A GENERAL APPROACH TO THE ESTIMATION AND CONTROL OF LINEAR SYSTEMS WITH UNCERTAIN STATISTICAL

Sergio Verdú and Y. Vincent Poor
Department of Electrical Engineering and
Center for Telecommunications Research
University of Illinois at Urbana-Champaign
Urbana, Illinois 61801

ABSTRACT

The problem of minimum design of linear observers and regulators for linear time-varying multivariable stochastic systems with uncertain models of their second-order statistics is treated in this paper. Complexity general classes of allow-
ance covariance matrices and means of the process and observation noises is of the random initial condition are considered. A game formulation of the problem, adopting an minmax approach, is shown to hold for each of the filtering situations analyzed. Also conditions satisfied by the saddle-point solutions are derived.

I. INTRODUCTION

In this paper we will consider the design of minimum deterministic linear observers and regula-
tors for linear time-varying stochastic systems in which the second-order statistics of the stochastic processes involved are not known exactly. Although several previous studies have considered problems of this type, the problem was not treated in this generality, and the available results remain to deal with particular cases of uncertainty classes, for the covariance matrices $[3]-[4]$, or with the steady-state solution $[1],[2]$. In the present work, we consider the general case in which the white processes and observation noises can be cross-cor-
related and have non-zero means. Also, general types of uncertainty are allowed in their second-
order statistics. Although we deal here with discrete-time systems, parallel proofs for the continuous-time case of the main results presented here can be found in [6].

In minimum filtering two results are sought. The first is a minmax theorem that gives a saddle-
point solution to the minimax optimal game by showing that the minmax inequality holds, and that the solution is the optimal filter for the least favor-
able uncertainty class. The second result is a procedure to find the least favor-
able uncertainty class for general uncertainty classes. In Sections III and IV we will present these kinds of results for the linear observer problem for one-
dimensional state prediction ($h_k|x_{k-1}$) and the regulator problem for linear quadratic optimal control. In these cases the payoff functions do not allow the use of well-known game-theoretic minmax theorems (note that [2] considers error only that the pay-
off function is convex in the Kalman gain), and thus a recently developed general formulation of minmax robust filters cannot be directly used here.

II. PRELIMINARIES

Consider the following linear discrete-time system

$$x_{k+1} = A_k x_k + B_k u_k + W_k, \quad x_0 = x_{-1} = 0 \quad (2.1)$$

$$z_k = C_k x_k + v_k, \quad z_k \in \mathbb{R}^n \quad (2.2)$$

with $x_k$ an $n \times 1$ state vector, $u_k$ an $m \times 1$ control vector and $z_k$ an $r \times 1$ output vector. The initial state $(x_0)$ is a random vector with mean $\mu_0$ and covariance $\Sigma_0$, and $(W_k, V_k, Y_k)$ are random sequences, independent of the initial state, representing the process and observation noises, respectively, and having means $\mu_k$ and $\Sigma_k$ and covariance $\Sigma_k$.

$$\text{cov}(x_k, x_{k-1}) = \Sigma_k, \quad \text{cov}(z_k, z_{k-1}) = \Sigma_k \quad (2.2)$$

$$(W_k, V_k, Y_k) \sim \mathcal{N}(0, \Sigma_k), \quad \text{cov}(W_k, V_k) = 0 \quad (2.3)$$

We suppose that the means and the covariance matrices are known only to belong to some uncertainty classes $\mathcal{N}(\mu_k, \Sigma_k^0)$. We will apply the aforementioned results for minimum robust filtering.

For the sake of clarity and because of the differing nature of results for various cases, we will present the cases of uncertain covariances separately before dealing with the most general case. In this particular situation, we will prove that the sought-after robust filter is the optimal filter for the least favorable set of covariances, and we will derive some of the equations fulfilled by such covariances for general uncertainty classes. For the uncertain mean case we will use the concept of worst minmax, and will give the equations describing the robust filter here as well.

