# A GENERAL APPROACH TO THE ESTIMATION AND CONTROL OF LINEAR SYSTEMS WITH UNCERTAIN STATISTICS

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ABSTRACT

The problem of minimax design of linear observers and regulators for linear time-varying multivariable stochastic systems with uncertain models of their second-order statistics is treated in this paper. Completely general classes of allowable covariance matrices and means of the process and observation noises and of the random initial condition are considered. A game formulation of the problem is adopted and minimax theorems are shown to hold for each of the filtering situations analyzed. Also conditions satisfied by the saddle-point solutions are derived.

## I. INTRODUCTION

In this paper we will consider the design of minimax deterministic linear observers and regulators for linear time-varying stochastic systems in which the second-order statistics of the stochastic processes involved are not known exactly. Although several previous studies have considered problems of this type, the problem has not been treated in full generality, and the available related results deal with particular cases of uncertainty classes for the covariance matrices [3]-[5], or with the steady-state solution [1],[2]. In the present work, we consider the general case in which the white process and observation noises can be cross-correlated and can have non-zero means. Also, general types of uncertainties are allowed in their second-order statistics. Although we deal here with discrete-time systems, parallel proofs for the continuous-time cases of the main results presented here can be found in [6].

In minimax filtering two results are sought. The first is a minimax theorem that gives a saddle-point solution to the corresponding game by showing that the minimax equality holds, and thus that the solution is the optimal filter for the least favorable among the uncertain situations. The second desired result is a procedure to find these least favorable situations for general uncertainty classes. In Sections III and IV we will present these kinds of results for the linear observer problem for one-step state prediction ($\hat{x}_{k/k-1}$) and the regulator problem for linear quadratic optimal control. In these cases the payoff functions do not allow the use of well-known game-theoretic minimax theorems (note that [12] considers erroneously that the payoff function is convex in the Kalman gain), and thus a recently developed general formulation of minimax robust filtering [6] is employed here.

## II. PRELIMINARIES

Consider the following linear discrete-time system

$$x_{k+1} = A_k x_k + B_k u_k + w_k + \bar{w}_k; \quad k_o \le k \le N-1 \quad (2.1)$$

$$z_k = C_k x_k + v_k + \bar{v}_k \quad ; \quad k_o \le k \le N-1 \quad (2.2)$$

with $x_k$ and $n \times 1$ state vector, $u_k$ an $m \times 1$ control vector and $z_k$ an $r \times 1$ output vector. The initial state ($x_{k_o}$) is a random vector with mean $m_o$ and covariance $\Sigma_o$, and $(w+\bar{w}_k)$, $(v_k+\bar{v}_k)$ are random sequences, independent of the initial state, representing the process and observations noises, respectively, and having means $\bar{w}_k$ and $\bar{v}_k$ and covariance

$$\mathrm{cov}\left(\begin{bmatrix} w_k \\ v_k \end{bmatrix}, \begin{bmatrix} w_\ell \\ v_\ell \end{bmatrix}\right) = \begin{bmatrix} \Xi_k & \Psi_k \\ \Psi_k^T & \theta_k \end{bmatrix} \delta_{k\ell}; \; k \ge k_o \quad \ell \ge k_o \quad (2.3)$$

We suppose that the means and the covariance matrices are known only to belong to some uncertainty classes, $\mathrm{col}(m_o, \bar{w}_{k_o}, \ldots, \bar{w}_{N-2}, \bar{v}_{k_o}, \ldots, \bar{v}_{N-2}) = y \in \mathcal{M}$, $(\Xi_k, \Psi_k, \theta_k, \Sigma_o)_{k=k_o}^{N-1} = \Lambda \in \mathcal{C}$, such that $\Xi_k$ and $\Sigma_k$ are positive semidefinite, $\theta_k$ is positive definite, and $\Psi_k$ is congruent with $\Xi_k$ and $\theta_k$. In the presence of these uncertainties the robust observer and regulator problems will be solved in a minimax sense, i.e., our goal is to find

$$h_R = \arg \min_{h \in \mathcal{K}} \sup_{(y,\Lambda) \in \mathcal{M} \times \mathcal{C}} \delta(h,(y,\Lambda)) \quad (2.4)$$

where $\mathcal{K}$ is some class of allowable filters and $\delta(\cdot,\cdot)$ is a penalty function. In order to solve this game $(\mathcal{M} \times \mathcal{C}, \mathcal{K}, \delta)$ we will apply the aforementioned results for minimax robust filtering.

