High-SNR Power Offset in Multi-Antenna Ricean Channels

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Abstract— **In the high-SNR regime, the multi-antenna mutual information behaves as an affine function of $\mathsf{SNR}|_{\mathrm{dB}}$, described by the *multiplexing gain*, which quantifies the multiplicative increase as function of the number of antennas, and the *power offset* (zero-order term in dB). The conventional high-SNR analysis that considers only the multiplexing gain is unable to assess the impact of channel features such as the Ricean factor since, irrespective thereof, the multiplexing gain equals the minimum of the number of transmit and receive antennas. The impact of the Ricean factor at high SNR can be conveniently quantified through the corresponding power offset, which this paper evaluates in closed-form.**

I. Introduction

In the high-power regime, the single-user multi-antenna mutual information with coherent reception has traditionally been expanded as

$$\begin{aligned}\mathcal{I}(\mathsf{SNR}) &= S_\infty \log_2 \mathsf{SNR} + O(1) \\ &= S_\infty \frac{\mathsf{SNR}|_{\mathrm{dB}}}{3\,\mathrm{dB}} + O(1)\end{aligned} \tag{1}$$

where SNR represents the signal-to-noise ratio whereas S_∞ denotes the *pre-log factor* or *multiplexing gain* in bits/s/Hz/(3 dB). This factor was first evaluated for a canonical channel having IID (independent identically distributed) Rayleigh-faded entries [1], [2], in which case

$$S_\infty = \min(n_{\mathrm{T}}, n_{\mathrm{R}}) \tag{2}$$

where n_{T} and n_{R} denote the number of transmit and receive antennas. As it turns out, (2) is upheld in very broad channel conditions and thus almost all channels of interest have the same S_∞ [3], [4]. The implications of (1) and (2) are that, at high SNR:

- Only $\min(n_{\mathrm{T}}, n_{\mathrm{R}})$ is relevant and thus any increase in $\max(n_{\mathrm{T}}, n_{\mathrm{R}})$ is immaterial.
- The mutual information exhibits no dependence on channel features such as the Ricean factor, antenna correlation, etc.

These observations are misleading at SNR levels of practical interest. This is illustrated by Fig. 1, which depicts the mutual information in a channel with 4 transmit and 4 receive antennas and IID entries, contrasting the distinct cases where the fading distribution is (*i*) Rayleigh, and (*ii*) strongly Ricean. Although, in either case, S_∞=4 bits/s/Hz, there is a sizeable disparity between the respective mutual informations. Hence, the potentially hefty impact of the Ricean factor—as well as that of other channel features—is not revealed by S_∞. Its characterization requires a more refined expansion of the high-SNR mutual information via the affine function

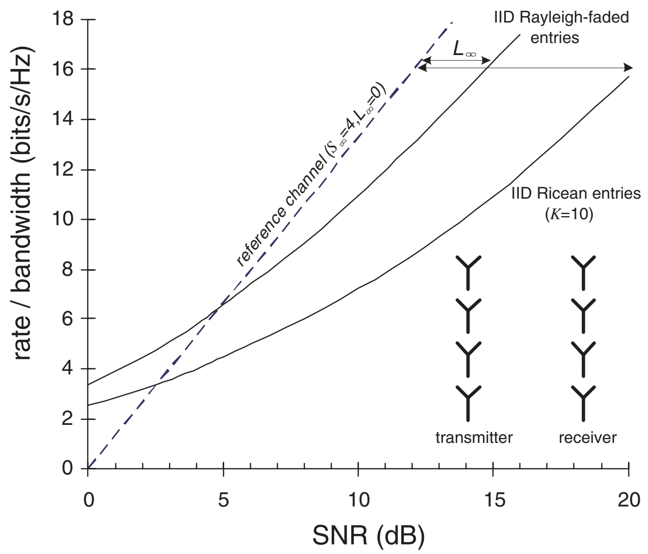

Fig. 1. In solid lines, simulated mutual information achieved by an isotropic input with n_{T}=n_{R}=4 for channels with (*i*) IID Rayleigh-faded entries, and (*ii*) IID Ricean-faded entries where the Ricean factor is 10. Also shown, in dashed, is the affine expansion for the corresponding reference channel (S_∞=4, $\mathcal{L}_\infty$=0 dB) with respect to which the power offset is measured.

$$\mathcal{I}(\mathsf{SNR}) = S_\infty \left(\frac{\mathsf{SNR}|_{\mathrm{dB}}}{3\,\mathrm{dB}} - \mathcal{L}_\infty \right) + o(1) \tag{3}$$

where $\mathcal{L}_\infty$ represents the zero-order term or *power offset*, in 3-dB units, with respect to a reference channel whose affine expansion has the same S_∞ and intersects the origin at SNR=0 dB (see Fig. 1). Although, as SNR→∞, the power offset becomes negligible relative to the leading term in (3), it is highly significant for SNR levels within the realm of practical systems.