In the remainder of this section, we outline the minimum robust filtering results [6] that will be needed in the sequel.

For an arbitrary minimum filtering gain $(P_k, K_k)$, we say that $h_k$ is a robust filter if

$$h_k = \text{arg min}_{h \in \mathcal{H}} \sup_{p \in \mathcal{P}} \text{inf}_{p \in \mathcal{P}} (P_k - h K_k) \quad (2.4)$$

From now on, we will make the following assumptions:

$\mathcal{P} = \{p \in \mathcal{F} : \text{inf}_{h \in \mathcal{H}} \text{sup}_{p \in \mathcal{P}} (P_k - h K_k) \leq \epsilon \}$

$\mathcal{H} = \{h \in \mathcal{H} : \text{inf}_{p \in \mathcal{P}} \text{sup}_{h \in \mathcal{H}} (P_k - h K_k) \leq \epsilon \}$

$\mathcal{F} = \{f \in \mathcal{F} : \text{inf}_{h \in \mathcal{H}} \text{sup}_{p \in \mathcal{P}} (P_k - h K_k) \leq \epsilon \}$

$\mathcal{K} = \{K_k \in \mathcal{K} \text{ is any set such that, for every } p \in \mathcal{P} \text{ we have :} \}$

$$\text{inf}_{h \in \mathcal{H}} \text{sup}_{p \in \mathcal{P}} (P_k - h K_k) \leq \epsilon \quad (2.5)$$

$\mathcal{K} = \{K_k \in \mathcal{K} : \}$

$\text{sup}_{h \in \mathcal{H}} \text{inf}_{p \in \mathcal{P}} (P_k - h K_k) \leq \epsilon \quad (2.6)$

$\text{For such a } \mathcal{K}, \text{ we define a function } f^* \text{ on } (P_k, \mathcal{H}, \mathcal{P})$,

$$f^* \equiv \text{inf}_{h \in \mathcal{H}} \text{sup}_{p \in \mathcal{P}} (P_k - h K_k) \quad (2.7)$$

and define $\mathcal{K}_f$ to be the least favorable reverting

$$\text{point } (\mathcal{K}_f, f) = \epsilon \quad (2.8)$$

330
\[ p = \text{arg max } \sigma^2(y) \quad \text{ s.t. } y \in \mathcal{Q} \]

Assumption (1) implies that a least favorable operating point exists for \((\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4)\). We see that \((h_1, h_2) \in \mathcal{X}_1 \times 0 \) is a regular point \( \iff \) for \((\mathcal{X}_1, \mathcal{X}_2) \rightarrow \mathcal{X}_2\), if for every \( \eta \in \mathcal{Q} \) and for every \( x \in \mathcal{X}_1 \) and every \( y \in \mathcal{Q} \), we have

\[ h_1[x(x) = 0] = h_2[0, 0, 0, 0] / \alpha = 0 \]  

(1.10)

Equipped with these definitions we state the following results.

**Lemma 1.** Suppose that \((h_1, h_2) \) is concave in \( \eta \) for every \( \eta \in \mathcal{Q} \). If a least favorable operating point \( \eta \in (\mathcal{X}_1, \mathcal{X}_2) \rightarrow \mathcal{X}_1 \) and its optimum filter \( h_1 \) is a regular point, then \( h_1 \) is a robust filter for \( (\mathcal{X}_2, \mathcal{X}_3) \).

**Lemma 2.** Suppose that \( \eta \) is a continuous product of sets \( \mathcal{Q} \times \mathcal{Q} \), and that the positivity function can be put as

\[ \delta(h_1, \delta_1, \ldots, \delta_L) = \sum \delta_1 \left( \begin{array}{c} \delta_1(x_1, \delta_1, \ldots, \delta_L) \\ \vdots \\ \delta_L(x_L, \delta_1, \ldots, \delta_L) \end{array} \right) \eta \left( \begin{array}{c} y_1 \\ \vdots \\ y_L \end{array} \right) \]  

then \( \eta \) underestimates \( \eta \) in each of its arguments.

