Problem 3.36 (Sharon Betz)

There are five possible relationships between $A_1$ and $A_2$:

| **Case** | **Probability** | $\boldsymbol{\eta_1^c}$ |
|---|---|---|
| $A_1 = A_2$ | $\frac{3}{8}$ | $(1-\lvert\rho\rvert)^2$ |
| $A_1 = \alpha A_2$ | $\frac{1}{4}$ | $(1-\alpha^{-1}\lvert\rho\rvert)^2 \quad \lvert\rho\rvert < \alpha$<br>$0 \quad else$ |
| $A_1 = \alpha^{-1} A_2$ | $\frac{1}{4}$ | $(1-\alpha\lvert\rho\rvert)^2 \quad \lvert\rho\rvert < \alpha^{-1}$<br>$0 \quad else$ |
| $A_1 = \alpha^2 A_2$ | $\frac{1}{16}$ | $(1-\alpha^{-2}\lvert\rho\rvert)^2 \quad \lvert\rho\rvert < \alpha^2$<br>$0 \quad else$ |
| $A_1 = \alpha^{-2} A_2$ | $\frac{1}{16}$ | $(1-\alpha^{2}\lvert\rho\rvert)^2 \quad \lvert\rho\rvert < \alpha^{-2}$<br>$0 \quad else$ |

Note that:

$$0 \le \alpha \le |\rho| \;\; \Rightarrow \;\; \begin{cases} \alpha^{-2} & > \; |\rho| \\ \alpha^{-1} & > \; |\rho| \\ \alpha^{2} & < \; |\rho| \end{cases} \tag{1}$$

$$|\rho| < \alpha^2 \;\; \Rightarrow \;\; \begin{cases} \alpha^{-2} & > \; |\rho| \\ \alpha^{-1} & > \; |\rho| \\ \alpha & > \; |\rho| \end{cases} \tag{2}$$

$$\alpha^2 \le |\rho| < \alpha < 1 \;\; \Rightarrow \;\; \begin{cases} \alpha^{-2} & > \; |\rho| \\ \alpha^{-1} & > \; |\rho| \end{cases} \tag{3}$$

$$0 \le \alpha^{-1} \le |\rho| \;\; \Rightarrow \;\; \begin{cases} \alpha^{2} & > \; |\rho| \\ \alpha & > \; |\rho| \\ \alpha^{-2} & < \; |\rho| \end{cases} \tag{4}$$

$$\alpha^{-2} \le |\rho| < \alpha^{-1} < 1 \;\; \Rightarrow \;\; \begin{cases} \alpha^{2} & > \; |\rho| \\ \alpha & > \; |\rho| \end{cases} \tag{5}$$

Thus,

$$\begin{aligned} \overline{\eta}_1^c \;\; = \;\; & \frac{1}{16}\big(6(1-|\rho|)^2 + 4(1-\alpha^{-1}|\rho|)^2 1_{\alpha>|\rho|} + 4(1-\alpha|\rho|)^2 1_{\alpha^{-1}>|\rho|} \\ & +(1-\alpha^{-2}|\rho|)^2 1_{\alpha^2>|\rho|} + (1-\alpha^2|\rho|)^2 1_{\alpha^{-2}>|\rho|}\big) \end{aligned} \tag{6}$$

$$
= \begin{cases} \frac{1}{16} & \big(11 - 2(6 + 4\alpha + \alpha^2)|\rho| \\ & +(6 + 4\alpha^2 + \alpha^4)|\rho|^2\big) & 0 \leq \alpha \leq |\rho|| \\ \frac{1}{16} & \big(16 - 2(\alpha^{-2} + 4\alpha^{-1} + 6 + 4\alpha + \alpha^2)|\rho| \\ & +(\alpha^{-4} + 4\alpha^{-2} + 6 + 4\alpha^2 + \alpha^4)|\rho|^2\big) & |\rho| < \alpha^2 \\ \frac{1}{16} & \big(15 - 2(4\alpha^{-1} + 6 + 4\alpha + \alpha^2)|\rho| \\ & +(4\alpha^{-2} + 6 + 4\alpha^2 + \alpha^4)|\rho|^2\big) & \alpha^2 \leq |\rho| \leq \alpha < 1 \\ \frac{1}{16} & \big(11 - 2(6 + 4\alpha^{-1} + \alpha^{-2})|\rho| \\ & +(6 + 4\alpha^{-2} + \alpha^{-4})|\rho|^2\big) & 0 \leq \alpha^{-1} \leq |\rho|| \\ \frac{1}{16} & \big(15 - 2(4\alpha + 6 + 4\alpha^{-1} + \alpha^{-2})|\rho| \\ & +(4\alpha^2 + 6 + 4\alpha^{-2} + \alpha^{-4})|\rho|^2\big) & \alpha^{-2} \leq |\rho| \leq \alpha^{-1} < 1 \end{cases} \tag{7}
$$