Multiuser Detection

PROBLEM 4.23.

(a)

Using the Hint, write:

$$f_{Y/+}(y) = 2^{1-K} \sum_{\mathbf{b}\in E_k^+} P[y/\mathbf{b}],$$

$$f_{Y/-}(y) = 2^{1-K} \sum_{\mathbf{b}\in E_k^-} P[y/\mathbf{b}] = 2^{1-K} \sum_{\mathbf{b}\in E_k^+} P[y/\phi(\mathbf{b})].$$

We have the equivalent binary hypothesis testing between two equiprobable hypothesis:

$$H_0 \ : \ Y \sim f_{Y/+},$$

$$H_1 \ : \ Y \sim f_{Y/-}.$$

From PROBLEM 3.1. we know that

$$P_k(\sigma) = \frac{1}{2} \int min\{f_{Y/+}(y), f_{Y/-}(y)\} dy.$$

So we have

$$P_k(\sigma) = \frac{1}{2} 2^{1-K} \int min\{ \sum_{\mathbf{b}\in E_k^+} P[y/\mathbf{b}], \sum_{\mathbf{b}\in E_k^+} P[y/\phi(\mathbf{b})]\} dy. \tag{1}$$

Using the fact that $min(\sum_{i=1}^n a_i, \sum_{i=1}^n a_i) \geq \sum_{i=1}^n min(a_i, b_i)$, we write

$$P_k(\sigma) \geq \frac{1}{2} 2^{1-K} \sum_{\mathbf{b}\in E_k^+} \int min\{P[y/\mathbf{b}], P[y/\phi(\mathbf{b})]\} dy. \tag{2}$$

We know that

$$\frac{1}{2} \int min\{P[y/\mathbf{b}], P[y/\phi(\mathbf{b})]\} dy = Q\Big(\frac{\|S(\mathbf{b} - \phi(\mathbf{b}))\|}{2\sigma}\Big), \tag{3}$$

since it is the probability of error with two fixed multiuser signals corresponding to $\mathbf{b}$ and $\phi(\mathbf{b})$. From (2) and (3) we get

$$P_k(\sigma) \geq 2^{1-K} \sum_{\mathbf{b}\in E_k^+} Q\Big(\frac{\|S(\mathbf{b} - \phi(\mathbf{b}))\|}{2\sigma}\Big). \tag{4}$$

(b)

We know that $d_{k,min}$ can be interpreted as one half of the minimum distance beween two multiuser signals that differ in the kth bit. Choose the one-to-one correspondance ϕ that maximizes the number of pairs $(\mathbf{b}, \phi(\mathbf{b}))$ such that $\|S(\mathbf{b} - \phi(\mathbf{b}))\| =$

$2d_{k,min}$. Then from the sommation in (4), keep only the pairs $(\mathbf{b}, \phi(\mathbf{b}))$ such that $\|S(\mathbf{b} - \phi(\mathbf{b}))\| = 2d_{k,min}$. We thus get the lower bound

$$P_k(\sigma) \geq \alpha_k 2^{1-K} Q\Big(\frac{d_{k,min}}{\sigma}\Big). \tag{5}$$

(c)

Write $\epsilon = \frac{1}{2}(\mathbf{b} - \phi(\mathbf{b}))$. The number of pairs $((\mathbf{b}, \phi(\mathbf{b}))$ such that $\|S(\epsilon)\| = d_{k,min}$ is greater or equal to the number of pairs $((\mathbf{b}, \phi(\mathbf{b}))$ such that $\|S(\epsilon)\| = d_{k,min}$, $\epsilon \in F_k$ and $w(\epsilon) = w_{k,min}$. The $w_{k,min}$ positions where $\epsilon_k = \pm 1$ being fixed, there are at least $2^{K-w_{k,min}}$ possible pairs $(\mathbf{b}, \phi(\mathbf{b}))$ Thus we have $\alpha_k \geq 2^{K-w_{k,min}}$ and therefore

$$P_k(\sigma) \geq 2^{1-w_{k,min}} Q\Big(\frac{d_{k,min}}{\sigma}\Big). \tag{6}$$