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The power offset was first introduced in [5], in the context of CDMA with random spreading.[1] The characterization of the high-SNR mutual information of multi-antenna channels through the power offset was advocated in [7], where closed-form expressions for Rayleigh-faded channels with antenna correlation were given. In this paper, we provide new closed-form expressions for the case of Ricean channels.

II. Definitions and Notation

With frequency-flat fading,[2] the baseband complex model we consider is

$$\mathbf{y} = \sqrt{g}\,\mathbf{H}\,\mathbf{x} + \mathbf{n}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the input and output vectors while $\mathbf{n}$ is AWGN (additive white Gaussian noise). The channel is represented by the $(n_\mathrm{R} \times n_\mathrm{T})$ zero-mean random matrix $\sqrt{g}\,\mathbf{H}$ with the scalar g such that

$$E[\mathrm{Tr}\{\mathbf{H}\mathbf{H}^\dagger\}] = n_\mathrm{R} n_\mathrm{T}.$$

The ergodic mutual information (average rate per unit bandwidth) achieved by an isotropic Gaussian input, in bits/s/Hz, is

$$\mathcal{I}(\mathsf{SNR}) = E\left[\log_2 \det\left(\mathbf{I} + \frac{\mathsf{SNR}}{n_\mathrm{T}}\,\mathbf{H}\mathbf{H}^\dagger\right)\right]$$

with

$$\mathsf{SNR} = g\,\frac{E[\|\mathbf{x}\|^2]}{\frac{1}{n_\mathrm{R}} E[\|\mathbf{n}\|^2]}$$

The multiplexing gain is found as

$$S_\infty = \lim_{\mathsf{SNR}\to\infty} \frac{\mathcal{I}(\mathsf{SNR})}{\log_2 \mathsf{SNR}}$$

while the power offset is given by [5], [7]

$$\mathcal{L}_\infty = \lim_{\mathsf{SNR}\to\infty} \log_2 \mathsf{SNR} - \frac{\mathcal{I}(\mathsf{SNR})}{S_\infty}$$

For notational compactness, we introduce specific variables to indicate the smaller and larger of the numbers of transmit and receive antennas:

$$n_\Downarrow = \min(n_\mathrm{T}, n_\mathrm{R})$$
$$n_\Uparrow = \max(n_\mathrm{T}, n_\mathrm{R})$$

[1]The zero-order counterpart of $\mathcal{L}_\infty$ for incoherent communication is the *fading number* introduced in [6].

[2]If the fading process is frequency selective, the channel can be decomposed into a number of parallel non-interacting subchannels, each experiencing frequency-flat fading and having the same ergodic capacity as the aggregate channel.

III. High-SNR Power Offset

A. IID Rayleigh-faded Channels

Before analyzing the more general Ricean channels, it is instructive to consider the canonical channel with IID Rayleigh-faded entries.

Example 1 *In a scalar channel, S_∞=1 bit/s/Hz/(3 dB). The power offset is 0 dB in the absence of fading while, with Rayleigh fading, the power offset in 3-dB units equals [8]*

$$\mathcal{L}_\infty = \gamma \log_2 e$$

where γ is the Euler-Mascheroni constant

$$\gamma = \lim_{n\to\infty}\left(\sum_{\ell=1}^{n} \frac{1}{\ell} - \log_e n\right) \approx 0.5772$$

Thus, $\mathcal{L}_\infty$ amounts to about 2.5 dB in a scalar Rayleigh-faded channel, where it reflects the cost of fading in terms of power.

The characterization extends straightforwardly to the multi-antenna scenario, for which $S_\infty = n_\Downarrow$ [1] and the power offset has been implicitly given by a number of authors [9], [10], [11].

