where

$$
\mu_j(b_j, x_j) = A_{\kappa(j)} b_j [2y_j - A_{\kappa(j)} b_j - 2 \sum_{n=1}^{K-1} x_j(n) A_{\kappa(j+n)} \rho_{\kappa(j), \kappa(j+n)}] \quad (6)
$$

$$
x_j = (x_j(1), x_j(2), \ldots, x_j(K-1))^T
$$

$$
x_{j-1} = (b_j, b_{j+1}, \ldots, b_{j+K-1})^T
$$

$$
x_{MK+K} = 0
$$

To maximize $\Omega(b)$ is equivalent to maximize $\tilde{\Omega}(b) = \sum_{j=1-MK}^{MK+K} \tilde{\mu}_j(b_j, x_j)
$, where:

$$
\tilde{\mu}_j(b_j, x_j) = A_{\kappa(j)} b_j [y_j - \sum_{n=1}^{K-1} x_j(n) A_{\kappa(j+n)} \rho_{\kappa(j), \kappa(j+n)}] \quad (10)
$$

Then we can write down backward dynamic programming algorithm as follows:

$$
J_{MK+K}(x_{MK+K}) = 0 \quad (11)
$$

$$
J_{MK+K-1}(x_{MK+K-1}) = \max \{ \tilde{\mu}_{MK+K}(1, x_{MK+K}), \tilde{\mu}_{MK+K}(-1, x_{MK+K}) \} \quad (12)
$$

$$
J_{j-1}(x_{j-1}) = \max \{ J_j(x_{j-1}(1), x_{j-1}(2), \ldots, x_{j-1}(K-1), 1) + \tilde{\mu}_j(x_{j-1}(1), x_{j-1}(2), \ldots, x_{j-1}(K-1), 1),
J_j(x_{j-1}(2), x_{j-1}(3), \ldots, x_{j-1}(K-1), -1) + \tilde{\mu}_j(x_{j-1}(1), x_{j-1}(2), \ldots, x_{j-1}(K-1), -1) \} \quad (13)
$$

As the same in the forward dynamic programming algorithm, since each level we need to calculate the $J_j$'s for $2^{K-1}$ states, so the time complexity per bit is $O(2^K)$

### 2 Problem 4.20

Solution: We first condition on $b_2$, so this becomes a simple binary hypothesis testing problem with the following hypotheses:

$$
H1 : y(t) = (1 + A_2 b_2)^2 s^2(t) + \sigma n(t) \quad (14)
$$

$$
H2 : y(t) = A_2^2 b_2^2 s^2(t) + \sigma n(t) \quad (15)
$$
The ML (Maximum Likelihood) test is to calculate the following equation and compare the result:

\[
e^{-\frac{1}{2\sigma^2}\int_0^T (y(t)-(1+A_2b_2)^2s^2(t))^2dt}
\]  

(16)

\[
e^{-\frac{1}{2\sigma^2}\int_0^T (y(t)-A_2^2\hat{b}_2^2s^2(t))^2dt}
\]  

(17)

To maximize the functions in equation (16) and equation (17), it is equivalent to minimize the following equation:

\[
\int_0^T (y(t)-(1+A_2b_2)^2s^2(t))^2dt
\]  

(18)

\[
\int_0^T (y(t)-A_2^2\hat{b}_2^2s^2(t))^2dt
\]  

(19)

To compute \(P_1(A_2, \sigma)\) conditioning on \(b_1 = 1\) is transmitted, then when an error occurs, \(\hat{b}_1 = 0\).

Equation (18) and (19) becomes:

\[
\int_0^T ((1+A_2b_2)^2s^2(t)+\sigma n(t)-(1+A_2b_2)^2s^2(t))^2 = \sigma^2 \int_0^T n^2(t)dt
\]  

(20)

\[
\int_0^T ((1+A_2b_2)^2s^2(t)+\sigma n(t) - A_2^2\hat{b}_2^2s^2(t))dt = \int_0^T [(1+2A_2b_2)^2s^4(t) + 2\sigma (1+2A_2b_2)s^2(t)n(t) + \sigma^2 n^2(t)] dt
\]  

(21)

let \(n = -\int_0^T n(t)s^2(t)dt\), then:

\[
\mathbb{E}n = 0
\]  

(22)

\[
\mathbb{E}n^2 = \int_0^T s^4(t)dt
\]  

(23)

Therefore to make an error, we have:

\[
P_1(A_2, \sigma| b_1 = 1, b_2) = \mathbb{P}\{(20) > (21)\}
\]

\[
= \mathbb{P}\{n > \frac{1 + 2A_2b_2}{2\sigma} \int_0^T s^4(t)dt\}
\]

\[
= Q\left(\frac{1 + 2A_2b_2}{2\sigma}\sqrt{\int_0^T s^4(t)dt}\right)
\]  

(24)

In the same way, we can get:

\[
P_1(A_2, \sigma| b_1 = 0, b_2) = Q\left(\frac{1 + 2A_2b_2}{2\sigma}\sqrt{\int_0^T s^4(t)dt}\right)
\]
Thus, we can get average bit error rate for user 1:

\[
P_1(A_2, \sigma) = 0.25P_1(A_2, \sigma | b_1 = 1, b_2 = 1) + 0.25P_1(A_2, \sigma | b_1 = 1, b_2 = 0) + 0.25P_1(A_2, \sigma | b_1 = 0, b_2 = 1) + 0.25P_1(A_2, \sigma | b_1 = 0, b_2 = 0)
\]

\[
= \frac{1}{2}Q\left(\frac{1 + 2A_2}{2\sigma} \sqrt{\int_0^T s^4(t) dt}\right) + \frac{1}{2}Q\left(\frac{1}{2\sigma} \sqrt{\int_0^T s^4(t) dt}\right)
\]

(25)

thus: \[ P_1(0, \sigma) = Q\left(\frac{1}{2\sigma} \sqrt{\int_0^T s^4(t) dt}\right) \]

(26)

Finally, we have: \[ \lim_{A_2 \to \infty} \frac{P_1(A_2, \sigma)}{P_1(0, \sigma)} = \frac{1}{2} \text{ by the property of Q-function} \]

3 Problem 4.58

Solution: Before we go on to part (a), we first explore the property of \((S(\epsilon'), S(\epsilon''))\). In this case, the correlation matrix is now (by the equivalence of SNR):

\[
R = \begin{pmatrix}
1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1K} \\
\rho_{12} & 1 & 0 & \cdots & 0 \\
\rho_{13} & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{1K} & 0 & 0 & \cdots & 1 \\
\end{pmatrix}
\]

(27)

We then have:

\[
H = ARA = A^2R
\]

(28)

Therefore, if \(\epsilon' + \epsilon''\) is a decomposition of some error vector \(\epsilon\), then

\[
\sum_{k=1}^K \epsilon'_k \epsilon''_k = 0 \quad \text{since corresponding components cannot be both nonzero}
\]

(29)