**Lemma 3.** Suppose, for \( i = 1, \ldots, L \)

\[ \eta_i \left( \begin{array}{c} x_i \end{array} \right) = \text{arg max } \eta \left( \begin{array}{c} x_i \end{array} \right) \quad \text{s.t. } x_i \in \mathcal{Q}_i \]

(1.11)

If \( h_i \) is a solution of the equation

\[ h_i = \text{arg min } \eta \left( \begin{array}{c} x_i \end{array} \right) \in \mathcal{X}_i \]

then \( h_i \) is a robust filter for \( (\mathcal{X}_1, \mathcal{X}_2) \).

**Lemma 4.** Under the assumptions of Lemma 2 and 1, \( P_s = \left( \begin{array}{c} (x_i, \delta_1, \ldots, \delta_L) \\ J \end{array} \right) \) is a least favorable operating point \( \eta \in (\mathcal{X}_1, \mathcal{X}_2) \rightarrow \mathcal{X}_1 \) \( \iff \) only if \( h_i \) is a solution to \( (1.11) \).

These results express the core of the aforementioned general formulation for various robust filtering algorithms. Their proofs can be found in [6].

III. THE NOISE OBSERVER PROBLEM

A. Formulation

The optimal linear observer of the state \((2.1)\) is the solution to

\[ \text{arg min } \eta \left( \begin{array}{c} x_i \end{array} \right) \in \mathcal{X}_i \]  

(1.12)

where \( \mathcal{X}_i \) is the set of linear filters of appropriate dimensionality with state \( x_i \left( \begin{array}{c} x_i \end{array} \right) \in \mathcal{X}_i \), \( f_i \left( \begin{array}{c} x_i \end{array} \right) = x_i \left( \begin{array}{c} x_i \end{array} \right) + b_i \left( \begin{array}{c} x_i \end{array} \right) + c_i \left( \begin{array}{c} x_i \end{array} \right) \), and \( b_i \) is a positive semidefinite weighting matrix. When the second-order statistics of the stochastic system \((3.1-3)\) are known, the solution for every time \( k \) is given by the Kalman filter:

\[ x_i[k] = \text{arg max } - \frac{1}{2} h_i[k-1] \left( \begin{array}{c} x_i[k-1] \end{array} \right) + h_i[k-1] \left( \begin{array}{c} x_i[k-1] \end{array} \right) + \eta \left( \begin{array}{c} x_i[k-1] \end{array} \right) + \text{arg max } \eta \left( \begin{array}{c} x_i[k-1] \end{array} \right) 

(3.2)

Subtracting (3.2) from (2.1) we get the error

\[ \eta = \text{arg max } \sigma^2(y) \quad \text{s.t. } y \in \mathcal{Q} \]

(2.9)

\[ \text{arg min } \eta \left( \begin{array}{c} x_i \end{array} \right) \in \mathcal{X}_i \]  

(2.10)

where \( \mathcal{X}_i \) is the set of linear filters of appropriate dimensionality with state \( x_i \left( \begin{array}{c} x_i \end{array} \right) \in \mathcal{X}_i \), \( f_i \left( \begin{array}{c} x_i \end{array} \right) = x_i \left( \begin{array}{c} x_i \end{array} \right) + b_i \left( \begin{array}{c} x_i \end{array} \right) + c_i \left( \begin{array}{c} x_i \end{array} \right) \), and \( b_i \) is a positive semidefinite weighting matrix. When the second-order statistics of the stochastic system \((3.1-3)\) are known, the solution for every time \( k \) is given by the Kalman filter:

\[ x_i[k] = \text{arg max } - \frac{1}{2} h_i[k-1] \left( \begin{array}{c} x_i[k-1] \end{array} \right) + h_i[k-1] \left( \begin{array}{c} x_i[k-1] \end{array} \right) + \eta \left( \begin{array}{c} x_i[k-1] \end{array} \right) + \text{arg max } \eta \left( \begin{array}{c} x_i[k-1] \end{array} \right) 