Example 2 *Consider a channel whose entries are IID Rayleigh-faded. The high-SNR power offset, in 3-dB units, is*

$$\mathcal{L}_\infty^{\mathsf{iid}} = \log_2 n_\mathrm{T} + \left(\gamma - \sum_{\ell=1}^{n_\Uparrow - n_\Downarrow} \frac{1}{\ell} - \frac{n_\Uparrow}{n_\Downarrow} \sum_{\ell=n_\Uparrow - n_\Downarrow + 1}^{n_\Uparrow} \frac{1}{\ell} + 1\right) \log_2 e$$

which, for n_T=n_R=n, reduces to

$$\mathcal{L}_\infty^{\mathsf{iid}} = \log_2 n + \left(\gamma - \sum_{\ell=2}^{n} \frac{1}{\ell}\right) \log_2 e$$

Using the expressions in this example, it is simple to unveil the role played by $n_\Uparrow$ on this canonical channel. To this end, in the ensuing observations we write $\mathcal{L}_\infty^{\mathsf{iid}}(n_\mathrm{T}, n_\mathrm{R})$ explicitly highlighting the dependence of the power offset on both n_T and n_R.

- Let n_T=n_R=n. Adding δ receive antennas, while not altering S_∞, would reduce $\mathcal{L}_\infty$ as

$$\mathcal{L}_\infty^{\mathsf{iid}}(n, n+\delta) = \mathcal{L}_\infty^{\mathsf{iid}}(n, n) - \left(\sum_{\ell=n+1}^{n+\delta} \frac{1}{\ell} + \frac{\delta}{n} \sum_{\ell=\delta+1}^{n+\delta} \frac{1}{\ell}\right) \log_2 e$$

  yielding a steady improvement with growing n_R because of the additional power captured by every new receive antenna.[3] Once $n_\mathrm{R} \gg n_\mathrm{T}$, the gain approaches 3 dB for every doubling of n_R.

[3]Note that, since this capacity growth is on a per-transmit-antenna basis, realizing this steady improvement demands increasingly larger signalling constellations.

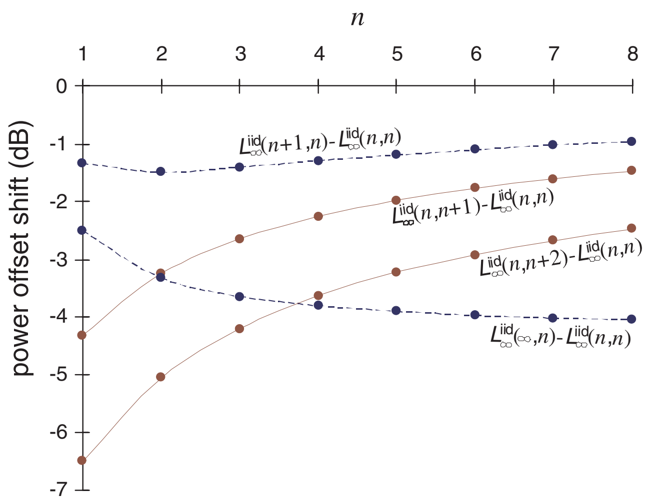

Fig. 2. High-SNR power offset reduction, in dB, obtained by adding either (*i*) one antenna to the transmitter, (*ii*) one antenna to the receiver, (*iii*) two antennas to the receiver, or (*iv*) by letting the number of transmit antennas grow without bound.

- Let $n_{\rm T}{=}n_{\rm R}{=}n$. Adding δ transmit antennas, while not altering $\mathcal{S}_\infty$, would reduce $\mathcal{L}_\infty$ as

$$\begin{aligned}\mathcal{L}^{\text{iid}}_\infty(n+\delta,n) &= \mathcal{L}^{\text{iid}}_\infty(n,n) + \log_2 \tfrac{n+\delta}{n} \\ &\quad - \left(\sum_{\ell=n+1}^{n+\delta} \tfrac{1}{\ell} + \tfrac{\delta}{n} \sum_{\ell=\delta+1}^{n+\delta} \tfrac{1}{\ell} \right) \log_2 e\end{aligned}$$

  This improvement cannot exceed 1.5 dB for $\delta{=}1$ and 4.3 dB for any δ.

These observations are exemplified in Fig. 2.

B. Ricean Channels

We now turn our attention to nonzero-mean channels of the kind

$$\mathbf{H} = \sqrt{\tfrac{K}{K+1}}\bar{\mathbf{H}} + \sqrt{\tfrac{1}{K+1}}\mathbf{W}$$

where $\bar{\mathbf{H}}$ is deterministic and arbitrary, normalized such that $\text{Tr}\{\bar{\mathbf{H}}\bar{\mathbf{H}}^\dagger\}{=}n_{\rm R}n_{\rm T}$, while the entries of $\mathbf{W}$ are zero-mean unit-variance IID complex Gaussian and $K{>}0$ is the Ricean factor between the unfaded (deterministic) and fading (random) components.

