Multiuser Detection

PROBLEM 4.18.

The K-user synchronous channel is

$$y(t) = \sum_{k=1}^{n} A_k b_k s_k(t) + \sigma(t)$$

Let $\mathbf{A} = diag(A_1, ...A_K)$, $\mathbf{b} = [b_1, ...b_K]^T$ and $\mathbf{y} = [y_1, ...y_K]^T$, where $y_k = \int_0^T y(t) s_k(t) dt$. As the amplitudes A_1,...,A_K are anknown to the receiver the jointly optimum decision rule gives

$$\widehat{\mathbf{A}}, \hat{\mathbf{b}} = \arg \max_{\mathbf{A} \in Diag^+(K), \mathbf{b} \in \{-1,1\}^K} p_{\mathbf{A},\mathbf{b}/Y}(\mathbf{A}, \mathbf{b}/\mathbf{y}) \tag{1}$$

where $Diag^+(K)$ denotes the $K \times K$ diagonal matrices with nonnegative elements. Using the Bayes rule we have

$$\widehat{\mathbf{A}}, \hat{\mathbf{b}} = \arg \max_{\mathbf{A} \in Diag^+(K), \mathbf{b} \in \{-1,1\}^K} p_{Y/\mathbf{A},\mathbf{b}}(\mathbf{y}/\mathbf{A}, \mathbf{b}) p_{A,\mathbf{b}}(A, \mathbf{b}). \tag{2}$$

We know that

$$p_{Y/\mathbf{A},\mathbf{b}}(\mathbf{y}/\mathbf{A}, \mathbf{b}) = \frac{1}{(2\pi\sigma^2)^{\frac{K}{2}}} \exp\left(-\frac{1}{2\sigma^2} \int_0^T [y(t) - \sum_{k=1}^{K} b_k A_k s_k(t)]^2 dt \right) \tag{3}$$

$$\begin{aligned} \ln(p_{Y/\mathbf{A},\mathbf{b}}(\mathbf{y}/\mathbf{A}, \mathbf{b})) &= -\frac{K}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \int_0^T [y(t) - \sum_{k=1}^{K} b_k A_k s_k(t)]^2 dt && (4) \\ &= -\frac{K}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \int_0^T y(t)^2 dt + \frac{2}{2\sigma^2} \mathbf{b}^T \mathbf{A} \mathbf{y} - \frac{1}{2\sigma^2} \mathbf{b}^T \mathbf{H} \mathbf{b} && (5) \end{aligned}$$

where $\mathbf{H} = \mathbf{ARA}$ with $\mathbf{R}$ being the normalized correlation matrix.

As the users bits are equiprobable and the amplitudes A_1,...,A_K are independent and independent of the bits, we have

$$p_{\mathbf{A},\mathbf{b}}(\mathbf{A}, \mathbf{b}) = \left(\frac{1}{2}\right)^K \prod_{i=1}^{K} p_{A_k}(A_k) \tag{6}$$

Using (2),(5) and (6) and keeping only the terms dependent on $\mathbf{b}$ and $\mathbf{A}$, we get

$$\widehat{\mathbf{A}}, \hat{\mathbf{b}} = \arg\max \mathbf{b}^T \mathbf{A} \mathbf{y} - \frac{1}{2} \mathbf{b}^T \mathbf{H} \mathbf{b} + \sigma^2 \sum_{i=1}^{K} \ln p_{A_k}(A_k) \tag{7}$$

Let $\mathbf{x} = \mathbf{Ab}$. As $\mathbf{A}$ contains only nonnegative elements, $\mathbf{b} = sgn(\mathbf{x})$ and $Ai = |x_i|$ (since $b_i = \pm 1$). We also have the $p_{x_i}(x_i) = \frac{1}{2} p_{A_i}(|x_i|)$. Equation (7) becomes:

$$\hat{\mathbf{x}} = \arg\max \mathbf{x}^T \mathbf{y} - \frac{1}{2}\mathbf{x}^T \mathbf{R} \mathbf{x} + \sigma^2 \sum_{i=1}^{K} \ln p_{x_i}(x_i) \tag{8}$$

If $\hat{\mathbf{x}}$ is optimum, then we must solve

$$\frac{d}{dx}\left(\mathbf{x}^T \mathbf{y} - \frac{1}{2}\mathbf{x}^T \mathbf{R} \mathbf{x} + \sigma^2 \sum_{i=1}^{K} \ln p_{x_i}(x_i) \right) = 0 \tag{9}$$

Thus the jointly optimum decisions satisfy for some $\mathbf{x} \in R^K$:

$$\hat{\mathbf{b}} = sgn(\mathbf{x}), \tag{10}$$

$$\mathbf{R}\mathbf{x} - \mathbf{y} - \sigma^2 \mathbf{v}(\mathbf{x}) = 0, \tag{11}$$

where $\mathbf{v}(\mathbf{x}) = [\frac{p'_{A_1}(x_1)}{p_{A_1}(x_1)}\hat{b_1}, ..., \frac{p'_{A_K}(x_K)}{p_{A_K}(x_K)}\hat{b_K}]^T$.