Random Matrix Transforms and Applications via Non-Asymptotic Eigenanalysis

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Abstract— **This work introduces an effective approach to derive the marginal density distribution of an unordered eigenvalue for finite-dimensional random matrices of Wishart and F type, based on which we give several examples of closed-form and series expressions for the Shannon and η transforms of random matrices with nonzero mean and/or dependent entries. The newly obtained results allow for a compact non-asymptotic characterization of MIMO and multiuser vector channels in terms of both ergodic capacity and minimum mean square error (MMSE). In addition, the derived marginal density distributions can be of interest on their own in other fields of applied statistics.**

I. INTRODUCTION

Random matrices have attracted great interest in the communications and information theory communities because of their applications to the fundamental limits of wireless communication vector channels. These channels are characterized by random matrices that admit various statistical descriptions depending on the actual application. Among the different types of transforms commonly adopted in random matrix theory to characterize the spectrum of random matrices [1][Chapter II], we focus herein on the Shannon and the η transforms, which are closely related to a more classical transform in random matrix theory, the Stieltjes transform, but provide more direct engineering insight [1]. These transforms were motivated by the intuition drawn from the application of random matrices to various problems in the information theory and signal processing of coherent noisy communication channels. Specifically, the Shannon transform gives the mutual information of various communication channels while the η transform characterizes the performance of linear multiuser detectors thereon [1]. In this paper, we exploit a new strategy to evaluate previously unavailable marginal density distributions of an unordered eigenvalue of finite-dimensional random matrices [2], [3] of particular interest in wireless communications. Based on these findings, we evaluate the corresponding Shannon and η transforms.

II. MATHEMATICAL BACKGROUND

This Section is aimed at providing some basic definitions from finite-dimensional random matrix theory. Specifically, we detail the statistical distributions of some random matrices of particular interest in wireless MIMO (multiple-input multiple-output) communication, whose marginal eigenvalue statistics will be then characterized in next Section.

A. Wishart Matrices

Definition 1 *The $m \times m$ random matrix $\boldsymbol{W} = \mathbf{H}\mathbf{H}^\dagger$ is a (central) complex Wishart matrix with n degrees of freedom and covariance matrix $\mathbf{\Theta}_m$, ($\mathbf{W} \sim \mathcal{W}_m(n, \mathbf{\Theta}_m)$), if the columns of the $m \times n$ matrix $\mathbf{H}$ are zero-mean independent complex Gaussian vectors with covariance matrix $\mathbf{\Theta}_m$. The p.d.f. of $\mathbf{W} \sim \mathcal{W}_m(n, \mathbf{\Theta}_m)$ for $n \geq m$ is [2]*

$$f_{\mathbf{W}}(\mathbf{B}) = \frac{\pi^{-m(m-1)/2} \det \mathbf{B}^{n-m}}{\det {\mathbf{\Theta}_m}^n \prod_{i=1}^m (n-i)!} \exp\left[-\mathrm{tr}\left\{{\mathbf{\Theta}_m}^{-1}\mathbf{B}\right\}\right]. \tag{1}$$

Definition 2 *The $m \times m$ random matrix $\boldsymbol{W} = \mathbf{H}\mathbf{H}^\dagger$ is a (central)* complex Wishart *matrix of second kind, with n degrees of freedom and covariance matrix $\mathbf{\Theta}_n$, ($\mathbf{W} \sim \tilde{\mathcal{W}}_m(n, \mathbf{\Theta}_n)$), if the rows of the $m \times n$ matrix $\mathbf{H}$ are zero-mean independent complex Gaussian vectors with covariance matrix $\mathbf{\Theta}_n$. The p.d.f. of $\mathbf{W} \sim \tilde{\mathcal{W}}_m(n, \mathbf{\Theta}_n)$ for $n \geq m$ is [3]*

$$f_{\mathbf{W}}(\mathbf{B}) = \frac{\det \mathbf{B}^{n-m} {}_0F_0(\mathbf{\Theta}_n^{-1}, -\mathbf{B})}{\pi^{m(m-1)/2} \prod_{i=1}^m (n-i)! \det \mathbf{\Theta}_n^m} \tag{2}$$

where ${}_0F_0(\cdot,\cdot)$ is the hypergeometric function of exponential type of two square matrix arguments of different dimensions [3].