It is easily verified that the multiplexing gain is $\mathcal{S}_\infty{=}n_\Downarrow$. The power offset, in turn, is characterized by the following result.

Proposition 1 *Consider a Ricean channel with factor K. Denoting by ϕ_j, $j \in (1,\ldots,n_\Downarrow)$, the eigenvalues of $K\bar{\mathbf{H}}\bar{\mathbf{H}}^\dagger$, the high-SNR power offset in 3-dB units is*

$$\mathcal{L}_\infty = \mathcal{L}^{\text{iid}}_\infty + \log_2(K+1) - \frac{\log_2 e}{n_\Downarrow} \frac{\sum_{i=1}^{n_\Downarrow} \det \boldsymbol{\Xi}_i}{\prod_{i<j}(\phi_i - \phi_j)}$$

where $\boldsymbol{\Xi}_i$, $i \in \{1,\ldots,n_\Downarrow\}$, is an $(n_\Downarrow \times n_\Downarrow)$ matrix whose (k,ℓ)-th entry is

$$(\boldsymbol{\Xi}_i)_{k,\ell} = \begin{cases} \phi_k^{n_\Downarrow-\ell} & \ell \neq i \\ \phi_k^{n_\Downarrow-\ell}\, \digamma(k,\ell) & \ell = i \end{cases}$$

with

$$\begin{aligned}\digamma(k,\ell) &= \log_e \phi_k + \text{E}_1(\phi_k) + \gamma - \sum_{q=1}^{n_\Uparrow-\ell} \tfrac{1}{q} \\ &\quad + \sum_{q=1}^{n_\Uparrow-\ell} (-\phi_k)^{-q} \left(e^{-\phi_k}(q-1)! - \tfrac{(n_\Uparrow-\ell)!}{q\,(n_\Uparrow-\ell-q)!} \right)\end{aligned}$$

where $\text{E}_1(\cdot)$ is the exponential integral

$$\text{E}_1(x) = \int_1^\infty \frac{e^{-x\,\xi}}{\xi} d\xi.$$

Proof: *See [12].*

The change in power offset induced by the Ricean component, $\mathcal{L}_\infty{-}\mathcal{L}^{\text{iid}}_\infty$, can be either positive or negative depending on the nature of such component and the numbers of antennas.

Example 3 *For a scalar Ricean channel with $n_{\rm T}{=}n_{\rm R}{=}1$,*

$$\mathcal{L}_\infty = \log_2 \tfrac{K+1}{K} - \text{E}_1(K)\log_2 e$$

which satisfies $0{\le}\mathcal{L}_\infty{\le}\mathcal{L}^{\text{iid}}_\infty$. This expression was already given in [6].

Example 4 *For $n_{\rm T}{=}n_{\rm R}{=}2$,*

$$\begin{aligned}\mathcal{L}_\infty &= 1 + \log_2 \frac{K+1}{\sqrt{\phi_1\phi_2}} \\ &\quad - \frac{\log_2 e}{2}\left(\text{E}_1(\phi_1) + \text{E}_1(\phi_2) - \frac{e^{-\phi_2} - e^{-\phi_1}}{\phi_2 - \phi_1} \right)\end{aligned}$$

Often, the unfaded component is associated with a line-of-sight or a diffracted wave and thus it is essentially unit-rank, i.e, $\bar{\mathbf{H}}{=}\mathbf{a}_{\rm R}\mathbf{a}_{\rm T}^\dagger$ where the vectors $\mathbf{a}_{\rm T}$ and $\mathbf{a}_{\rm R}$ are the transmit and receive array responses to a plane wave.

Corollary 1 *For a Ricean channel with factor K and unit-rank unfaded component,*

$$\begin{aligned}\mathcal{L}_\infty &= \mathcal{L}^{\text{iid}}_\infty + \log_2 \frac{K+1}{(Kn_\Downarrow n_\Uparrow)^{1/n_\Downarrow}} \\ &\quad - \frac{\log_2 e}{n_\Downarrow} \left(\gamma - \sum_{\ell=1}^{n_\Uparrow-1} \tfrac{1}{\ell} + \text{E}_1(Kn_\Downarrow n_\Uparrow) \right. \\ &\quad \left. + \sum_{\ell=1}^{n_\Uparrow-1} \frac{e^{-Kn_\Downarrow n_\Uparrow}(\ell-1)! - \frac{(n_\Uparrow-1)!}{\ell\,(n_\Uparrow-1-\ell)!}}{(-Kn_\Downarrow n_\Uparrow)^\ell} \right)\end{aligned}$$

Example 5 *For a Ricean channel with $n_{\mathrm{T}}{=}n_{\mathrm{R}}{=}2$ and unit-rank unfaded component, Corollary 1 yields*

$$\mathcal{L}_\infty = 1{+}\log_2\left(\frac{K+1}{2\sqrt{K}}\right){-}\frac{\log_2 e}{2}\left(\mathrm{E}_1(4K) - \gamma - \frac{e^{-4K}-1}{4K}\right)$$

which satisfies $\mathcal{L}_\infty{\geq}\mathcal{L}_\infty^{\mathsf{iid}}$.

