Problem 3.40 (Solution by Dongning Guo, June 20, 2001)

Let $X_K = \sum_{j=2}^{K} \rho_{1j}$. I show that $X_K$ is asymptotically Gaussian in distribution and then the result follows.

For every $K$, $\{\rho_{12}, \cdots, \rho_{1K}\}$ is a set of zero-mean independent random variables. The variance of $X_K$ is

$$E \left[ X_K^2 \right] = E \left[ \sum_{j=2}^{K} \rho_{1j}^2 \right]$$

(1)

$$= (K - 1) \cdot E \left[ \rho_{12}^2 \right]$$

(2)

$$= (K - 1) \cdot E \left[ \left( \frac{1}{N} \sum_{n=1}^{N} s_{n1}s_{n2} \right)^2 \right]$$

(3)

$$= (K - 1) \cdot E \left[ \left( \frac{1}{N^2} \sum_{n_1=1}^{N} \sum_{n_2=1}^{N} s_{n1}s_{n2}^2s_{n12}s_{n21} \right) \right]$$

(4)

$$= \frac{K - 1}{N}$$

(5)

which converges to $\beta$ as $K \to \infty$. Also, for every $\epsilon > 0$,

$$E \left[ \sum_{j=2}^{K} \rho_{1j}^2 \cdot 1_{\{\rho_{1j} > \epsilon\}} \right] \leq E \left[ \sum_{j=2}^{K} \frac{\rho_{1j}^2}{\epsilon^2} \right]$$

(6)

$$= \frac{K - 1}{\epsilon^2} E \left[ \rho_{12}^4 \right]$$

(7)

$$= \frac{(K - 1)(3N^2 - 2N)}{\epsilon^2 N^4}$$

(8)

$$\to 0.$$  

(9)

Hence the Lindeberg condition is satisfied. By the Lindeberg-Feller Central Limit Theorem, $X_K$ converge weakly to a zero-mean Gaussian distribution with a variance of $\beta$. Hence,

$$E \left[ 1_{\{X_K > 1\}} \right] = P \left[ X_K > 1 \right]$$

(10)

converges to $Q \left( \frac{1}{\sqrt{\beta}} \right)$ as $K \to \infty$. Obviously,

$$\text{var} \left\{ 1_{\{X_K > 1\}} \right\} = E \left[ 1_{\{X_K > 1\}} \right] - \left( E \left[ 1_{\{X_K > 1\}} \right] \right)^2$$

(11)

converges to

$$Q \left( \frac{1}{\sqrt{\beta}} \right) \left( 1 - Q \left( \frac{1}{\sqrt{\beta}} \right) \right).$$

(12)