Let now $\mathbf{H}$ be a $m \times n$ matrix

$$\mathbf{H} = \bar{\mathbf{H}} + \mathbf{H}_w \tag{3}$$

where the columns of $\mathbf{H}_w$ are zero mean, independent complex Gaussian vectors with covariance matrix $\mathbf{\Sigma}$ while $\bar{\mathbf{H}}$ is a deterministic $m \times n$ matrix.

Definition 3 *Defined $\mathbf{H}$ as in (3), the $m \times m$ random matrix $\boldsymbol{W} = \mathbf{H}\mathbf{H}^\dagger$ is a noncentral complex Wishart matrix with n degrees of freedom and noncentrality matrix $\boldsymbol{M} = \bar{\mathbf{H}}\bar{\mathbf{H}}^\dagger$, ($\mathbf{W} \sim \mathcal{W}_m(n, \boldsymbol{M}, \mathbf{\Sigma})$). Its p.d.f. for $n \geq m$ is [2]*

$$f_{\mathbf{W}}(\mathbf{B}) = K \frac{e^{-\mathrm{tr}\left\{\mathbf{\Sigma}^{-1}\mathbf{B}\right\}}}{\det \mathbf{B}^{m-n}} {}_0F_1\big(n; \mathbf{\Sigma}^{-1}\boldsymbol{M}\mathbf{\Sigma}^{-1}\mathbf{B}\big),$$

where ${}_0F_1(\cdot;\cdot)$ *is the Bessel hypergeometric function of matrix argument [2] and* $K = \frac{e^{-\mathrm{tr}\left\{\boldsymbol{\Sigma}^{-1}\mathbf{M}\right\}}}{\det \boldsymbol{\Sigma}^n \pi^{m(m-1)/2} \prod_{i=1}^m (n-i)!}$

B. Matrix-variate F

Henceforth, for sake of compactness, we let $\tau = n - m$.

Definition 4 *Let* $\boldsymbol{\Phi}$ *and* $\boldsymbol{W}$ *be central Wishart matrices,* $\boldsymbol{\Phi} \sim \mathcal{W}_m(L, \boldsymbol{\Sigma})$*,* $\boldsymbol{W} \sim \mathcal{W}_m(n, \boldsymbol{\Theta}_m)$*, as in Definition 1. The* $m \times m$ *random matrix* $\mathbf{A} = \boldsymbol{W}\boldsymbol{\Phi}^{-1}$ *is a central F matrix. The p.d.f. of a complex central F matrix with* $L \geq n$ *is [2]*

$$f_{\mathbf{A}}(\mathbf{B}) = \prod_{\ell=1}^{m} \frac{\pi^{-\frac{(m-1)}{2}}(n+L-\ell)!}{\omega_\ell^n (n-\ell)!(L-\ell)!} \frac{\det \mathbf{B}^\tau}{\det(\mathbf{I}_m + \boldsymbol{\Omega}^{-1}\mathbf{B})^{n+L}} \quad (4)$$

where ω_ℓ *is the* ℓ*-th eigenvalue of* $\boldsymbol{\Omega} = \boldsymbol{\Theta}_m \boldsymbol{\Sigma}^{-1}$.