(3.2)

Subtracting (3.2) from (2.1) we get the error

\[ \eta = \text{arg max } \sigma^2(y) \quad \text{s.t. } y \in \mathcal{Q} \]

(2.9)

\[ \text{arg min } \eta \left( \begin{array}{c} x_i \end{array} \right) \in \mathcal{X}_i \]  

(2.10)

where \( \mathcal{X}_i \) is the set of linear filters of appropriate dimensionality with state \( x_i \left( \begin{array}{c} x_i \end{array} \right) \in \mathcal{X}_i \), \( f_i \left( \begin{array}{c} x_i \end{array} \right) = x_i \left( \begin{array}{c} x_i \end{array} \right) + b_i \left( \begin{array}{c} x_i \end{array} \right) + c_i \left( \begin{array}{c} x_i \end{array} \right) \), and \( b_i \) is a positive semidefinite weighting matrix. When the second-order statistics of the stochastic system \((3.1-3)\) are known, the solution for every time \( k \) is given by the Kalman filter:

\[ x_i[k] = \text{arg max } - \frac{1}{2} h_i[k-1] \left( \begin{array}{c} x_i[k-1] \end{array} \right) + h_i[k-1] \left( \begin{array}{c} x_i[k-1] \end{array} \right) + \eta \left( \begin{array}{c} x_i[k-1] \end{array} \right) + \text{arg max } \eta \left( \begin{array}{c} x_i[k-1] \end{array} \right) 

(3.2)

Subtracting (3.2) from (2.1) we get the error
Although all the matrices involved are possibly time-varying, the reminder of the discussion will omit their explicit dependence on time in order to simplify the notation except when this could be ambiguous.

In order to prove the minimum theorem we will make use of the following sensitivity result.

**Lemma 4.** Suppose \( D^* \) is a convex set and \( (R_i, r_i, p_{ij}) \) and \( G = (G_{ij}) \) both belonging to \( D^* \). Let \( G^*_{ij}/ \alpha \) be the state estimate of the Kalman filter designed for \( G^*_{ij} \) when \( \alpha \) describes the true statistics. With \( 0 < \delta < 1 \) and \( \alpha \) we have for all \( i, j \),

\[
\sum_{k=0}^{\infty} \alpha_{ij}(k+1)\frac{\alpha_{ij}(k)}{\alpha_{ij}(k+1)} = 0 \tag{1.14}
\]

**Proof.** \( \delta_k \) and \( \delta_{k-1} < \delta \) (1.8) and (1.11) for the optimum Kalman gains \( K^* \) and \( K^* \) designed for \( \alpha \) and \( \alpha \) respectively. Note that in both cases the least mean of covariances \( E(F^2, \alpha, \alpha) \) is present. Therefore, we have

\[
\sum_{k=0}^{\infty} \alpha_{ij}(k+1)\frac{\alpha_{ij}(k)}{\alpha_{ij}(k+1)} = 0 \tag{1.15}
\]

where in order to get the last equality the expression (1.12) for \( \alpha \) has been taken into account. It is worth noting that (1.15) gives the variation of the error variance due to an arbitrary modification (up to here the value of \( \alpha^* \) has not been used) of the optimum Kalman gains of \( \alpha \). Hence

\[
\sum_{k=0}^{\infty} \alpha_{ij}(k+1)\frac{\alpha_{ij}(k)}{\alpha_{ij}(k+1)} \text{ exists for every } \alpha \text{ and for all } \alpha \in \alpha^* \text{ (see 4.1)}, \text{ is the limit (1.12) reduces to a homogeneous equation, and, since } E(0,0) = 0, \text{ we have the sought-after result.}
\]

