where

$$\begin{aligned}
\mu_j(b_j, \mathbf{x_j}) &= A_{\kappa(j)} b_j [2y_j - A_{\kappa(j)} b_j - 2 \sum_{n=1}^{K-1} x_j(n) A_{\kappa(j+n)} \rho_{\kappa(j),\kappa(j+n)}] && (6) \\
\mathbf{x_j} &= (x_j(1), x_j(2), \dots, x_j(K-1))^T \\
&= (b_{j+1}, b_{j+2}, \dots, b_{j+K-1})^T && (7) \\
\mathbf{x_{j-1}} &= (b_j, b_{j+1}, \dots, b_{j+K-2})^T \\
&= f(b_j, \mathbf{x_j}) && (8) \\
\mathbf{x_{MK+K}} &= \mathbf{0} && (9)
\end{aligned}$$

To maximize $\Omega(\mathbf{b})$ is equivalent to maximize $\tilde{\Omega}(\mathbf{b}) = \sum_{j=1-MK}^{MK+K} \tilde{\mu_j}(b_j, \mathbf{x_j})$ ,where:

$$\tilde{\mu_j}(b_j, \mathbf{x_j}) = A_{\kappa(j)} b_j [y_j - \sum_{n=1}^{K-1} x_j(n) A_{\kappa(j+n)} \rho_{\kappa(j),\kappa(j+n)}] \quad (10)$$

Then we can write down backward dynamic programming algorithm as follows:

$$\begin{aligned}
J_{MK+K}(\mathbf{x_{MK+K}}) &= 0 && (11) \\
J_{MK+K-1}(\mathbf{x_{MK+K-1}}) &= \max\{\tilde{\mu}_{MK+K}(1, \mathbf{x_{MK+K}}), \\
&\quad \tilde{\mu}_{MK+K}(-1, \mathbf{x_{MK+K}})\} && (12) \\
J_{j-1}(\mathbf{x_{j-1}}) &= \max\big\{ J_j(x_{j-1}(2), x_{j-1}(3), \dots, x_{j-1}(K-1), 1) + \\
&\quad \tilde{\mu_j}(x_{j-1}(1), x_{j-1}(2), \dots, x_{j-1}(K-1), 1), \\
&\quad J_j(x_{j-1}(2), x_{j-1}(3), \dots, x_{j-1}(K-1), -1) + \\
&\quad \tilde{\mu_j}(x_{j-1}(1), x_{j-1}(2), \dots, x_{j-1}(K-1), -1)\big\} && (13)
\end{aligned}$$

As the same in the forward dynamic programming algorithm, since each level we need to calculate the $J_j$'s for $2^{K-1}$ states, so the time complexity per bit is $O(2^K)$

# 2 Problem 4.20

Solution: We first condition on $b_2$, so this becomes a simple binary hypothesis testing problem with the following hypotheses:

$$\begin{aligned}
H1 : y(t) &= (1 + A_2 b_2)^2 s^2(t) + \sigma n(t) && (14) \\
H2 : y(t) &= A_2^2 b_2^2 s^2(t) + \sigma n(t) && (15)
\end{aligned}$$

The ML(Maximum Likelihood) test is to calculate the following equation and compare the result:

$$e^{-\frac{1}{2\sigma^2}\int_0^T (y(t)-(1+A_2b_2)^2s^2(t))^2\mathrm{d}t} \tag{16}$$

$$e^{-\frac{1}{2\sigma^2}\int_0^T (y(t)-A_2^2b_2^2s^2(t))^2\mathrm{d}t} \tag{17}$$