Definition 5 *Let* $\mathbf{H}$ *be a* $m \times n$ *complex matrix (with* $n \geq m$ *), whose entries are zero-mean i.i.d Gaussian random variables, and let* $\boldsymbol{\Phi} \sim \widetilde{\mathcal{W}}_n(L, \boldsymbol{\Theta}_L)$ *be a central Wishart Matrix of second kind as in Definition 2. The* $m \times m$ *random matrix* $\mathbf{A} = \mathbf{H}\boldsymbol{\Phi}^{-1}\mathbf{H}^\dagger$ *is a central F matrix of the second kind. Its p.d.f. when* $L \geq n$ *is [3]*

$$f_{\mathbf{A}}(\mathbf{B}) = \prod_{i=1}^{n} \frac{(m+L-i)!}{(L-i)!} \frac{\det \boldsymbol{\Theta}_L^{-n}}{\pi^{m(m-1)/2}\prod_{i=1}^m (n-i)!} \quad (5)$$
$$\times \frac{q^{n(m+L)} \det \mathbf{B}^\tau}{\det(\mathbf{I}_m + q\mathbf{B})^{m+L}} {}_1F_0(m+L; \mathbf{I}_L - q\boldsymbol{\Theta}_L^{-1}, \widetilde{\mathbf{F}})$$

with $q > 0$ *and*

$$\widetilde{\mathbf{F}} = \begin{pmatrix} (\mathbf{I}_m + q\mathbf{B})^{-1} & 0 \\ 0 & \mathbf{I}_\tau \end{pmatrix}.$$

Definition 6 *Let* $\boldsymbol{W}$ *be a noncentral Wishart matrix,* $\boldsymbol{W} \sim \mathcal{W}_n(L, \mathbf{I}_n)$*, as in Definition 1. Defined* $\mathbf{H}$ *as in (3) with* $\boldsymbol{\Sigma} = \mathbf{I}_m$*, the* $m \times m$ *random matrix* $\mathbf{A} = \mathbf{H}\mathbf{W}^{-1}\mathbf{H}^\dagger$ *is a noncentral F matrix. Its p.d.f. when* $L \geq n \geq m$ *is [2]*

$$f_{\mathbf{A}}(\mathbf{B}) = \frac{e^{-\mathrm{tr}\{\mathbf{M}\}} \det \mathbf{B}^\tau}{\pi^{m(m-1)/2} \det(\mathbf{I}+\mathbf{B})^{m+L}} \prod_{\ell=1}^{m} \frac{(L+m-\ell)!}{(-\tau+L-\ell)!(n-\ell)!}$$
$${}_1F_1\left(m+L, n, \boldsymbol{M}(\mathbf{I}_m + \mathbf{B}^{-1})^{-1}\right) \quad (6)$$

with $\boldsymbol{M} = \bar{\mathbf{H}}\bar{\mathbf{H}}^\dagger$*, and* ${}_1F_1(a, b, \cdot)$ *the confluent hypergeometric function of matrix argument [2]*[1].

III. Marginal density distribution characterizations

This main section is devoted to the formulation of a general strategy to obtain the marginal density distribution of the unordered eigenvalues of finite-dimensional random matrices conforming to the models described in the previous section. The method is based on two main results that are stated as Lemmas in the following.

[1]The results for the case of $m \geq n$ can be obtained from those relative to $m \leq n$ by conveniently defining $r = \max\{n, m\}$, $t = \min\{n, m\}$, $\widetilde{\tau} = r - t$ and $\bar{L} = L + \widetilde{\tau}$ and substituting these parameters in place of n, m and τ and L, respectively.

Lemma 1 *[6] A hypergeometric function of two square matrix arguments* $\mathbf{A}$ *and* $\mathbf{B}$ *of the same*[2] *dimension* m *and with all distinct eigenvalues can be expressed as a ratio of determinants*

$${}_pF_q\left(\begin{matrix} x_1+m, \ldots, x_p+m \\ y_1+m, \ldots, y_q+m \end{matrix} \; \mathbf{A}, \mathbf{B} \right) = \frac{c_m \det \boldsymbol{F}}{\prod_{k<\ell}^{m}(a_k - a_\ell) \prod_{k<\ell}^{m}(b_k - b_\ell)} \quad (7)$$

with a_k *(resp.* b_k*) the* k*-th eigenvalue of* $\mathbf{A}$ *(resp.* $\mathbf{B}$*),*

$$c_m = \prod_{t=1}^{m-1} \left[\frac{\prod_{i=1}^p (x_i + t)}{t \prod_{j=1}^q (y_j + t)} \right]^{t-m}$$

and with the (i,j)*-th entry of* $\boldsymbol{F}$ *given by*

$$(\boldsymbol{F})_{i,j} = {}_pF_q\left(\begin{matrix} x_1+1, \ldots, x_p+1 \\ y_1+1, \ldots, y_q+1 \end{matrix} \; ; a_i b_j \right)$$

