{R}{2(1+z)}\Big(\frac{1}{s}-1\Big) - \frac{R_0}{2(1+z_1)}\Big(\frac{1}{s}-1\Big) \geq 0 \tag{A.25}$$

from Lemma 1 it follows that it suffices to show that $\int_0^1 \tilde{f}_1(s,z,R)\tilde{g}(s,z_1,R_0)ds > 0$ to be able to conclude that $\int_0^1 f_1(s,z,R)\tilde{g}(s,z,R)ds > 0$. In addition,

$$\frac{\partial f_1(s,z,R)}{\partial z} = \frac{s^{-\alpha}}{\sqrt{1-s}}\frac{-1+(R-1)s}{R}$$

and, therefore,

$$\int_0^1 \Big(\frac{-1+(R-1)s}{R}\Big)\frac{s^{-\alpha}}{\sqrt{1-s}}ds = \frac{(R-1)\Gamma(0.5)\Gamma(2-\alpha)}{\Gamma(2.5-\alpha)} - \frac{\Gamma(0.5)\Gamma(1-\alpha)}{\Gamma(1.5-\alpha)} = \frac{\Gamma(0.5)\Gamma(1-\alpha)}{\Gamma(1.5-\alpha)}\Big(R(1-\alpha)-2.5+\alpha\Big) > 0$$

if (A.19) holds. Consequently, under condition (A.19), by Lemma 1,

$$\int_0^1 \frac{\partial \tilde{f}(s,z,R)}{\partial z}\tilde{g}(s,z_1,R_0)ds \geq 0$$

and it suffices to show $\int_0^1 \tilde{f}(s,z_0,R)\tilde{g}(s,z_1,R_0)ds > 0$ to conclude that, for all $z \in [z_0,z_1]$, inequality (A.20) and, therefore, (A.13) holds.

Finally, note that

$$\frac{\partial \tilde{f}(s,z_0,R)}{\partial R} = \frac{s^{-\alpha}}{\sqrt{1-s}}\frac{z+s(2+z)}{R^2} \geq 0$$

so that it, under (A.19), it suffices to show

$$\int_0^1 \tilde{f}(s,z_0,R_0)\tilde{g}(s,z_1,R_0)ds > 0 \tag{A.26}$$

to conclude that for all $z \in [z_0,z_1]$ and $R \geq R_0$, inequality (A.13) is satisfied.

It is easy to check that conditions (A.16) and (A.19) are satisfied for all $R \geq \tilde{R}$ and $z \geq 0$ if $\alpha = 5/2 - \tilde{R}/2 = (5-\sqrt{5})/4 \approx 0.691$.

From the above discussion we conclude that, in order to show that (A.13) holds for $R \geq \tilde{R}$ and $0 \leq z \leq \tilde{z}$, it suffices to check that, for a sufficiently small Δ and all $1 \leq m \leq \tilde{z}/\Delta + 1$, $\mathcal{I}(m,\Delta) = \int_0^1 \tilde{f}(s,(m-1)\Delta,\tilde{R})\tilde{g}(s,m\Delta,\tilde{R})ds > 0$.

Using numerical computation, one can check that, for $\Delta = 0.1$ and all $1 \leq m \leq \tilde{z}/\Delta + 1$, $\mathcal{I}(m,\Delta) > \mathcal{I}(0,\Delta) \approx 0.0076 > 0$. This implies that (A.13) is satisfied for $R \geq \tilde{R}$ and $0 \leq z \leq \tilde{z}$, and, therefore, for all $z \geq 0$.

From the above arguments it follows that it remains to consider the case where $x_i = R/(k-R)$, $i = 1,...,k$ and $x_i = 0$, $i = k+1,...,n$ for some $1 \leq k \leq n$. That is, as in Bakirov (1989), we need to compare the quantities

$$\Phi_k = \Phi_k(R) = P\Big(|T_k| > \sqrt{\frac{R(k-1)}{k-R}}\Big) = \frac{1}{\pi}\int_0^1 \frac{s^{k/2-1}ds}{\sqrt{(z_k+s)^{k-1}(1-s)}}, \quad R < k \leq n, \tag{A.27}$$

where $z_k = \frac{R}{k-R}$. We have that

$$\Phi_k' = \frac{1}{\pi}\int_0^1 f_1^*(s,z_k,R)g(s,z_k,R)ds,$$

where

$$f_1^*(s,z,R) = \tilde{f}_1^*(s,z,R)\tilde{g}_1^*(s,z,R),$$

$$\tilde{f}_1^*(s,z,R) = \frac{s^{1-\alpha}}{\sqrt{1-s}}\Big(z + z^2\big(1-\frac{1}{R}\big) - (z+s)\ln\big(1+\frac{z}{s}\big)\Big),$$

$$\tilde{g}_1^*(s,z,R) = \exp\Big[R\frac{\ln(s)-s}{2(1+z)}\Big],$$

$$g_1^*(s,z,R) = \frac{s^{k/2-2+\alpha}}{(z+s)^{(k+1)/2}}\exp\Big[-R\frac{\ln(s)-s}{2(1+z)}\Big].$$

