Solution to Text problem 4.21 by Ankit Gupta

Statement of the problem
Find lower and upper bounds on the minimum bit error rates for user1 in the four user equal amplitude channel of problem 4.17

Solution

According to problem 4.17 the covariance matrix is

$$H = \begin{bmatrix} 1 & 1/3 & -1/3 & -1/3 \\ 1/3 & 1 & 1/3 & 1/3 \\ -1/3 & 1/3 & 1 & -1/3 \\ -1/3 & 1/3 & -1/3 & 1 \end{bmatrix} \tag{1}$$

Using this covariance matrix and using proposition 4.1 in the textbook the upper bound is given as

$$P_k(\sigma) \leq \sum_{\epsilon \in F_k} 2^{-w(\epsilon)} Q\left(\frac{||S(\epsilon)||}{\sigma}\right) \tag{2}$$

For the given covariance matrix and using the criterion for decomposability of error vectors the following error vectors and their antipodal images are found to be indecomposable using a computer program. Thus the following array together with its antipodal set makes up the set F_k

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

thus the probability of error is upper bounded by. Where the signals are supposed to have unit energy.

$$\begin{aligned} P_k(\sigma) &\leq \sum_{\epsilon \in F_k} 2^{-w(\epsilon)} Q\left(\frac{||S(\epsilon)||}{\sigma}\right) \\ &< 2(\frac{1}{2}Q\left(\frac{1}{\sigma}\right) + \frac{1}{4}Q\left(\frac{2}{\sqrt{3}\sigma}\right) + \frac{1}{8}Q\left(\frac{1}{\sigma}\right) + \frac{1}{8}Q\left(\frac{1}{\sigma}\right) \qquad (3) \\ &\quad + \frac{1}{16}Q\left(\frac{0}{\sigma}\right) + \frac{1}{4}Q\left(\frac{2}{\sqrt{3}\sigma}\right) + \frac{1}{4}Q\left(\frac{2}{\sqrt{3}\sigma}\right) + \frac{1}{8}Q\left(\frac{1}{\sigma}\right)) \\ &= \frac{1}{16} + \frac{3}{2}Q\left(\frac{2}{\sqrt{3}\sigma}\right) + \frac{7}{4}Q\left(\frac{1}{\sigma}\right) \qquad (4) \end{aligned}$$

Equation 4.89 in the textbook gives a lower bound for the k user problem as

$$P_k(\sigma) \geq 2^{1-w_{k,min}} Q\left(\frac{d_{k,min}}{\sigma}\right) \tag{5}$$

now since $d_{k,min}$ was found to be 0 for the vector $x = [1, -1, 1, 1]$ we have the lower bound as

$$P_k(\sigma) \geq 2^{1-w_{k,min}} Q\left(\frac{d_{k,min}}{\sigma}\right) \tag{6}$$

$$\geq \frac{1}{16} \tag{7}$$

Finally

$$\frac{1}{16} \leq P_k(\sigma) \leq \frac{1}{16} + \frac{3}{2} Q\left(\frac{2}{\sqrt{3}\sigma}\right) + \frac{7}{4} Q\left(\frac{1}{\sigma}\right) \tag{8}$$