Lemma 2 *[7] Let* $\mathbf{F}$ *and* $\boldsymbol{G}$ *be two* $(n \times n)$ *matrices whose* (i,j)*-th entries are, respectively,* $(\mathbf{F})_{i,j} = f_j(w_i)$ *and* $(\boldsymbol{G})_{i,j} = g_j(w_i)$ *where* f_j *and* g_j*,* $j = 1, \ldots, n$*, are functions defined on* $\Re^+$*. Then, for* $b > a > 0$*,*

$$\int_{[a,b]^n} \det \mathbf{F} \det \boldsymbol{G} \prod_{k=1}^{n} h(w_k)\, dw_1 \ldots dw_n = n! \det \mathbf{A}$$

where h *is a function defined on* $\Re^+$ *and* $\mathbf{A}$ *is another* $(n \times n)$ *matrix whose* (i,j)*-th entry is*

$$\mathsf{A}_{i,j} = \int_a^b f_i(w) g_j(w) h(w)\, dw$$

Using these lemmas, we can now state the following

Theorem 1 *Let* ψ_i *be a function defined on* $\Re^+$ *and consider a random matrix whose unordered eigenvalues admit a joint distribution of the form*

$$f(\boldsymbol{\Lambda}) = K \prod_{k=1}^{m} \zeta(\lambda_k) \prod_{k<\ell}^{m} (\lambda_k - \lambda_\ell) \det(\boldsymbol{\Psi}(\boldsymbol{\Lambda})) \quad (8)$$

with $\boldsymbol{\Lambda} = diag\{\lambda_1, \ldots, \lambda_m\}$*,* K *a normalization constant and* $\boldsymbol{\Psi}(\boldsymbol{\Lambda})$ *a matrix whose* (i,j)*th entry is* $\psi_i(\lambda_j)$*. The marginal probability density function of the unordered eigenvalues can be written as*

$$f(\lambda) = \widetilde{K} \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda^{j-1} \zeta(\lambda) \psi_i(\lambda) \mathcal{D}(i,j) \quad (9)$$

where $\widetilde{K}$ *is a proper normalizing constant and* $\mathcal{D}(i,j)$ *is the* (i,j)*-cofactor of the* $(m \times m)$ *matrix* $\mathbf{A}$ *whose* (ℓ, k)*-th entry equals*

$$\int_0^\infty \lambda^{\ell-1} \zeta(\lambda) \psi_k(\lambda) d\lambda$$

[2]For a generalization of this formula to square matrices of different dimensions, or with nondistinct eigenvalues, see [6].

IV. NONASYMPTOTIC RESULTS

A. η Transforms

Definition 7 *[1] Given a nonnegative definite random matrix* $\mathbf{A}$*, its* η*-transform is*

$$\eta(\gamma) = \mathbb{E}\left[\frac{1}{1+\gamma X}\right] \tag{10}$$

where X *is a nonnegative random variable whose distribution is the marginal density distribution of an unordered eigenvalue of* $\mathbf{A}$ *while* γ *is a nonnegative real number.*

Example 1 *[10] The* η *transform of the marginal density distribution of an unordered eigenvalue of a central Wishart matrix (cf. Definition 1) is*

$$\eta(\gamma) = K\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\mathcal{D}(i,j)}{(-\gamma)^{\tau+j}}\left[-\frac{E_i\ -\frac{1}{\gamma\theta_i}}{e^{-\frac{1}{\gamma\theta_i}}} + \sum_{k=1}^{\tau+j-1}\frac{(k-1)!}{(-\gamma\theta_i)^{-k}}\right]$$