We have

$$\frac{\partial[\ln g_1^*(s,z_k,R)]}{\partial s} = \frac{k-4+2\alpha}{2s} - \frac{k+1}{2(z_k+s)} - \frac{R}{2s(z_k+1)} + \frac{R}{2(z_k+1)} =$$

$$\frac{\alpha-5/2}{s}+\frac{k+1}{2}\Big[\frac{1}{s}-\frac{1}{z_k+s}\Big]-\frac{R}{2s(z_k+1)}=$$
$$\frac{\alpha-5/2}{s}+\frac{(k+1)z_k}{2s(z_k+s)}-\frac{R}{2(z_k+1)s}+\frac{R}{2(z_k+1)}.$$

Using again (A.14), we obtain that

$$\frac{\partial[\ln g_1^*(s,z_k,R)]}{\partial s}\geq\frac{\alpha-5/2}{s}+\frac{k+1}{2s}\Big[\frac{z_k}{z_k+1}+(1-s)\frac{z_k}{(z_k+1)^2}\Big]-$$
$$\frac{R}{2(z_k+1)s}+\frac{R}{2(z_k+1)}=\frac{\alpha-5/2}{s}+\frac{k+1}{2s}\Big[\frac{R}{k}+(1-s)\frac{R}{k(z_k+1)}\Big]-\frac{(1-s)R}{2(z_k+1)s}.$$

Evidently, the last expression is nonnegative for $R\geq 5-2\alpha$.

As in Bakirov (1989), we have that

$$\int_0^1\tilde{f}_1^*(s,z,R)\geq\frac{\Gamma(0.5)\Gamma(2-\alpha)}{\Gamma(2.5-\alpha)}h(z),$$

where $h(z)=2(1-\alpha)\Big[z+z^2\big(1-\frac{1}{R}\big)-(1+z)\ln(1+z)\Big]-z^2/2$. It is not difficult to check that, for $R\geq\tilde{R}_1=3.7$ and $\alpha=5/2-\tilde{R}_1/2=0.65$,

$$h''(z)=-1+4(1-\alpha)\big(1-\frac{1}{R}\big)-\frac{2(1-\alpha)}{1+z}\geq 0$$

for

$$z\geq\frac{2(1-\alpha)}{4(1-\alpha)\big(1-\frac{1}{\tilde{R}_1}\big)-1}-1\approx 31.38.$$

Since the function $\tilde{g}_1^*(s,z,R)$ is nondecreasing, from Lemma 1 we conclude that

$$\int_0^1\tilde{f}_1^*(s,z,R)>0$$

for all $z\geq 32$ and $R\geq\tilde{R}_1$.

Let now z_0 and z_1 be such that $0\leq z_0\leq z\leq z_1$ and $R_0\leq R$. Using (A.25), we conclude, as before that it suffices to show that

$$\int_0^1\tilde{f}_1(s,z,R)\tilde{g}_1(s,z_1,R_0)ds>0$$

for all $0\leq z_0\leq z\leq z_1$ and $R_0\leq R$ to be able to conclude that

$$\int_0^1 \tilde{f}_1(s,z,R)\tilde{g}_1(s,z,R)ds > 0 \tag{A.28}$$

for all $0 \le z_0 \le z \le z_1$ and $R_0 \le R$.

In addition, since

$$\frac{\partial \tilde{f}_1^*(s,z,R)}{\partial R} = \frac{s^{1-\alpha}z^2}{R^2\sqrt{1-s}} > 0$$

we conclude that (A.28) is satisfied if

$$\mathcal{I}(z) = \int_0^1 \tilde{f}_1(s,z,R_0)\tilde{g}_1(s,z_1,R_0)ds > 0$$

for all $0 \le z_0 \le z \le z_1$.

Finally, we note that it suffices to show that

$$\mathcal{I}''(z_0) = \int_0^1 \frac{\partial^2 \tilde{f}_1(s,z_0,R_0)}{\partial z^2}\tilde{g}_1(s,z_1,R_0)ds =$$

$$\int_0^1 \frac{s^{1-\alpha}}{\sqrt{1-s}}\Big(2\big(1-\frac{1}{R_0}\big) - \frac{1}{z_0+s}\Big)\tilde{g}_1(s,z_1,R_0)ds > 0 \tag{A.29}$$

to be able to conclude that

$$\mathcal{I}(z) > \mathcal{I}(z_0) + \mathcal{I}'(z_0)(z-z_0).$$

Since, as is easy to see, $\mathcal{I}(0) = \mathcal{I}'(0) = 0$, from the above discussion we conclude that, in order to show that (A.28) holds for $R \ge \tilde{R}_1$ and $0 \le z \le \tilde{z}$, it suffices to check that, for a sufficiently small Δ and all $1 \le m \le \tilde{z}/\Delta + 1$,

$$\mathcal{I}(m,\Delta) = \int_0^1 \Big(2\big(1-\frac{1}{\tilde{R}}\big) - \frac{1}{(m-1)\Delta+s}\Big)\tilde{g}_1(s,m\Delta,\tilde{R})ds > 0.$$