If, when the second-order statistics are uncertain, we use the weighted square error (3.11) as the penalty function, then it can be shown that the least favorable covariance sequence can be derived for every \( k \). Thus, it is possible that the minimax filter is given by a recursion that does not solve the problem for previous times, and therefore we are forced to build a different filter for every \( k \). The feasibility of this solution leads to the consideration of the penalty function (cf. [4]).

\[
L(R, \alpha) = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} 2\alpha_{ij}(k+1) \frac{\alpha_{ij}(k)}{\alpha_{ij}(k+1)} \tag{3.16}
\]

which has the same minimizing filter as (3.11) when the statistics model is known. Furthermore, note that the penalty function of (3.11) can be considered a particular case of (3.16).

We assume that \( C \) is an uncertainty class such that there exists a convex set \( C^* \subset C \) that fulfills (2.6). Given that the means are known and \( X \) is the least favorable covariance sequence, the optimality condition (3.10) follows from Lemma 6 and Lemma 1 and results in the following theorem.

**Theorem 2.** For the problem of one-step state prediction the Kalman filter for the least favorable collection of covariances for one-step state prediction is robust for the class \( C \).

This result reduces the original minimax design problem to the search for least favorable sets of covariances. In this point the application of Lemma 3 has proven to be successful in dealing with important uncertainty classes in other contexts [10]. In other cases, we can define

\[
\gamma(k) = \sum_{k=0}^{\infty} \alpha_{ij}(k+1) \frac{\alpha_{ij}(k)}{\alpha_{ij}(k+1)} \tag{3.17}
\]

where \( \gamma(k) = \beta(k) \gamma(0) \) if \( \beta > 0 \), and \( \gamma(0) > 0 \) is the state transition matrix of the fundamental matrix \( A(C, C) \). Using this definition and interchanging the order of summation, (3.17) reduces to

\[
\gamma(k) = 2 \gamma(k) \sum_{k=0}^{\infty} \alpha_{ij}(k+1) \frac{\alpha_{ij}(k)}{\alpha_{ij}(k+1)} = \sum_{k=0}^{\infty} \gamma(k+1) \sum_{k=0}^{\infty} \alpha_{ij}(k+1) \frac{\alpha_{ij}(k)}{\alpha_{ij}(k+1)}
\]

where

\[
\gamma(k) = \arg \max_{\gamma(0)} \gamma(k) \sum_{k=0}^{\infty} \alpha_{ij}(k+1) \frac{\alpha_{ij}(k)}{\alpha_{ij}(k+1)} \tag{3.18}
\]

In the above expression, the following results follow straightforwardly.

**Theorem 2.** If \( C \) can be put as the Cartesian product of two uncertainty classes \( A(C, C) \), then

\[
\sum_{k=0}^{\infty} \alpha_{ij}(k+1) \frac{\alpha_{ij}(k)}{\alpha_{ij}(k+1)} \text{ is the least favorable collection of covariances for one-step state prediction if and only if the following inequality is satisfied.}
\]

\[
\sum_{k=0}^{\infty} \alpha_{ij}(k+1) \frac{\alpha_{ij}(k)}{\alpha_{ij}(k+1)} \leq \gamma(k) \sum_{k=0}^{\infty} \alpha_{ij}(k+1) \frac{\alpha_{ij}(k)}{\alpha_{ij}(k+1)} \tag{3.19}
\]

where

\[
\gamma(k) = \arg \max_{\gamma(0)} \gamma(k) \sum_{k=0}^{\infty} \alpha_{ij}(k+1) \frac{\alpha_{ij}(k)}{\alpha_{ij}(k+1)} \tag{3.20}
\]

and

\[
\sum_{k=0}^{\infty} \alpha_{ij}(k+1) \frac{\alpha_{ij}(k)}{\alpha_{ij}(k+1)} \leq \gamma(k) \sum_{k=0}^{\infty} \alpha_{ij}(k+1) \frac{\alpha_{ij}(k)}{\alpha_{ij}(k+1)} \tag{3.21}
\]