with θ_i *the* i*-th eigenvalue of* $\mathbf{\Theta}_m$*,* K *a proper normalizing constant given by* $K = \frac{\det(\mathbf{\Theta}_m)^{-n}}{m\prod_{\ell=1}^{m}(n-\ell)!\prod_{k<\ell}^{m}(\frac{1}{\theta_\ell}-\frac{1}{\theta_k})}$ *and* $\mathcal{D}(i,j)$ *the* (i,j)*-cofactor of the* $(m\times m)$ *matrix* $\boldsymbol{D}$ *such that*

$$(\mathbf{D})_{\ell,k} = \frac{(\tau+k-1)!}{\theta_\ell^{-\tau-k}}. \tag{11}$$

Example 2 *[10] The* η *transform of the marginal density distribution of an unordered eigenvalue of a central Wishart matrix of second kind (cf. Definition 2) is*

$$\eta(\gamma)=K\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\mathcal{D}(i,j)\theta_{\tau+i}^{\tau-1}}{(-\gamma)^{j}}\left\{\left[-\frac{E_i\ -\frac{1}{\gamma\theta_{\tau+i}}}{e^{-\frac{1}{\gamma\theta_{\tau+i}}}} + \sum_{p=1}^{j-1}\frac{(p-1)!}{(-\gamma\theta_{\tau+i})^{-p}}\right]\right.$$
$$\left.-\sum_{l=1}^{\tau}\sum_{k=1}^{\tau}\frac{(\mathbf{\Psi}^{-1})_{k,\ell}}{\theta_{\tau+i}^{1-k}\theta_\ell^{1-\tau}}\left[-\frac{E_i\ -\frac{1}{\gamma\theta_\ell}}{\exp\ -\frac{1}{\gamma\theta_\ell}} + \sum_{p=1}^{j-1}\frac{(p-1)!}{(-\gamma\theta_\ell)^{-p}}\right]\right\}$$

where $K = \frac{\det(\mathbf{\Psi})}{m\prod_{k<\ell}^{n}(\theta_\ell-\theta_k)\prod_{\ell=1}^{m-1}\ell!}$*,* $\mathbf{\Psi}$ *is the* $\tau\times\tau$ *matrix*

$$\mathbf{\Psi} = \begin{bmatrix} 1 & \theta_1 & \ldots & \theta_1^{\tau-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \theta_\tau & \ldots & \theta_\tau^{\tau-1}\end{bmatrix},$$

with θ_i *the* i*-th eigenvalue of* $\mathbf{\Theta}_n$ *and* $\mathcal{D}(i,j)$ *the* (i,j)*-cofactor of the* $(m\times m)$ *matrix* $\boldsymbol{D}$ *whose* (ℓ,k)*-th entry equals*

$$(k-1)!\left(\theta_{\tau+\ell}^{\tau+k-1} - \sum_{p=1}^{\tau}\sum_{q=1}^{\tau}(\mathbf{\Psi}^{-1})_{p,q}\theta_{\tau+\ell}^{p-1}\theta_q^{\tau+k-1}\right).$$

Example 3 *[10] The* η *transform of the marginal density distribution of an unordered eigenvalue of a noncentral Wishart matrix,* $\mathbf{W}\sim\mathcal{W}_m(n,\boldsymbol{M},\mathbf{I})$ *as per Definition 3, is*

$$\eta(\gamma) = \sum_{i=1}^{m}\sum_{j=1}^{m}\frac{K\mathcal{D}(i,j)}{(-\gamma)^{\tau+j}}\sum_{p=0}^{\infty}\frac{(\mu_i/\gamma)^p}{p![\tau+1]_p}\left[-\frac{E_i\left(-\frac{1}{\gamma}\right)}{e^{\left(-\frac{1}{\gamma}\right)}} + \sum_{k=1}^{\tau+j+p-1}\frac{(k-1)!}{(-\gamma)^{-k}}\right]$$