For the sake of clarity and because of the differing nature of results for various cases, we will present the case of uncertain covariances separately before dealing with the most general case. In this particular situation we will prove that the sought-after robust filter is the optimal filter for the least favorable set of covariances, and we will derive sets of equations fulfilled by such covariances for general uncertainty classes. For the uncertain means case we will use the concept of soft minimax, and will give the equations describing the robust filter here as well.

In the remainder of this section, we outline the minimax robust filtering results [6] that will be needed in the sequel.

For an arbitrary minimax filtering game $(P, \mathcal{K}, \delta)$, we say that $h_R$ is a robust filter if

$$h_R = \arg \min_{h \in \mathcal{K}} \sup_{p \in P} \delta(h,p) . \quad (2.5)$$

From now on, we will make the following assumptions:

i) There exists a convex set $Q \supset P$ such that

$$\sup_{p \in P} \inf_{h \in \mathcal{K}} \delta(h,p) = \max_{p \in Q} \inf_{h \in \mathcal{K}} \delta(h,p) \quad (2.6)$$

ii) The set $\mathcal{K}^*$ is any set such that, for every $p \in Q$, we have

$$\inf_{h \in \mathcal{K}^*} \delta(h,p) = \inf_{h \in \mathcal{K}} \delta(h,p) . \quad (2.7)$$

For such a set $\mathcal{K}^*$, we define a function $\delta^*$ on Q by

$$\delta^*(p) = \inf_{h \in \mathcal{K}^*} \delta(h,p) , \quad (2.8)$$

and define $p_L$ to be the least favorable operating point -- for $(Q, \mathcal{K}^*, \delta)$ -- if

$$p_L = \arg \max_{p \in Q} \delta^*(p) . \quad (2.9)$$

Assumption (i) implies that a least favorable operating point exists for $(Q,\mathcal{H}^*,\delta)$.

We say that $(h_L,p_L) \in \mathcal{H}^* \times Q$ is a regular pair -- for $(Q,\mathcal{H}^*,\delta)$ -- if, for every $p \in Q$ such that $(1-\alpha)p_L + \alpha p = X_\alpha \in Q$ for all $0 \le \alpha \le 1$, we have

$$\lim_{\alpha \to 0^+} [\delta^*(X_\alpha) - \delta(h_L,X_\alpha)]/\alpha = 0 . \quad (2.10)$$

Equipped with these definitions we state the following results.

Lemma 1. Suppose that $\delta(h,\cdot)$ is concave in Q for every $h \in \mathcal{H}^*$. If a least favorable operating point -- for $(Q,\mathcal{H}^*,\delta)$ --, $p_L$, and its optimum filter, $h_L$, form a regular pair, then $h_L$ is a robust filter for $(P,\mathcal{H},\delta)$.

Lemma 2. Suppose that Q is a Cartesian product of sets, $Q = Q_1 \times \ldots \times Q_k$ and that the penalty function can be put as

$$\delta(h,(p_1,\ldots,p_k)) = F(f_1(h,p_1),\ldots,f_k(h,p_k)), \quad (2.11)$$

with F nondecreasing in each of its arguments. Denote, for $i = 1,\ldots,k$

$$p_i^*(h) = \arg \max_{p_i \in Q_i} f_i(h,p) . \quad (2.12)$$

If $h_L$ is a solution of the equation

$$h_L = \arg \min_{h \in \mathcal{H}^*} \delta(h,p_1^*(h_L),\ldots,p_k^*(h_L)) \quad (2.13)$$

then $h_L$ is a robust filter for $(P,\mathcal{H},\delta)$.