To maximize the functions in equation (16) and equation (17),
it is equivalent to minimize the following equation:

$$\int_0^T (y(t)-(1+A_2b_2)^2s^2(t))^2\mathrm{d}t \tag{18}$$

$$\int_0^T (y(t)-A_2^2b_2^2s^2(t))^2\mathrm{d}t \tag{19}$$

To compute $P_1(A_2,\sigma)$ conditioning on $b_2$, we first assume $b_1 = 1$ is transmitted, then when an error occurs, $\hat{b}_1 = 0$
Equation (18) and (19) becomes:

$$\int_0^T ((1+A_2b_2)^2s^2(t)+\sigma n(t)-(1+A_2b_2)^2s^2(t))^2 \;=\; \sigma^2\int_0^T n^2(t)\mathrm{d}t \tag{20}$$

$$\begin{aligned}\int_0^T ((1+A_2b_2)^2s^2(t)+\sigma n(t)-A_2^2b_2^2s^2(t))\mathrm{d}t \;&=\; \int_0^T \big[(1+2A_2b_2)^2s^4(t)+\\ &2\sigma(1+2A_2b_2)s^2(t)n(t)+\sigma^2n^2(t)\big]\mathrm{d}t\end{aligned} \tag{21}$$

let $n = -\int_0^T n(t)s^2(t)\mathrm{d}t$, then:

$$\mathbb{E}n \;=\; 0 \tag{22}$$

$$\mathbb{E}n^2 \;=\; \int_0^T s^4(t)\mathrm{d}t \tag{23}$$

Therefore to make an error, we have:

$$\begin{aligned} P_1(A_2,\sigma|\; b_1=1,b_2) \;&=\; \mathbb{P}\big\{(20)>(21)\big\}\\ &=\; \mathbb{P}\big\{n>\frac{1+2A_2b_2}{2\sigma}\int_0^T s^4(t)\mathrm{d}t\big\}\\ &=\; Q\left(\frac{1+2A_2b_2}{2\sigma}\sqrt{\int_0^T s^4(t)\mathrm{d}t}\right)\end{aligned} \tag{24}$$

In the same way, we can get:

$$P_1(A_2,\sigma|\; b_1=0,b_2) = Q\left(\frac{1+2A_2b_2}{2\sigma}\sqrt{\int_0^T s^4(t)\mathrm{d}t}\right)$$

Thus, we can get average bit error rate for user 1:

$$
\begin{aligned}
P_1(A_2,\sigma) &= 0.25P_1(A_2,\sigma|\ b_1=1,b_2=1)+ \\
&\quad 0.25P_1(A_2,\sigma|\ b_1=1,b_2=0)+ \\
&\quad 0.25P_1(A_2,\sigma|\ b_1=0,b_2=1)+ \\
&\quad 0.25P_1(A_2,\sigma|\ b_1=0,b_2=0) \\
&= \frac{1}{2}Q\left(\frac{1+2A_2}{2\sigma}\sqrt{\int_0^T s^4(t)\mathrm{d}t}\right)+ \\
&\quad \frac{1}{2}Q\left(\frac{1}{2\sigma}\sqrt{\int_0^T s^4(t)\mathrm{d}t}\right) \qquad (25) \\
\text{thus:}\quad P_1(0,\sigma) &= Q\left(\frac{1}{2\sigma}\sqrt{\int_0^T s^4(t)\mathrm{d}t}\right) \qquad (26) \\
\text{Finally, we have:}\quad \lim_{A_2\to\infty}\frac{P_1(A_2,\sigma)}{P_1(0,\sigma)} &= \frac{1}{2} \quad \text{by the property of Q-function}
\end{aligned}
$$

# 3 Problem 4.58

Solution: Before we go on to part (a), we first explore the property of $\langle S(\epsilon'), S(\epsilon'')\rangle$. In this case, the correlation matrix is now(by the equivalence of SNR):

$$
\mathbf{R} = \begin{pmatrix}
1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1K} \\
\rho_{12} & 1 & 0 & \cdots & 0 \\
\rho_{13} & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \\
\rho_{1K} & 0 & 0 & \cdots & 1
\end{pmatrix} \qquad (27)
$$

We then have:

$$
\mathbf{H} = \mathbf{ARA} = A^2\mathbf{R} \qquad (28)
$$

Therefore, if $\epsilon' + \epsilon''$ is a decomposition of some error vector $\epsilon$, then

$$
\sum_{k=1}^{K}\epsilon'_k\epsilon''_k = 0 \qquad \text{since corresponding components cannot be both nonzero} \qquad (29)
$$