with μ_i *the* i*-th eigenvalue of* $\boldsymbol{M}$*,* $[a]_p = \Gamma(a+p)/\Gamma(a)$ *the Pochhammer's symbol of order* p *[5],* $K = \frac{e^{-\phi_m}}{m\,((n-m)!)^m\prod_{k<\ell}^{m}(\mu_\ell-\mu_k)}$ *and* $\mathcal{D}_{i,j}$ *the* (i,j)*-cofactor of the* $(m\times m)$ *matrix* $\boldsymbol{D}$ *whose* (ℓ,k)*-th entry is*

$$(\mathbf{D})_{\ell,k} = (\tau+k-1)!\ {}_1F_1(\tau+k,\tau+1,\mu_\ell).$$

Example 4 *[10] The* η *transform of the marginal density distribution of an unordered eigenvalue of a central F matrix (cf. Definition 4) is*

$$\eta(\gamma) = K\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\mathcal{D}(i,j)}{\gamma^{\tau+j}}\frac{{}_2F_1\ \ \chi,\tau+j,\chi+1;1-\frac{1}{\gamma\omega_i}\ \ (\tau+j-1)!}{\chi!/(L-j+1)!}$$

with $\chi=\tau+L+1$*,* ω_i *the* i*-th eigenvalue of* $\mathbf{\Theta}_m\mathbf{\Sigma}^{-1}$*,* K *given by*

$$K \ = \ \frac{\prod_{\ell=1}^{m-1}\left(\frac{n+L+\ell}{\ell}\right)^{\ell-m}}{m\prod_{k<\ell}^{m}(\frac{1}{\omega_\ell}-\frac{1}{\omega_k})}\prod_{\ell=1}^{m}\frac{\omega_\ell^{-n}(\tau+L-\ell)!}{(n-\ell)!(m-\ell)!(L-\ell)!} \tag{12}$$

and with $\mathcal{D}(i,j)$ *the* (i,j)*-cofactor of the* $(m\times m)$ *matrix whose* (ℓ,k)*-th entry equals*

$$\frac{(\tau+\ell-1)!(L-\ell)!}{(\tau+L)!}\omega_k^{\tau+\ell}.$$

Example 5 *[10] The* η *transform of the marginal density distribution of an unordered eigenvalue of a central F matrix of second kind (cf. Definition 5) is*

$$\eta(\gamma) = K\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\mathcal{D}(i,j)}{\gamma^j}\frac{(j-1)!(m-j+1)!}{(m+1)!}$$
$$\left[\omega_{\xi+i}^{\delta+m}{}_2F_1\left(m+1,j,m+2;1-\frac{\omega_{\xi+i}}{\gamma}\right)\right.$$
$$\left.-\sum_{k=1}^{\xi}\mathcal{R}_{i,k}\omega_k^{\delta+m}{}_2F_1\left(m+1,j,m+2;1-\frac{\omega_i}{\gamma}\right)\right] \tag{13}$$

with ω_i *the* i*-th eigenvalue of* $\mathbf{\Theta}_L$*,* $\mathcal{D}(i,j)$ *the* (i,j)*-cofactor of the* $(m\times m)$ *matrix* $\boldsymbol{D}$ *whose* (k,ℓ)*-th entry equals*

$$\frac{(\ell-1)!(m-\ell)!}{m!}\left(\omega_{\xi+k}^{\delta-\ell+m} - \sum_{q=1}^{\xi}\mathcal{R}_{k,q}\omega_q^{\delta-\ell+m}\right)$$

$$\mathcal{R}_{i,k} = \sum_{\ell=1}^{\delta}(\mathbf{\Psi}^{-1})_{\ell,k}\omega_{\xi+i}^{\ell-1} + \sum_{\ell=\delta+1}^{\xi}(\mathbf{\Psi}^{-1})_{\ell,k}\omega_{\xi+i}^{\ell+m-1},$$

$$\mathbf{\Psi} = \begin{bmatrix} 1 & \omega_1 & \ldots & \omega_1^{\delta-1} & \omega_1^{\delta+m} & \ldots & \omega_1^{L-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_\xi & \ldots & \omega_\xi^{\delta-1} & \omega_\xi^{\delta+m} & \ldots & \omega_\xi^{L-1}\end{bmatrix}$$

$$K = \frac{(-1)^{\delta m+L(L-1)/2}(m!)^{m-1}(m-1)!\det(\mathbf{\Psi})}{\prod_{k<\ell}^{L}(\omega_\ell-\omega_k)\prod_{\ell=1}^{m}(\ell-1)!(m-\ell)!} \tag{14}$$

$\delta = L-n$ *and* $\xi = L-m$.