Lemma 3. Under the supositions of Lemmas 1 and 2, $p_L = (p_1^*(h_L),\ldots,p_k^*(h_L))$ is a least favorable operating point -- for $(Q,\mathcal{H}^*,\delta)$ -- if and only if $h_L$ is a solution to (2.13).

These results represent the core of the aforementioned general formulation for minimax robust filtering and their proofs can be found in [6].

## III. THE ROBUST OBSERVER PROBLEM

### A. Formulation

The optimal linear observer of the state of (2.1) is the solution to

$$\arg \min_{H \in \mathcal{H}} E[e_{k/k-1}^T Q_k e_{k/k-1} | U_{k-1}, Z_{k-1}] \quad (3.1)$$

where $\mathcal{H}$ is the set of linear filters of appropriate dimensionality with state $\hat{x}_{k/k-1}, e_{k/k-1} \triangleq x_k - \hat{x}_{k/k-1}$, $U_{k-1} = \{u_{k_o},\ldots,u_{k-1}\}$, $Z_{k-1} = \{z_{k_o},\ldots,z_{k-1}\}$, and $Q_k$ is a positive semidefinite weighting matrix. When the second-order statistics of the stochastic system (2.1-3) are known, the solution for every time k is given by the Kalman filter:

$$\hat{x}_{k+1/k} = A_k\hat{x}_{k/k-1} + B_k u_k + \bar{w}_{Lk} + K_k(z_k - \bar{v}_{Lk} - C_k\hat{x}_{k/k-1}) \quad (3.2)$$

$$\hat{x}_{k_o/k_o-1} = m_{Lo} . \quad (3.3)$$

Subtracting (3.2) from (2.1) we get the error dynamical equations for arbitrary $K_k, E_k, m_{Lo}, \bar{w}_{Lk}, \bar{v}_{Lk}$

$$e_{k+1/k} = (A_k - K_kC_k)e_{k/k-1} - K_k v_k + w_k + \bar{w}_k - \bar{w}_{Lk} - K_k(\bar{v}_k - \bar{v}_{Lk}) \quad (3.4)$$

$$e_{k_0/k_o-1} = x_{k_o} - m_{Lo} \quad (3.5)$$

Denoting $\Sigma^a_{k/k-1} = \mathrm{var}[e_{k/k-1}]$, the vector of means in the estimator by $y_L$ (defined analogously to y), and the state-transition matrix of the fundamental matrix $(A - KC)$ by $\Phi(\cdot,\cdot)$ and taking into account that the white sequences $w_k$, $v_k$ are independent of $e_{k/k-1}$, (3.4-5) lead to the following expressions for the error:

$$E[e_{k/k-1}e^T_{k/k-1}] = \Sigma^a_{k/k-1} + \Phi_k(IK)(y-y_L)(y-y_L)^T(IK)^T\Phi_k^T \quad (3.6)$$

where

$$\Sigma^a_{k/k-1} = \Phi(k,k_o)\Sigma_o\Phi^T(k,k_o) + \sum_{j=k_o}^{k-1} \Phi(k,j+1)M_j\Phi^T(k,j+1), \quad (3.7)$$

$$M_k = [I : -K_k] \begin{bmatrix} \Xi_k & \Psi_k \\ \Psi_k^T & \Theta_k \end{bmatrix} \begin{bmatrix} I \\ -K_k^T \end{bmatrix} , \quad (3.8)$$

$$\Phi_k = [\Phi(k,k_o) : \ldots : \Phi(k,k) : 0_{n \times (N-1-k)n}], \quad (3.9)$$

and

$$IK = \left[ \begin{array}{c|c} I_{(N-k_o)n} & \begin{array}{c} 0_{n \times (N-1-k_o)r} \\ \begin{array}{ccc} -K_{k_o} & & \\ & \ddots & \\ & & -K_{N-2} \end{array} \end{array} \end{array} \right] \quad (3.10)$$

where the subindices of the null and identity matrices indicate their respective size.

