Mercury/Waterfilling for Fixed Wireless OFDM Systems

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***Abstract*— Fixed wireless systems are designed to provide broadband connectivity to residential and enterprise customers. Since they typically operate in frequency-selective low-Doppler conditions, a preferred signalling choice is OFDM (orthogonal frequency-division multiplexing). In contrast with mobile systems, where the channel variations are often too rapid to be followed by the transmitter, in fixed wireless such variations can be accurately tracked. This provides the added flexibility of allocating power over the OFDM tones. For ideal Gaussian signals, the allocation of power over parallel channels is solved by the classical waterfilling policy. For the discrete constellations used in practice, however, waterfilling is no longer optimal. Rather, the power allocation that maximizes the mutual information is then given by the more general mercury/waterfilling policy. This paper illustrates the usance of mercury/waterfilling on frequency-selective OFDM channels with QAM constellations and it quantifies the extend to which conventional waterfilling curtails the mutual information.**

I. Introduction

An intuitive approach to the problem of communicating over a linear frequency-selective channel is to decompose it into a number of parallel non-interacting subchannels, each sufficiently narrowband so as to have an approximately flat frequency response. This approach naturally leads to multitone techniques such OFDM (orthogonal frequency-division multiplexing) [1]. A fitting application for OFDM is that of fixed wireless access to residential and enterprise customers, an application where frequency-selective channel conditions are usually encountered. In contrast with mobile systems, which may be subject to high Doppler rates, in fixed wireless communication the channel variations are slow and can be accurately tracked by both transmitter and receiver. This enables the added flexibility of allocating power and rate over the OFDM tones.

Since the largest spectral efficiency achievable with arbitrary reliability is given by the mutual information, an enticing optimality criterion to allocate power is precisely the maximization of the mutual information. For the capacity-achieving Gaussian signals, such optimum allocation is solved by the classical *waterfilling* policy [2][3, Sec. 10.4]. In practice, however, the ideal Gaussian signals must be forsaken in favor of discrete constellations for which the waterfilling policy ceases to be optimal. Rather, the power allocation that maximizes the mutual information is then given by the more general *mercury/waterfilling* policy presented in [4].

This paper illustrates the use of mercury/waterfilling in an exemplary frequency-selective OFDM channel with QAM (quadrature amplitude modulation) constellations. The objective is twofold:

- To cast insight on how the non-ideal nature of the constellations impacts the process of optimally allocating power.
- To quantify the extent to which conventional waterfilling, applied in conjunction with QAM constellations, is suboptimal when the mutual information is the driving performance measure.

Note that, in contrast with power allocation solutions tailored to specific coding schemes, the information-theoretic problem of allocating power in order to maximize the mutual information is technology-nonspecific. (In fact, it furnishes the power allocation to which any sound code will tend as its blocklength grows.)

II. Definitions

Given n orthogonal tones, the input-output relationship on the ith tone is

$$Y_i = h_i \sqrt{p_i}\, S_i + W_i \tag{1}$$

where $\{h_i\}_{i=1}^n$ are complex scalar gains that do not change appreciably during the span of a codeword. These gains are known by both receiver and transmitter.[1] Each noise W_i is a zero-mean σ_i^2-variance complex Gaussian random variable independent of the noise on the other channels. The complex input signals $\{S_i\}_{i=1}^n$ are also mutually independent and unit-variance, such that the transmission powers are given by $\{p_i\}_{i=1}^n$ with average

$$\frac{1}{n}\sum_{i=1}^{n} p_i = P. \tag{2}$$

For convenience, we define

$$\gamma_i = \frac{|h_i|^2}{\sigma_i^2} \tag{3}$$

such that $p_i \gamma_i$ is the receive signal-to-noise ratio on the ith tone. The average receive signal-to-noise ratio thus depends

[1] The transmitter, more precisely, need only know the amplitudes $\{|h_i|\}_{i=1}^n$.

on the power allocation. For a uniform allocation, it equals

$$\mathsf{SNR} = \frac{P}{n}\sum_{i=1}^{n}\gamma_i. \tag{4}$$

We shall let the input signals conform to a square m-QAM constellation (equal on all tones) having $m = 2^b$ discrete points with $b = 2, 4$ or 6. Moreover:

- For a given cardinality m, we denote the composing discrete points by $\{s_\ell\}_{\ell=1}^{m}$.
- Square QAM constellations are quadrature symmetric. The real and imaginary parts of $\{s_\ell\}_{\ell=1}^{m}$ take the values,

$$\sqrt{\frac{1.5}{m-1}}\,(2k-1-\sqrt{m}) \qquad k = 1,\ldots,\sqrt{m} \tag{5}$$

- The cardinality is not to exceed 64-QAM.

