An Application of Random Matrix Theory: Asymptotic Capacity of Ergodic and Non-ergodic Multiantenna Channels

Antonia M. Tulino Universitá di Napoli   Sergio Verdú Princeton University

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1 Summary

Random matrices have fascinated mathematicians and physicists since they were first introduced in mathematical statistics by Wishart in 1928. The landmark contributions of Wishart, Wigner and later Marčenko-Pastur (in 1967) were motivated to a large extent by practical problems. Nowadays, random matrices find applications in fields as diverse as the Riemann hypothesis, stochastic differential equations, statistical physics, chaotic systems, numerical linear algebra, neural networks, etc.

Currently random matrices are also finding an increasing number of applications in the context of information theory and signal processing which include, among others:

- Wireless communications channels
- Learning and neural networks
- Capacity of ad hoc networks
- Speed of convergence of iterative algorithms for multiuser detection
- Direction of arrival estimation in sensor arrays
- Maximal entropy methods.

The earliest applications to wireless communication were the pioneering works of Foschini [2] and Telatar [3] in the mid-90s on characterizing the capacity of multi-antenna fading channels, which can be modeled as random matrices whose distribution is determined by the fading characteristics.

In the last ten years, a considerable body of work has emerged on the fundamental information-theoretic limits of various wireless communication channels that makes substantial use of random matrix theory.

In the same way that the original contributions of Wishart and Wigner were motivated by their applications, such is also the driving force behind the efforts by information-theoreticians and engineers. The Shannon and the η transforms, introduced for the first time in [5, 6], are prime examples: they characterize the spectrum of a random matrix while providing direct engineering insight [4]. In fact, these transforms were motivated by the intuition drawn from the application of random matrices to various problems in the information theory of noisy communication channels. Specifically, the rational for introducing the Shannon and η transforms is that the Shannon transform is directly of interest as it gives the mutual information of various noisy coherent communication channels, while the η transform yields the performance of linear multiuser detectors [4]. Both these quantities are key measures of performance in communications. Specifically, the mutual information quantifies the data rate per unit of bandwidth that can be conveyed reliably.

In this paper we characterize, using the η and Shannon transforms, the ergodic and non-ergodic capacity of a general class of noisy multi-input multi-output communication channels which are characterized by random matrices that admit various statistical descriptions of interest in wireless communications problems. The ergodic capacity, obtained by averaging the mutual information over the channel fading coefficients, represents the fundamental operational limit in the regime in which a codeword spans many realizations of the fading coefficients.

Results on large random matrices, (e.g. [4]), show that the normalized (to the number of dimensions) mutual information converges almost surely to its expectation as the number of dimensions goes to infinity (with a given ratio of input to output dimensions). Thus, as the number of input and output dimension grows, a self-averaging mechanism hardens the mutual information to its expected value.

In the non-ergodic regime, where the fading variations are not fast enough during the duration of a codeword to ensure that the statistics of the fading coefficients are revealed to the receiver, the outage capacity (cumulative distribution function) of the mutual information is of interest. Using the result in [1], we show the asymptotic normality of the random fluctuations that non-normalized mutual information suffers with respect to the mean and we characterize their mean and variance for arbitrary signal-to-noise ratios. These random fluctuations although small with respect to the mean are of vital interest in the study of outage capacity.

For both the ergodic and non-ergodic regimes, we illustrate the applicability of our results to real-world problems, where the asymptotic behaviors are shown to be excellent approximations of the behavior of actual systems with very modest numbers of antennas.

References

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