### B. Uncertainty in Covariances

We suppose now that the means are known, i.e. $y = y_L$, and therefore that the estimates are unbiased. By the definition of the state-transition matrix it is easy to check [8] that $\Sigma^a_{k/k-1}$ satisfies the equation

$$\Sigma^a_{k+1/k} = [A_k - K_kC_k]\Sigma^a_{k/k-1}[A_k - K_kC_k]^T + M_k. \quad (3.11)$$

The optimal gains are given by

$$K_k^* = \arg \min_{K_k \in \mathbb{R}^{n \times r}} \Sigma^a_{k/k-1} = [A_k\Sigma_{k/k-1}C_k^T + \Psi_k][C_k\Sigma_{k/k-1}C_k^T + \Theta_k]^{-1} , \quad (3.12)$$

where

$$\Sigma_{k+1/k} = A_k\Sigma_{k/k-1}A_k^T + \Xi_k - [A_k\Sigma_{k/k-1}C_k^T + \Psi_k][C_k\Sigma_{k/k-1}C_k^T + \Theta_k]^{-1}[A_k\Sigma_{k/k-1}C_k^T + \Psi_k]^T , \quad (3.13)$$

and

$$\Sigma_{k_o/k_o-1} = \Sigma_o .$$

Although all the matrices involved are possibly time-varying, in the remainder of the discussion we will omit their explicit dependence on time in order to simplify the notation except when this could be ambiguous.

In order to prove the minimax theorems we will make use of the following sensitivity result.

Lemma 4. Suppose $\mathcal{C}'$ is a convex set and $(\Xi,\Psi,\theta,\Sigma_o) = \Lambda = (1-\alpha)\Lambda_L + \alpha\Omega$ with $\Lambda_L = (\Xi_L,\Psi_L,\theta_L,\Sigma_{oL})$ and $\Omega = (V,X,Y,Z)$ both belonging to $\mathcal{C}'$. Let $\hat{x}^L_{k/k-1}$ be the state estimate of the Kalman filter designed for $\Lambda_L$ when $\Lambda$ describes the true statistics. With $\bar{e}_{k/k-1} \triangleq x_k - \hat{x}^L_{k/k-1}$ and $\Gamma_{k/k-1} \triangleq E[\bar{e}_{k/k-1}\,\bar{e}^T_{k/k-1}]$, we have that for all $\Lambda_L,\Omega \in \mathcal{C}'$, and for each time k,

$$\lim_{\alpha \to 0} \frac{1}{\alpha}(\Sigma_{k/k-1} - \Gamma_{k/k-1}) = 0 \tag{3.14}$$

Proof. $\Sigma_{k/k-1}$ and $\Gamma_{k/k-1}$ satisfy (3.8) and (3.11) for the optimal Kalman gains $K^*$ and $K^*_L$ designed for $\Lambda$ and $\Lambda_L$ respectively. Note that in both cases the same set of covariances $(\Xi,\Psi,\theta,\Sigma_o)$ is present. Therefore, we have

$$\begin{aligned}\Sigma_{k+1/k} - \Gamma_{k+1/k} &= [A-K^*_LC](\Sigma_{k/k-1}-\Gamma_{k/k-1})[A-K^*_LC]^T \\ &\quad - \Psi(K^*-K^*_L)^T - (K^*-K^*_L)\Psi^T + K^*\theta K^{*T} - K^*_L\theta K^{*T}_L \\ &\quad - (K^*-K^*_L)C\Sigma_{k/k-1}A^T - A\Sigma_{k/k-1}C^T(K^*-K^*_L)^T \\ &\quad - K^*_LC\Sigma_{k/k-1}C^TK^{*T}_L + K^*C\Sigma_{k/k-1}C^TK^{*T} \\ &= [A-K^*_LC](\Sigma_{k/k-1}-\Gamma_{k/k-1})[A-K^*_LC]^T \\ &\quad - (K^*-K^*_L)[C\Sigma_{k/k-1}C^T+\theta](K^*-K^*_L)^T\end{aligned} \tag{3.15}$$