Denote by $\mathcal{I}_i(\rho) = I(S_i; \sqrt{\rho}S_i + W_i)$ the input-output mutual information on the ith tone. With a square m-QAM input, the function $\mathcal{I}(\cdot)$ on a generic tone can be expressed as [3]

$$\mathcal{I}(\rho) = -2\log(\pi e) - 2\int f(y,\rho)\log\left(f(y,\rho)\right)\,dy \tag{6}$$

where the integration is over the complex plane and the base of the logarithms determines the information units while

$$f(y,\rho) = \frac{1}{\sqrt{m}\,\pi}\sum_{\ell=1}^{\sqrt{m}} e^{-|y-\sqrt{\rho/2}\,s_\ell|^2}. \tag{7}$$

For QPSK, 16-QAM and 64-QAM the function $\mathcal{I}(\cdot)$ is plotted in Fig. 1. Also shown, for reference, is the mutual information achieved by the ideal unit-variance Gaussian signal. For later use, we tabulate $\mathcal{I}(\cdot)$ for each of these constellations.

The mercury/waterfilling power allocation policy rests on the differential relationship between mutual information and MMSE (minimum mean-square error) in Gaussian-noise channels [5]. A basic element on its formulation is thus the function

$$\mathsf{MMSE}_i(\rho) = E\left[\left|S_i - \hat{S}_i(\sqrt{\rho}S_i + W_i, \rho)\right|^2\right] \tag{8}$$

which returns the MMSE incurred in the estimation of the symbol S_i from its noisy observation $\sqrt{\rho}S_i + W_i$ via the (in general nonlinear) MMSE estimator [6]

$$\hat{S}_i(y,\rho) = E\left[S_i | \sqrt{\rho}S_i + W_i = y\right]. \tag{9}$$

For a generic tone loaded with a square m-QAM constellation, (8) and (9) lead to

$$\mathsf{MMSE}(\rho) = 1 - \frac{1}{m\pi}\int \frac{1}{f(y,\rho)}\left|\sum_{\ell=1}^{\sqrt{m}} s_\ell\, e^{-|y-\sqrt{\rho/2}\,s_\ell|^2}\right|^2 dy \tag{10}$$

with $f(\cdot,\cdot)$ given in (7) and with the integration extending to the complex field. Since the inputs are unit power, $\mathsf{MMSE}(\cdot) \in (0,1]$. The inverse of $\mathsf{MMSE}(\cdot)$ with respect to the composition of functions is denoted by $\mathsf{MMSE}^{-1}(\cdot) \in [0,\infty)$. Again for later use, we tabulate this inverse function for each of the constellations to be considered (QPSK, 16-QAM and 64-QAM).

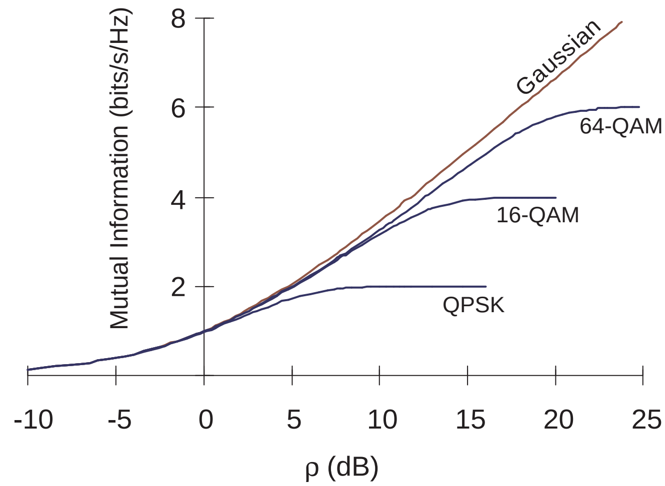

Fig. 1. $\mathcal{I}(\rho)$ as function of ρ in dB for QPSK, 16-QAM, 64-QAM and Gaussian input distributions.