Using numerical computation, one can check that, for $\Delta = 0.1$ and all $1 \le m \le \tilde{z}/\Delta + 1$, $\mathcal{I}(m,\Delta) > \mathcal{I}(0,\Delta) \approx 0.012 > 0$. This implies that (A.13) is satisfied for $R \ge \tilde{R}_1$ and $0 \le z \le \tilde{z}$, and, therefore, for all $z \ge 0$.

Using Lemma 1 we conclude that (A.27) is increasing in $R < k \le n$ for all $R \ge \tilde{R}_1$. This also implies that

$$P\Big(|T_{q-1}| > \sqrt{\frac{\tilde{R}_1(q-1)}{q-\tilde{R}_1}}\Big) > P\Big(|T_{14}| > \sqrt{\frac{13\tilde{R}_1}{14-\tilde{R}_1}}\Big) \approx 0.0503$$

for all $q \geq 15$. Since, as we discussed before, (A.13) is satisfied for all $z \geq 0$ if $R \geq \tilde{R}$, where $\tilde{R} = (5+\sqrt{5})/2 < \tilde{R}_1 = 3.7$, in view of (A.12), this completes the proof of the theorem for all $q \geq 14$.

Let us now show that (A.13) is satisfied for $5 \leq q \leq 13$ and $R \geq \tilde{\tilde{R}}(q) = \frac{q(5-2\alpha)}{q+1}$, where $\alpha = \tilde{\tilde{\alpha}}(q) = \frac{-2+5q-\sqrt{4+5q^2}}{4q}$.

Note that condition (A.5) implies that $\max\limits_{1 \leq k \leq q} x_k/(x_k+1) \geq R/q$, and, therefore, since $s/(s+1)$ is increasing in $s \geq 0$, $z = \max\limits_{1 \leq k \leq q} z_k \geq R/(q-R) = z_{min}$. Consequently, conditions (A.16) and (A.19) are satisfied if and $R \geq \tilde{\tilde{R}}(q)$.

For $\alpha = 2 - \tilde{\tilde{R}}(q)/2 + 1/(2(1+z))$, condition (A.16) and (A.22) are satisfied for $R \geq \tilde{\tilde{R}}(q)$ and $z > \frac{-\tilde{\tilde{R}}^2(q)+6\tilde{\tilde{R}}(q)-6}{\tilde{\tilde{R}}^2(q)-4\tilde{\tilde{R}}(q)+3} = \tilde{\tilde{z}}(q)$. This implies that (A.13) holds for $R \geq \tilde{\tilde{R}}(q)$ and $z > \tilde{\tilde{z}}(q)$.

In view of the above arguments, for completion of the proof, it remains to check that, for $5 \leq q \leq 13$, (A.26) is satisfied with $z_0 = z_0(q) = \tilde{\tilde{R}}(q)/(q - \tilde{\tilde{R}}(q))$, $z_1 = \tilde{\tilde{z}}(q)$, $\alpha = \tilde{\tilde{\alpha}}(q)$ and $R_0 = \tilde{\tilde{R}}(q)$.

Numerical calculations shows that, indeed, for $5 \leq q \leq 13$, with the above-defined α, z_0, z_1 and R_0,

$$I(q) = \int_0^1 \tilde{f}(s, z_0(q), \tilde{\tilde{R}}(q))\tilde{g}(s, \tilde{\tilde{z}}(q), \tilde{\tilde{R}}(q))ds \geq I(13) > 0. \tag{A.30}$$

We have that $\tilde{\tilde{R}}(5) = 3.19648$, $\tilde{\tilde{R}}(6) = 3.25462$, $\tilde{\tilde{R}}(7) = 3.29873$, $\tilde{\tilde{R}}(8) = 3.33333$, $\tilde{\tilde{R}}(9) = 3.36119$, $\tilde{\tilde{R}}(10) = 3.38409$, $\tilde{\tilde{R}}(11) = 3.40325$, $\tilde{\tilde{R}}(12) = 3.41951$ and $\tilde{\tilde{R}}(13) = 3.43349$.

Numerical calculations show that

$$P\Big(|T_{q-1}| > \sqrt{\frac{R(q)(q-1)}{q-R(q)}}\Big) \geq P\Big(|T_5| > \sqrt{\frac{4R(5)}{5-R(5)}}\Big) \approx 0.056$$

for $5 \leq q \leq 13$. Consequently, the theorem holds for all $q \geq 5$. The proof is complete.

References

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