Example 6 *[10] The* η *transform of the marginal density distribution of an unordered eigenvalue of a noncentral F matrix (cf. Definition 6) is*

$$\eta(\gamma)=K\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\mathcal{D}(i,j)}{\gamma^{\tau+j}}\sum_{p=0}^{\infty}\frac{\frac{{}_2F_1\ \ L+p+1,\tau+j+p,L+p+2;1-\frac{1}{\gamma}}{(L+p+1)!p![\tau+1]_p}}{(\phi_i/\gamma)^p[L+1]_p(\tau+j+p-1)!(L-\tau-j+1)!},$$

with ϕ_i the i-th eigenvalue of $\boldsymbol{M}$ and with $\mathcal{D}(i,j)$ the (i,j)-cofactor of the $(m \times m)$ matrix whose (ℓ,k)-th entry equals

$$\sum_{t=0}^{L-\tau} \binom{L-\tau}{t} \frac{(\tau+t+\ell-1)!(L-\tau-\ell)!}{\phi_k^{-t}[\tau+1]_t(t+L)!} {}_1F_1\left(\tau+t+\ell, t+L+1, \phi_k\right)$$

with K

$$K = \frac{e^{-\sum_i \mu_i}}{m\prod_{k<\ell}^m(\mu_k-\mu_\ell)} \prod_{\ell=1}^{m} \frac{(n+L-\tau-\ell)!}{(n-\ell)!(m-\ell)!(L-\tau-\ell)!} \prod_{\ell=1}^{m-1} \frac{(L+\ell)^{\ell-m}}{(\ell(\tau+\ell))^{\ell-m}}$$

B. Shannon Transforms

Definition 8 *[1] Given a nonnegative definite random matrix $\mathbf{A}$, its Shannon transform is defined as*

$$\mathcal{V}(\gamma) = \mathbb{E}[\log(1+\gamma X)] \tag{15}$$

where X is a nonnegative random variable whose distribution is the marginal density distribution of an unordered eigenvalue of $\mathbf{A}$ while γ is a nonnegative real number.

Example 7 *The Shannon transform of the marginal density distribution of an unordered eigenvalue of a central Wishart matrix (cf. Definition 1) is*

$$\mathcal{V}(\gamma) = K \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\mathcal{D}(i,j)(\tau+j-1)!}{\gamma^{\tau+j}} e^{\frac{1}{\gamma\theta_i}} \sum_{k=1}^{\tau+j} \frac{\Gamma\ \ k-\tau-j, \frac{1}{\gamma\theta_i}}{(\gamma\theta_i)^{-k}}$$

with K, θ_i and $\mathcal{D}(i,j)$ defined as in Example 1.

Example 8 *The Shannon transform of the marginal density distribution of an unordered eigenvalue of a central Wishart matrix of the second kind (cf. Definition 2) is*

$$\mathcal{V}(\gamma) = K \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\mathcal{D}(i,j)(j-1)!}{\gamma^{j}} \left[\frac{\theta_{\tau+i}^{\tau-1}}{e-\frac{1}{\gamma\theta_{\tau+i}}} \sum_{k=1}^{j} \frac{\Gamma\ \ k-j, \frac{1}{\gamma\theta_{\tau+i}}}{(\gamma\theta_{\tau+i})^{-k}} \right.$$
$$\left. - \sum_{\ell=1}^{\tau}\sum_{k=1}^{\tau} \frac{(\boldsymbol{\Psi}^{-1})_{k,\ell}\theta_{\tau+i}^{k-1}\theta_\ell^{\tau-1}}{e^{-\frac{1}{\gamma\theta_\ell}}} \sum_{k=1}^{j} \frac{\Gamma\ \ k-j, \frac{1}{\gamma\theta_\ell}}{(\gamma\theta_\ell)^{-k}} \right] \tag{16}$$

with K, θ_i, $\mathcal{D}(i,j)$ and $\boldsymbol{\Psi}$ as in Example 2.