III. Mercury/waterfilling

A. Description and Implementation

Denote by $\{p_i^\star\}_{i=1}^{n}$ the power allocation that either maximizes the mutual information $\frac{1}{n}\sum_{i=1}^{n}\mathcal{I}_i(p_i\gamma_i)$ given a constraint on the power P or, equivalently, that minimizes the power given a constraint on the mutual information. The mercury/waterfilling policy states that [4]

$$p_i^\star = 0 \qquad \gamma_i < \eta \tag{11}$$

$$p_i^\star = \frac{1}{\gamma_i}\mathsf{MMSE}_i^{-1}\left(\frac{\eta}{\gamma_i}\right) \qquad \gamma_i \geq \eta \tag{12}$$

with η such that the applicable constraint (either apower or mutual information) is satisfied.

Henceforth, this policy is implemented using the tabulated functions $\mathcal{I}(\cdot)$ and $\mathsf{MMSE}^{-1}(\cdot)$ as follows:

(a) Obtain η by enforcing the appropriate constraint. This entails solving a single nonlinear equation, either

$$\sum_{\substack{i=1\\ \gamma_i>\eta}}^{n}\frac{1}{n\gamma_i}\mathsf{MMSE}_i^{-1}\left(\frac{\eta}{\gamma_i}\right) = P^\star \tag{13}$$

if the power is constrained to equal $P^\star$, or

$$\frac{1}{n}\sum_{\substack{i=1\\ \gamma_i>\eta}}^{n}\mathcal{I}_i\left(\mathsf{MMSE}_i^{-1}\left(\frac{\eta}{\gamma_i}\right)\right) = \mathcal{I}^\star \tag{14}$$

if the mutual information is constrained to equal $\mathcal{I}^\star$.

(b) Use η to identify $\{p_i^\star\}_{i=1}^{n}$ via (11)–(12).

Some limiting behaviors of the mercury/waterfilling policy for square m-QAM constellations are [4]:

- In the low-power regime (i.e., for $P^\star \to 0$ or $\mathcal{I}^\star \to 0$), mercury/waterfilling reverts to conventional waterfilling.

- In the high-power regime (i.e., for $P^\star \to \infty$ or $\mathcal{I}^\star \to \log m$), mercury/waterfilling behaves as

$$p^\star \approx \frac{\alpha}{\gamma_i} \tag{15}$$

where α is a constant ensuring that the applicable constraint is satisfied. In stark constrast with classical waterfilling, which tends to equalize the *transmit* powers, this channel-inverting behavior equalizes the *received* signal-to-noise ratios on all of the high-power tones.

The cardinality of the constellation critically determines when the high-power behavior comes into play. The smaller the constellation, the less power needed to bring it about.

B. Graphical Interpretation

Mercury/waterfilling, like its special case of waterfilling, is a parametric procedure that owes its name to a graphical interpretation. With the aid of the auxiliary function

$$G_i(\zeta) = \begin{cases} 1/\zeta - \mathsf{MMSE}_i^{-1}(\zeta) & \zeta \in (0,1] \\ 1 & \zeta > 1 \end{cases} \tag{16}$$

such interpretation is as follows.

(a) For each of the n tones, set up a unit-base vessel solid up to a height $1/\gamma_i$.
(b) Compute η. Pour mercury onto each of the vessels until its height (including the solid) reaches $G_i(\eta/\gamma_i)/\gamma_i$.
(c) Waterfill, keeping identical upper level of water in all vessels, with a volume of water equal to n (or, equivalently, until the water level reaches $1/\eta$).
(d) The water height over the mercury on the ith vessel gives $p_i^\star$.

C. Waterfilling as a Special Case

What distinguishes mercury/waterfilling from conventional waterfilling is the mercury pouring stage, which regulates the water admitted by each vessel thereby tailoring the process to arbitrary signal distributions. Thus, skipping step (b) directly turns mercury/waterfilling into waterfilling. Indeed, for a Gaussian signal

$$\mathsf{MMSE}^{-1}(\zeta) = \frac{1}{\zeta} - 1 \tag{17}$$

which results in $G_i(\cdot) = 1$ and thus in no poured mercury. Correspondingly, plugging (17) into (11) and (12) we obtain, as a special case of mercury/waterfilling, the conventional waterfilling policy

$$p_i^{\mathsf{WF}} = 0 \qquad \gamma_i < \eta \tag{18}$$

$$p_i^{\mathsf{WF}} = \frac{1}{\eta} - \frac{1}{\gamma_i} \qquad \gamma_i \geq \eta \tag{19}$$

with the conditions determining η obtained by also plugging (17) into either (13) or (14). Well established limiting behaviors of waterfilling are:

- In the low-power regime, only the tone(s) with the highest γ_i are allocated power.
- In the high-power regime, the power allocation becomes uniform.

