Multiuser Detection

PROBLEM 4.23.

(a)

Using the Hint, write:

\[ f_{Y/+}(y) = 2^{1-K} \sum_{b \in E_k^+} P[y/b], \]

\[ f_{Y/-}(y) = 2^{1-K} \sum_{b \in E_k^+} P[y/b] = 2^{1-K} \sum_{b \in E_k^+} P[y/\phi(b)]. \]

We have the equivalent binary hypothesis testing between two equiprobable hypothesis:

\[ H_0 : Y \sim f_{Y/+}, \]

\[ H_1 : Y \sim f_{Y/-}. \]

From PROBLEM 3.1. we know that

\[ P_k(\sigma) = \frac{1}{2} \int min\{f_{Y/+}(y), f_{Y/-}(y)\} dy. \]

So we have

\[ P_k(\sigma) = \frac{1}{2} 2^{1-K} \int min\{ \sum_{b \in E_k^+} P[y/b], \sum_{b \in E_k^+} P[y/\phi(b)] \} dy. \]

(1)

Using the fact that \( min(\sum_{i=1}^{n} a_i, \sum_{i=1}^{n} a_i) \geq \sum_{i=1}^{n} min(a_i, b_i) \), we write

\[ P_k(\sigma) \geq \frac{1}{2} 2^{1-K} \sum_{b \in E_k^+} \int min\{P[y/b], P[y/\phi(b)]\} dy. \]

(2)

We know that

\[ \frac{1}{2} \int min\{P[y/b], P[y/\phi(b)]\} dy = Q\left( \frac{\|S(b - \phi(b))\|}{2\sigma} \right), \]

(3)

since it is the probability of error with two fixed multiuser signals corresponding to \( b \) and \( \phi(b) \). From (2) and (3) we get

\[ P_k(\sigma) \geq 2^{1-K} \sum_{b \in E_k^+} Q\left( \frac{\|S(b - \phi(b))\|}{2\sigma} \right), \]

(4)

(b)

We know that \( d_{k, min} \) can be interpreted as one half of the minimum distance between two multiuser signals that differ in the kth bit. Choose the one-to-one correspondence \( \phi \) that maximizes the number of pairs \((b, \phi(b))\) such that \( \|S(b - \phi(b))\| = \)
2d_{k,\text{min}}. Then from the summation in (4), keep only the pairs \((b, \phi(b))\) such that \(\|S(b - \phi(b))\| = 2d_{k,\text{min}}\). We thus get the lower bound

\[ P_k(\sigma) \geq \alpha_k 2^{1-K} Q\left(\frac{d_{k,\text{min}}}{\sigma}\right). \] (5)

(c)

Write \(\epsilon = \frac{1}{2}(b - \phi(b))\). The number of pairs \(((b, \phi(b))\) such that \(\|S(\epsilon)\| = d_{k,\text{min}}\) is greater or equal to the number of pairs \(((b, \phi(b))\) such that \(\|S(\epsilon)\| = d_{k,\text{min}}\), \(\epsilon \in F_k\) and \(w(\epsilon) = w_{k,\text{min}}\). The \(w_{k,\text{min}}\) positions where \(\epsilon_k = \pm 1\) being fixed, there are at least \(2^{K-w_{k,\text{min}}}\) possible pairs \((b, \phi(b))\) Thus we have \(\alpha_k \geq 2^{K-w_{k,\text{min}}}\) and therefore

\[ P_k(\sigma) \geq 2^{1-w_{k,\text{min}}} Q\left(\frac{d_{k,\text{min}}}{\sigma}\right). \] (6)