Example 9 *[8] The Shannon transform of the marginal density distribution of an unordered eigenvalue of a noncentral Wishart matrix, $\mathbf{W} \sim \mathcal{W}_m(n, \boldsymbol{M}, \mathbf{I})$, as in Definition 3, is*

$$\mathcal{V}(\gamma) = K \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\mathcal{D}(i,j)}{e^{-\frac{1}{\gamma}}} \sum_{p=0}^{\infty} \frac{\mu_i^p(\tau+j+p-1)!}{p![\tau+1]_p\gamma^{\tau+j+p}} \sum_{k=1}^{\tau+j+p} \frac{\Gamma\left(k-\tau-j-p, \frac{1}{\gamma}\right)}{\gamma^{-k}}$$

with K, μ_i and $\mathcal{D}(i,j)$ defined as in Example 3.

Example 10 *[9] The Shannon transform of the marginal density distribution of an unordered eigenvalue of a central F matrix (cf. Definition 4) is*

$$\mathcal{V}(\gamma) = K \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\mathcal{D}(i,j)}{\gamma} \sum_{\nu=0}^{\tau+j-1} \frac{(\tau+j-1)!(L-j)!\omega_i^{\tau+j}}{(\tau+L)!(\gamma\omega_i)^\nu}$$
$$\frac{{}_2F_1\ \ \tau+L, \nu+1, \tau+L+1, 1-\frac{1}{\gamma\omega_i}}{(\tau+L-\nu)} \tag{17}$$

with K, ω_i and $\mathcal{D}(i,j)$ as in Example 4.

Example 11 *The Shannon transform of the marginal density distribution of an unordered eigenvalue of a central F matrix of second kind (cf. Definition 5) is*

$$\mathcal{V}(\gamma) = \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{K\mathcal{D}(i,j)}{\gamma} \left[\sum_{\nu=0}^{j-1} \frac{(j-1)!\,(m-j)!\,{}_2F_1\left(m,\nu+1,m+1,1-\frac{\omega_{\xi+i}}{\gamma}\right)}{(m-\nu)\,m!\,\gamma^\nu\,\omega_{\xi+i}^{-\delta-m+j-\nu}} \right.$$
$$\left. - \sum_{\nu=0}^{j-1}\sum_{k=1}^{\xi} \frac{(j-1)!(m-j)!{}_2F_1\left(m,\nu+1,m+1,1-\frac{\omega_k}{\gamma}\right)\mathcal{R}_{i,k}}{(m-\nu)\,m!\,\gamma^\nu\,\omega_k^{-\delta-m+j-\nu}} \right],$$

with δ, ξ, $\mathcal{D}(i,j)$, $\mathcal{R}_{i,k}$ and K as in Example 4.

Example 12 *[9] The Shannon transform of the marginal density distribution of an unordered eigenvalue of a noncentral F matrix (cf. Definition 6) is*

$$\mathcal{V}(\gamma) = K \sum_{i=1}^{m}\sum_{j=1}^{m} \frac{\mathcal{D}(i,j)}{\gamma} \sum_{p=0}^{\infty} \frac{\phi_i^p\,[L+1]_p}{\gamma^p\,p![\tau+1]_p} \sum_{\nu=0}^{\tau+p+j-1} \frac{(\tau+j+p-1)!(L-\tau-j)!}{(L+p)!}$$
$$\frac{{}_2F_1\left(L+p, \nu+1, L+1+p, 1-\frac{1}{\gamma}\right)}{(L+p-\nu)\gamma^\nu}$$

with K, ϕ_i and $\mathcal{D}(i,j)$ as in Example 6.

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