IV. An Exercise in Mercury/Waterfilling: Frequency-Selective OFDM Channel

The frequency response of the exemplary channel we shall use, sampled into $n = 128$ orthogonal tones, is portrayed in Fig. 2. This response was obtained by realizing a Rayleigh fading process with delay spread (in μs) equal to the reciprocal of the signal bandwidth (in MHz). This corresponds, for example, to:

- A 1.25-MHz system subject to a 0.8-μs spread.
- A 5-MHz system exposed to a 0.2-μs spread.

On this channel, for which (4) yields[2]

$$\mathsf{SNR}|_{\mathrm{dB}} = P|_{\mathrm{dB}} - 7.32 \text{ dB}, \tag{20}$$

an evaluation of the relative performances of mercury/waterfilling and waterfilling can be conducted. Specifically, given an average power $P^\star$:

(a) Apply mercury/waterfilling via (11)–(13) and compute the resulting mutual information

$$\mathcal{I}^\star = \frac{1}{n}\sum_{i=1}^{n} \mathcal{I}_i(p_i^\star \gamma_i). \tag{21}$$

(b) Using (14) and (17)–(19), calculate the corresponding P^{WF} required for waterfilling to achieve this same mutual information.
(c) $(P^{\mathsf{WF}}/P^\star)|_{\mathrm{dB}}$ is the excess power required by waterfilling over mercury/waterfilling at this operating point.

For a wide range of operating points, the excess power is depicted in the bottom chart of Fig. 3, parameterized by the constellation. The top chart, in turn, shows the corresponding values for $\mathcal{I}^\star$. Altogether, the figure prompts some immediate observations:

- The smaller the constellation, the uniformly higher the excess power. This reflects the fact that constellations with smaller cardinality are a further departure from the Gaussian signal idealization to which conventional waterfilling is tailored.
- Regardless of the constellation, the excess power vanishes for $P^\star \to 0$, consistent with the converged behaviors of mercury/waterfilling and waterfilling in this regime.
- As $\mathcal{I}^\star$ approaches the maximum value that can be supported by each constellation, i.e., for $\mathcal{I}^\star \to \log m$, excess powers of several dB are observed. The excess powers at the representative operating points where $\mathcal{I}^\star$ equals $1/2$, $3/4$ and $9/10$ its maximum value are summarized in Table I.

V. Conclusions

Using an exemplary channel, we have applied the mercury/waterfilling power allocation policy to the problem of OFDM transmission with discrete QAM constellations. In contrast with traditional waterfilling, which is optimum only for ideal Gaussian signals, mercury/waterfilling adjusts to

[2] $x|_{\mathrm{dB}} = 10\log_{10} x$.

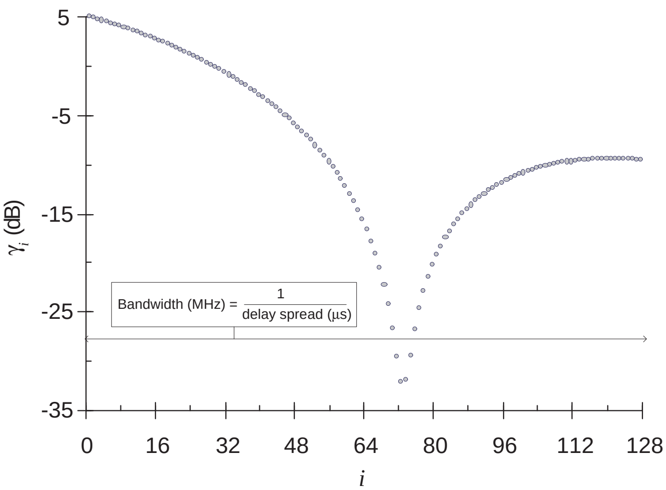

Fig. 2. Wireless channel frequency response sampled into $n = 128$ orthogonal tones and scaled by the noise variance.

