Multiuser Detection

Problem 4.17 Qing Wang

(a) $\Omega(\mathbf{b}) = \mathbf{2b}^T\mathbf{Ay} - \mathbf{b}^T\mathbf{Hb}$, where $\mathbf{A}$ is diagonal, $\mathbf{H} = \mathbf{ARA}$. We assume these two matrices are both given.
To evaluate $\Omega(\mathbf{b})$ for each $\mathbf{b}$, first since $2\mathbf{b}^T\mathbf{Ay} = 2A\mathbf{b}^T\mathbf{Iy} = 2A\mathbf{b}^T\mathbf{y}$, then it involves 3 sums and 6 multiplications; to calculate $\mathbf{b}^T\mathbf{Hb}$, first $\mathbf{b}^T\mathbf{H}$ takes 16 multiplications and 12 sums, and $\mathbf{b}^T\mathbf{Hb}$ takes 4 multiplications and 3 sums, then altogether it takes 20 multiplications and 15 sums.
In all, we have to do 6+16=22 multiplications and 3+15+1=19 sums. Actually, here multiplication is just change of sign since b_i =+1 or –1 except for the case with multiplying $2A$. So doing multiplication in this case can be greatly simplified and thus won't entail much hardware or software cost. Then we don't really need to differentiate between multiplications and sums.
Then, the number of all the arithmetic operations involved is 22+19=41.
In order to get all the sixteen possible values of $\Omega(\mathbf{b})$, we have to repeat this calculations sixteen times. So altogether, we have to do $41 \times 16 = 656$ arithmetic operations.

(b) To show this decision rule is maximum likelihood, we have to show that the $\hat{\mathbf{b}}$ decided in each case does maximize $\Omega(\mathbf{b})$.
Denote y_k as matched filter outputs, i.e. $y_k = \int_0^1 y(t)s_k(t)dt$
According to the definition of the chip matched filter outputs, we have

$$y_1 = A(r_1 + r_2 + r_3)$$

$$y_2 = A(r_1 - r_2 + r_3)$$

$$y_3 = A(r_1 - r_2 - r_3)$$

$$y_4 = A(-r_1 - r_2 + r_3)$$

Actually, since $\left\{ p_{1/3}(t), p_{1/3}(t-\frac{1}{3}), p_{1/3}(t-\frac{2}{3}) \right\}$ is a set of orthogonal functions which spans the space of signature waveform, $\{r_1, r_2, r_3\}$ provides sufficient statistics to demodulate the received signals.

And by calculation, we get $\mathbf{H} = A^2 \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & 1 & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 1 \end{bmatrix}$

Substitute the above results into $\Omega(\mathbf{b})$:

$$\Omega(\mathbf{b}) = 2A^2\left[(b_1+b_2+b_3-b_4)r_1 + (b_1-b_2-b_3-b_4)r_2 + (b_1+b_2-b_3+b_4)r_3\right]$$
$$-\frac{2}{3}A^2(b_1b_2-b_1b_3-b_1b_4+b_2b_3+b_2b_4-b_3b_4) - 4A^2$$

Note that $(b_1b_2-b_1b_3-b_1b_4+b_2b_3+b_2b_4-b_3b_4)$ depends on whether each pair (b_i, b_j) $i \neq j$ are equal or different rather than on the exact value of b_i. So if we can fix the values of $(b_1+b_2+b_3-b_4)$, $(b_1-b_2-b_3-b_4)$ and $(b_1+b_2-b_3+b_4)$, we can determine the relative values among b_i's and thus we can determine the value of $(b_1b_2-b_1b_3-b_1b_4+b_2b_3+b_2b_4-b_3b_4)$.

Intuitively, we can maximize $\Omega(\mathbf{b})$ by (i) maximizing the absolute value of the coefficient of r_i, which has the largest absolute value among $\{r_1, r_2, r_3\}$ or by (ii) maximizing the sum of the absolute values of the coefficients of r_i's. or by (iii) minimizing $(b_1b_2-b_1b_3-b_1b_4+b_2b_3+b_2b_4-b_3b_4)$. We can evaluate $\Omega(\mathbf{b})$ respectively in these three cases and find the largest one.