where

\[
\gamma(k) = \arg \max_{\gamma(0)} \gamma(k) \sum_{k=0}^{\infty} \alpha_{ij}(k+1) \frac{\alpha_{ij}(k)}{\alpha_{ij}(k+1)} \tag{3.22}
\]

and

\[
\sum_{k=0}^{\infty} \alpha_{ij}(k+1) \frac{\alpha_{ij}(k)}{\alpha_{ij}(k+1)} \leq \gamma(k) \sum_{k=0}^{\infty} \alpha_{ij}(k+1) \frac{\alpha_{ij}(k)}{\alpha_{ij}(k+1)} \tag{3.23}
\]

Notice that while the optimal observer for fixed covariances is independent of the error weighting matrix \( Q \), this is not the case here where the least favorable covariances may depend on \( Q \). Moreover, the least favorable covariances
where

\[ P(\theta|X) = E[\theta^T R^{-1} \theta^T X] \]

\[ \theta = \sum \delta X \theta^T \delta X \delta^T \]

Here \( X \) is the state of the system (2.1-2) with zero-mean process noise, \( X \) is the set of linear filters with input \( x_k \) and output \( u_k \), and where \( P \) and \( Q_k \) are positive semidefinite and \( Z_k \) is positive definite. The optimal filter when the noise covariances are fixed is given by

\[ \hat{X} = \hat{X}_k - \hat{X}_k \cdot X \]

\[ \hat{X} = (x_k + x_k) (x_k + x_k)^T + Z_k \]

and \( \hat{X} \) is the optimal predictor estimate given by (3.2), (3.4), (3.12), (3.13). This is the consequence of the separation principle of stochastic control which cancels the optimality of the feedback of the state estimate with the same gain as that in the human state case. Since the feedback gain \( P \) does not depend on the statistics of the noise, it should be expected that when there are unknown the optimal value of \( G \) is unchanged, and thus the control and extra estimation problems can still be solved separately. We will prove this statement rigorously and show that in general the state estimates are not given by the minimum observer of section III. These phenomena were demonstrated for the situation of a noisy-state, continuous time control by Levine et al. [4].

In the following sections, we will use the following identity (4.3).

\[ \phi^2 = (x_k + x_k) (x_k + x_k)^T + Z_k \]

Then the Kalman filter given by (3.12-25) and with statistical means known for some \( k \leq N \) is

IV. THE KALMAN-ESTIMATION PROBLEM

A. Formulation

The optimal regulator for linear quadratic optimal control is the solution to

\[ \arg \min J(\theta, u) = 0 , \]

\[ K \subset \mathbb{K} \]

(4.1)
\[ J(\mathbf{u}, \mathbf{x}) = \sum_{i=0}^{n-1} x_i^2 + \sum_{j=0}^m \sum_{i=0}^n \tau_i (x_i x_j) \]

\[ = \sum_{i=0}^{n-1} x_i^2 \]  
\[ \text{w}_0 \]

(4.5)

Thus, we have the following result which is analogous to Theorem 1 (see [6]).

Theorem 4. The regulator consisting of the feedback for complete state information of the state estimates produced by the Kalman filter for the least favorable collection of covariances for linear quadratic optimal control is robust for the game \( \mathcal{G}(x, u) \).

Note that, as the following theorem states, the least favorable set of covariances for this problem may be different from that for state estimation, hence our previous assertion that the Kalman filter for the minimum regulator is not generally a minimum observer. Using the same kind of manipulation that led to (2.18), the penalty function in this case can be expressed as

\[ J(\mathbf{u}, \mathbf{x}) = \frac{1}{2} \mathbf{x}^T (\mathbf{K}_0 + \mathbf{M})^{-1} \mathbf{x} + \mathbf{u}^T (\mathbf{K}_0 + \mathbf{M})^{-1} \mathbf{u} \]

\[ \text{w}_0 \]  

(4.10)

Lemma 1 can be applied to this expression to yield the following result.