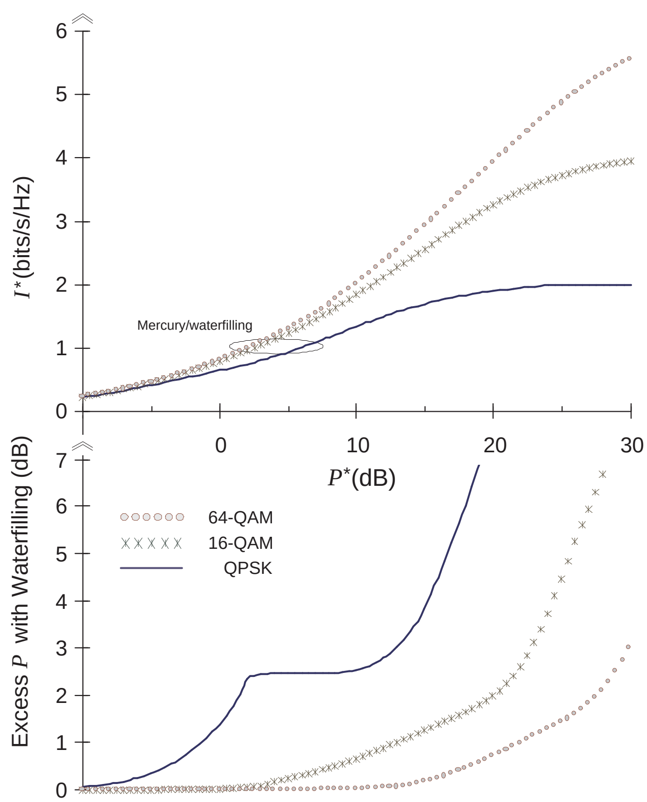

Fig. 3. Top chart: mutual information (bits/s/Hz) achieved by mercury/waterfilling as function of $P^{\star}$, parameterized by the constellation. Bottom chart: additional power required by waterfilling to achieve the same mutual information with the same constellation.

TABLE I
EXCESS POWER REQUIRED BY WATERFILLING AT REPRESENTATIVE OPERATING POINTS.

Constellation	$\mathcal{I}^{\star} = \frac{1}{2}\log m$	$\mathcal{I}^{\star} = \frac{3}{4}\log m$	$\mathcal{I}^{\star} = \frac{9}{10}\log m$
QPSK	2.5 dB	3 dB	4.4 dB
16-QAM	0.8 dB	1.6 dB	3.7 dB
64-QAM	0.2 dB	1.1 dB	2.5 dB

whichever signalling distributions are being utilized. This results in:

- Qualitatively dissimilar behaviors in those regimes where a constellation departs, in terms of mutual information, from the Gaussian idealization. In the case of QAM constellations, this departure takes place once the power has exceeded a certain level (which depends on the cardinality of the constellation).
- Quantitative differences in the amount of power required to achieve a certain mutual information. We have reported the excess power demanded by waterfilling to achieve any specific mutual information.

The performed evaluation sheds light on the extent to which waterfilling curtails the mutual information when applied with discrete constellations. Specifically, its suboptimality is seen to depend critically on the interplay of:

- The dynamic range spanned by $\{\gamma_i\}_{i=1}^n$. For a given constellation and operating point, a larger dynamic range is likely to heighten the suboptimality.
- The cardinality of the constellation. On a given channel, a richer constellation stretches the range of operating points on which the waterfilling is effective.
- The operating point. Regardless of the dynamic range and constellation cardinality, waterfilling is near-optimal if the operating point is sufficiently low. Conversely, it incurs substantial excess powers at operating points that are sufficiently close to the maximum achievable mutual information.

REFERENCES

[1] J. A. C. Bingham, "Multicarrier modulation for data transmission: An idea whose time has come," *IEEE Commun. Magazine*, vol. 28, no. 5, pp. 5–14, May 1990.

[2] C. E. Shannon, "Communication in the presence of noise," *Proc. IRE*, vol. 37, pp. 10–21, Jan. 1949.

[3] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, New York, Wiley, 1990.

[4] A. Lozano, A. M. Tulino, and S. Verdú, "Mercury/waterfilling: Optimum power allocation with arbitrary input constellations," *Proc. of IEEE Intern. Symp. on Inform. Theory (ISIT'05)*, Sept. 2005.

[5] D. Guo, S. Shamai, and S. Verdú, "Mutual information and minimum mean-square error in Gaussian channels," *IEEE Trans. on Inform. Theory*, vol. 51, no. 4, pp. 1261–1283, Apr. 2005.

[6] H. V. Poor, *An Introduction to Signal Detection and Estimation*, Springer-Verlag, New York, 1994.