(i) To evaluate $\Omega(\mathbf{b})$ in this case, we set 4sgn(r_i) =4ξ_i to the coefficient corresponding to $\max|r_i|$. Then all the values of b_i's are determined.
For example, if $b_1+b_2+b_3-b_4 = 4\xi_1$, then $b_1 = \xi_1, b_2 = \xi_1, b_3 = \xi_1, b_4 = -\xi_1$
(Upon deriving b_i's under the other two conditions, we can get the **list** in the Textbook)
Then $\Omega(\mathbf{b}) = 8A^2|r_1| - \frac{16}{3}A^2 = 4A^2\left(2|r_1| - \frac{4}{3}\right)$.
With simple calculation, we can generalize that
$$\Omega(\mathbf{b}) = 4\left(2A^2 \max|r_i| - \frac{4}{3}A^2\right) = 4\varsigma_1$$

(ii) In this case, we find that we can always set
$$b_1+b_2+b_3-b_4 = 2\xi_1, b_1-b_2-b_3-b_4 = 2\xi_2, b_1+b_2-b_3+b_4 = 2\xi_3$$

$\Rightarrow b_1 = b_4 + \xi_1 + \xi_2, b_2 = -b_4 - \xi_2 + \xi_3, b_3 = b_4 + \xi_1 - \xi_3$

And we assert that b_4 can be fully decided using the above equations.

If $\xi_1 = \xi_2$, to ensure that $b_1 = +1$ or -1, we must have $b_4 = -\xi_1$. With b_4, we can determine all the other three bits.

Otherwise,

If $\xi_1 = \xi_3$, then $\xi_2 \neq \xi_3$. Then to ensure that $b_2 =$ + 1 or -1, we must have $b_4 = \xi_3 = \xi_1$.

Otherwise, $\xi_1 \neq \xi_3$, then to ensure the validity of b_3, we must have $b_4 = -\xi_1$.

Using the above rules, we can have the **table** in the Textbook.

Evaluating $\Omega(\mathbf{b})$ in the case, we get

$\Omega(\mathbf{b}) = 4A^2(|r_1| + |r_2| + |r_3| - 1) = 4\varsigma_2$

(iii) To minimize $(b_1b_2 - b_1b_3 - b_1b_4 + b_2b_3 + b_2b_4 - b_3b_4)$, we should have

$b_1b_2 = -1, b_1b_3 = 1, b_1b_4 = 1, b_2b_3 = -1, b_2b_4 = -1, b_3b_4 = 1$,

then we have **TWO** possibilities

$(b_1, b_2, b_3, b_4) = (+1, -1, +1, +1)\, or\, (-1, +1, -1, -1)$

and $\Omega(\mathbf{b}) = 0$ under either condition

In fact, we can NEVER differentiate between these two possibilities, because $y(t) = \sigma n(t)$ under both conditions.

After elaborating on the above three cases, we know which decision rule to choose by comparison. Namely, if ς_1 is the largest, we resort to the decision rule given by (i) (for the detailed, please refer to the book. Omitted here). And similarly, if ς_2 is the largest, we use the decision rule given by (ii) (Table is in the book). Finally, is 0 is the largest, we arrive at two possibilities between which we cannot decide.

We can conclude that the above decision rule is maximum likelihood by maximizing $\Omega(\mathbf{b})$.

(c) To calculate ς_1, it takes 3+2=5 comparisons to get $\max|r_i|$ and one multiplication and one sum to get the final number.

With the information obtained in calculating ς_1, we only need to do 3 sums to get ς_2.

With 2 comparisons, we can find the largest among ς_1, ς_2 and 0. Then we know which rule we should use. In the worse case (ii), we only need to do 3 comparisons to arrive at the final decision.

So the number of operations of the decision rule in (b) is 5+3+2+3=13 at most (5+3+2=10 at least). We see that the decision rule in (b) achieves a great improvement in efficiency compared to the exhaustive testing in (a).