While Example 3 dictates that the presence of a unit-rank unfaded component can only reduce the power offset (and hence increase the high-SNR mutual information) on a scalar channel, Example 5 shows that this is no longer the case in a matrix channel. In fact, the behavior of Example 5 extends to any unit-rank Ricean channel with $n_\Downarrow{\geq}2$ [12]. It is worth contrasting this finding with the result put forth in [13] for arbitrary SNR, namely, that if the power received over the fading portion of the channel is held constant, then an unfaded component can only increase the mutual information. If the total received power is held constant, then this is not the case (at least at high SNR). The impact of Ricean components at high SNR is thus seen to depend critically on the coupling between the faded and unfaded components in terms of power gain.

As the number of antennas grows, the excess power offset caused by a unit-rank unfaded component adopts a remarkably simple form:

$$\lim_{n_{\mathrm{T}},n_{\mathrm{R}}\to\infty} \mathcal{L}_\infty - \mathcal{L}_\infty^{\mathsf{iid}} = \log_2(K+1)$$

indicating that, for sufficiently many antennas, the presence of a Ricean component has a deleterious effect at high SNR. Specifically, the mutual information in a channel whose unfaded component is unit-rank behaves, as the number of antennas grows large, as if only the fading portion of the channel were present. This behavior is a direct manifestation of the fact that only a single eigenvalue of $\mathbf{H}\mathbf{H}^\dagger$ is perturbed by the presence of the deterministic component. For $n_{\mathrm{T}}, n_{\mathrm{R}}{\to}\infty$, this perturbation is not reflected in the empirical eigenvalue distribution of $\mathbf{H}\mathbf{H}^\dagger$, which determines $\mathcal{L}_\infty$. Moreover, this is the case even if the rank of $\bar{\mathbf{H}}\bar{\mathbf{H}}^\dagger$ is $r{>}1$ as long as [14], [15]

$$\lim_{n_{\mathrm{T}},n_{\mathrm{R}}\to\infty} \frac{r}{n_\Downarrow} = 0.$$

IV. Conclusions

The mutual information dictates the interplay between power, bandwidth and rate. With transmit power P and bandwidth B, the rate R than can be conveyed with arbitrary reliability must satisfy $R/B{\leq}\mathcal{I}$ where, in the high-SNR regime,

$$\mathcal{I} \approx S_\infty\left(\frac{g\frac{P}{BN_0}|_{\mathrm{dB}}}{3\,\mathrm{dB}} - \mathcal{L}_\infty\right)$$

with N_0 the noise spectral density per receive antenna and with g the average channel gain. Given a desired rate and available bandwidth, the minimum required transmit power is therefore

$$P|_{\mathrm{dB}} \approx \frac{BN_0}{g}|_{\mathrm{dB}} + \left(\frac{R}{BS_\infty} + \mathcal{L}_\infty\right) 3\,\mathrm{dB} \tag{4}$$

where $P|_{\mathrm{dB}}$ has the same reference as $(BN_0)|_{\mathrm{dB}}$ and the approximation sharpens as SNR$\to\infty$. Since almost all channels of interest have the same S_∞, the power required to sustain a certain rate R with a given bandwidth B is essentially defined, at high R/B, by $\mathcal{L}_\infty$. In this paper, we have presented closed-form expressions for $\mathcal{L}_\infty$ from which the impact of the Ricean factor can be assessed. These expressions complement those put forth in [7], [12] addressing other various channel and input features.

To conclude, it is worth remarking that the expressions presented have direct operational significance in ergodic channel conditions. In non-ergodic channels, the power offset becomes a random quantity whose expected value is precisely given by these expressions. Although characterizing the outage distribution of such random quantity appears to be analytically intractable, recent results strongly suggest that this distribution becomes rapidly Gaussian as the number of antennas grows large [16, Sec. 3.3.8]. Thus, the expressions given herein, complemented by those for the variance of the limiting Gaussian distribution, would suffice to provide accurate evaluations thereof.

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