Theorem 5. Under the same assumptions and definitions as in Theorem 1, \( \mathcal{G}(x, u, \mathbf{K}_0) \) is the least favorable collection of covariances for linear quadratic optimal control if and only if (2.23-25) and the following are satisfied:

\[ R = \arg \max \left\{ \mathbf{K}_0 + \mathbf{M} \right\} \]

\[ \mathbf{x}^0 \in \mathcal{D} \]  

(4.11)

\[ \mathbf{x}_0^0 = \arg \max \left\{ \mathbf{K}_0 + \mathbf{M} \right\} \]

\[ \mathbf{K}_0 + \mathbf{M} \]  

(4.12)

\[ \mathbf{x}_0^0 = \arg \max \left\{ \mathbf{K}_0 + \mathbf{M} \right\} \]

\[ \mathbf{K}_0 + \mathbf{M} \]  

(4.13)

\[ \mathbf{x}_0^0 = \arg \max \left\{ \mathbf{K}_0 + \mathbf{M} \right\} \]

\[ \mathbf{K}_0 + \mathbf{M} \]  

(4.14)

C. Uncertainty in Means and Covariances

We now consider the case, analogous to the treated in subsection 2.2.3, in which the means of the initial state and of the observation noise are only uncertain. In this situation an appropriate penalty function is, using both minimums,

\[ J(\mathbf{u}, \mathbf{x}) = \frac{1}{2} \mathbf{x}^T (\mathbf{K}_0 + \mathbf{M})^{-1} \mathbf{x} + \mathbf{u}^T (\mathbf{K}_0 + \mathbf{M})^{-1} \mathbf{u} \]

\[ \text{w}_0 \]

(4.15)

where \( \mathbf{x} = \text{diag}(x_0^0, \ldots, x_0^0) \).

Theorem 6. The regulator consisting of the feedback for complete state information of the state estimates produced by the Kalman filter given by (3.25-25) and (4.11-14) and with means \( \mathbf{x}_0^0 \) that satisfy

\[ \mathbf{x}_0^0 = \arg \max \left\{ \mathbf{K}_0 + \mathbf{M} \right\} \]

\[ \mathbf{K}_0 + \mathbf{M} \]  

(4.16)

is robust for the game \( \mathcal{G}(x, u) \).

V. CONCLUSIONS

The application of the general formulation of minimax robust filtering has allowed us to present minimax theorems for state estimation and for quadratic control under general classes of uncertainty in the second-order statistical of the linear stochastic system. These results represent a generalization of earlier works [11,12] derived to the deterministic case for invariant autoparametric systems with uncorrelated noise. The minimax theorems state that the minimax filter is the optimal for the least favorable collection of noise covariances; this saddlepoint property implies a very attractive feature, namely, that when the actual covariances differ from the least favorable case, the performance of the robust filter is upgraded. Sets of necessary and sufficient conditions are given for the least invariability of the four types of covariance matrices involved. These conditions lend themselves to a recursive solution, and are applied successfully to deviation classes of practical interest. It is worthwhile to endorse the fact that under the same type of uncertainty the worst case covariances do not necessarily coincide with the types of filters treated here and that the observer used for the minimax regulator is not, in general, a minimum prediction state estimator. Given a particular expected deviation behavior, the crossover between the decrease of performance (with respect to the original filter) in the nominal model and the improvement of the worst case, should be assessed for the practical application of the minimax filters presented. The null minimax approach has been used to deal with unknown means, and if the uncertainty uncertainty classes can also be modeled using models that are more likely to occur, then a corresponding soft minimax solution is straightforward from the previous results.

ACKNOWLEDGMENT

This research was supported in part by the Army Research Office under contract DAAH04-68-K-0067 and in part by the office of Naval Research under contract N00014-61-A-0054.
REFERENCES


