ELE 533 Multiuser Communication Theory

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Selected Problems: 4.12, 4.20, 4.58

1 Problem 4.12

Solution: According to the construction in the book, we 'convert' the asynchronous CDMA model to equivalent synchronous CDMA model, and we will compute $\Omega(\mathbf{b})$, which is the payoff function:

$$\begin{aligned}
\mathbf{b^T H b} &= \sum_{j=1-MK}^{MK+K} \sum_{l=1-MK}^{MK+K} b_j b_l h_{j,l} \\
&= \sum_{j=1-MK}^{MK+K} b_j [A_{\kappa(j)}^2 b_j + 2 \sum_{l=j+1}^{j+K-1} b_l h_{l,j}] \\
&= \sum_{j=1-MK}^{MK+K} b_j [A_{\kappa(j)}^2 b_j + 2 \sum_{n=1}^{K-1} b_{j+n} h_{j+n,j}] \\
&= \sum_{j=1-MK}^{MK+K} b_j [A_{\kappa(j)}^2 b_j + 2 \sum_{n=1}^{K-1} b_{j+n} A_{\kappa(j)} A_{\kappa(j+n)} \rho_{\kappa(j),\kappa(j+n)}] \qquad (1)
\end{aligned}$$

This comes from the property of symmetric matrix: $\forall\ \mathbf{Q} \in \mathbb{R}^{N\times N}$ and $\mathbf{x} \in \mathbb{R}^{N\times 1}$, we have:

$$\begin{aligned}
\mathbf{x^T Q x} &= \sum_{i=1}^{N}\sum_{j=1}^{N} Q_{ij}x_i x_j \\
&= \sum_{i=1}^{N}[Q_{ii}x_i^2 + \sum_{j=1, j\neq i}^{N} Q_{ij}x_i x_j] \\
&= \sum_{i=1}^{N}[Q_{ii}x_i^2 + \sum_{j=1}^{i-1} Q_{ij}x_i x_j + \sum_{j=i+1}^{N} Q_{ij}x_i x_j] \qquad (2)
\end{aligned}$$

We further note that:

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{i-1} Q_{ij}x_i x_j &= \sum_{j=1}^{N-1}\sum_{i=j+1}^{N} Q_{ij}x_i x_j \\
&= \sum_{j=1}^{N}\sum_{i=j+1}^{N} Q_{ij}x_i x_j \\
&= \sum_{i=1}^{N}\sum_{j=i+1}^{N} Q_{ji}x_i x_j \quad \text{by interchanging i and j} \\
&= \sum_{i=1}^{N}\sum_{j=i+1}^{N} Q_{ij}x_i x_j \quad \text{by the symmetry of Q} \qquad (3)
\end{aligned}$$

Plug (3) into (2) we have:

$$\mathbf{x^T Q x} = \sum_{i=1}^{N}[Q_{ii}x_i^2 + 2\sum_{j=i+1}^{N} Q_{ij}x_i x_j] \qquad (4)$$

From equation (4) we can get equation (1)

Then we can compute $\Omega(\mathbf{b})$:

$$\begin{aligned}
\Omega(\mathbf{b}) &= 2\mathbf{b^T A_M y} - \mathbf{b^T H b} \\
&= 2\sum_{j=1-MK}^{MK+K} A_{\kappa(j)} b_j y_j - \sum_{j=1-MK}^{MK+K} b_j[A_{\kappa(j)}^2 b_j + 2\sum_{n=1}^{K-1} b_{j+n}A_{\kappa(j)}A_{\kappa(j+n)}\rho_{\kappa(j),\kappa(j+n)}] \\
&= \sum_{j=1-MK}^{MK+K} A_{\kappa(j)} b_j[2y_j - A_{\kappa(j)}b_j - 2\sum_{n=1}^{K-1} b_{j+n}A_{\kappa(j+n)}\rho_{\kappa(j),\kappa(j+n)}] \\
&= \sum_{j=1-MK}^{MK+K} \mu_j(b_j, \mathbf{x_j}) \qquad (5)
\end{aligned}$$

where

$$\begin{aligned}
\mu_j(b_j, \mathbf{x_j}) &= A_{\kappa(j)} b_j [2y_j - A_{\kappa(j)} b_j - 2 \sum_{n=1}^{K-1} x_j(n) A_{\kappa(j+n)} \rho_{\kappa(j),\kappa(j+n)}] && (6) \\
\mathbf{x_j} &= (x_j(1), x_j(2), \ldots, x_j(K-1))^T \\
&= (b_{j+1}, b_{j+2}, \ldots, b_{j+K-1})^T && (7) \\
\mathbf{x_{j-1}} &= (b_j, b_{j+1}, \ldots, b_{j+K-2})^T \\
&= f(b_j, \mathbf{x_j}) && (8) \\
\mathbf{x_{MK+K}} &= \mathbf{0} && (9)
\end{aligned}$$

To maximize $\Omega(\mathbf{b})$ is equivalent to maximize $\tilde{\Omega}(\mathbf{b}) = \sum_{j=1-MK}^{MK+K} \tilde{\mu_j}(b_j, \mathbf{x_j})$,where:

$$\tilde{\mu_j}(b_j, \mathbf{x_j}) = A_{\kappa(j)} b_j [y_j - \sum_{n=1}^{K-1} x_j(n) A_{\kappa(j+n)} \rho_{\kappa(j),\kappa(j+n)}] \quad (10)$$

Then we can write down backward dynamic programming algorithm as follows:

$$\begin{aligned}
J_{MK+K}(\mathbf{x_{MK+K}}) &= 0 && (11) \\
J_{MK+K-1}(\mathbf{x_{MK+K-1}}) &= \max\{\tilde{\mu}_{MK+K}(1, \mathbf{x_{MK+K}}), \\
&\quad \tilde{\mu}_{MK+K}(-1, \mathbf{x_{MK+K}})\} && (12) \\
J_{j-1}(\mathbf{x_{j-1}}) &= \max\big\{J_j(x_{j-1}(2), x_{j-1}(3), \ldots, x_{j-1}(K-1), 1) + \\
&\quad \tilde{\mu_j}(x_{j-1}(1), x_{j-1}(2), \ldots, x_{j-1}(K-1), 1), \\
&\quad J_j(x_{j-1}(2), x_{j-1}(3), \ldots, x_{j-1}(K-1), -1) + \\
&\quad \tilde{\mu_j}(x_{j-1}(1), x_{j-1}(2), \ldots, x_{j-1}(K-1), -1)\big\} && (13)
\end{aligned}$$

As the same in the forward dynamic programming algorithm, since each level we need to calculate the J_j's for 2^{K-1} states, so the time complexity per bit is $O(2^K)$

2 Problem 4.20

Solution: We first condition on b_2, so this becomes a simple binary hypothesis testing problem with the following hypotheses:

$$\begin{aligned}
H1 : y(t) &= (1 + A_2 b_2)^2 s^2(t) + \sigma n(t) && (14) \\
H2 : y(t) &= A_2^2 b_2^2 s^2(t) + \sigma n(t) && (15)
\end{aligned}$$