where in order to get the last equality the expression (3.12) for $K^*$ has been taken into account. It is worth noting that (3.15) gives the variation of the error variance due to an arbitrary modification (up to here the value of $K^*_L$ has not been used) of the optimal Kalman gain (cf. [9]). Because $\lim_{\alpha\to 0} \frac{1}{\alpha}[K^*-K^*_L]$ exists for every time k and for all $\Lambda_L,\Omega \in \mathcal{C}'$ (see 4.1 [6]), in the limit (3.15) reduces to a homogeneous equation, and, since $\Sigma_{k_o/k_o-1} = \Sigma_o = \Gamma_{k_o/k_o-1}$, we have the sought-after result.

If, when the second-order statistics are uncertain, we use the weighted square error (3.1) as the penalty function, then it can be shown that the least favorable covariance sequence can be different for every k. Thus, it is possible that the minimax filter is given by a recursion that does not solve the problem for previous times, and therefore we are forced to build a different filter for every k. The infeasibility of this solution leads to the consideration of the penalty function (cf. [4])

$$L(H,y,\Lambda) = \sum_{k=k_o}^{N-1} E[e^T_{k/k-1}Q_ke_{k/k-1}|U_{k-1},Z_{k-1}] \tag{3.16}$$

which has the same minimizing filter as (3.1) when the statistical model is known. Furthermore, note that the penalty function of (3.1) can be considered a particular case of (3.16).

We assume that C is an uncertainty class such that there exists a convex set $C' \supset C$ that fulfills (2.6). Given that the means are known and $\mathcal{K}$ is the set of linear filters, then the regularity condition (2.10) follows from Lemma 4 and Lemma 1 results on the following theorem.

Theorem 1. For the problem of one-step state prediction the Kalman filter for the least favorable collection of covariances for one-step state prediction is robust for the game $(C,\mathcal{K},L)$.

This result reduces the original minimax filter design problem to the search for least favorable sets of covariances. In this point the application of Lemma 3 has proven to be successful in dealing with important uncertainty classes in other contexts [10]. In the present case, let us define

$$W_j(R) \triangleq \sum_{k=j}^{N-1} \Phi^T_L(k,j)R_k\Phi_L(k,j) \tag{3.17}$$

where $R_k \in \mathbb{R}^{n\times n}$ (if $R \geq 0$ then $W(R) \geq 0$), and $\Phi_L(\cdot,\cdot)$ is the state transition matrix of the fundamental matrix $(A-K_LC)$. Using this definition and interchanging the order of summation, (3.7) results in

$$\sum_{k=k_o}^{N-1} tr\{Q_k\Sigma^a_{k/k-1}\} = tr\{\Sigma_oW_{k_o}(Q) + \sum_{j=k_o}^{N-2} M_jW_{j+1}(Q)\}. \tag{3.18}$$

Applying Lemma 3 to this expression, the following result follows straightforwardly.

Theorem 2. If C' can be put as the Cartesian product of uncertainty classes $\mathcal{X} \times \mathcal{P} \times \mathcal{N} \times \mathcal{S}$, then $(\Xi_L,\Psi_L,\theta_L,\Sigma_{oL})$ is the least favorable collection of covariances for one-step state prediction if and only if the following are satisfied

$$\Xi_L = \arg\max_{\Xi\in\mathcal{X}} tr\{\sum_{j=k_o}^{N-2} \Xi_jW_{j+1}(Q)\}, \tag{3.19}$$

$$\Psi_L = \arg\min_{\Psi\in\mathcal{P}} tr\{\sum_{j=k_o}^{N-2} \Psi_jK^T_{Lj}W_{j+1}(Q)\}, \tag{3.20}$$

$$\theta_L = \arg\max_{\theta\in\mathcal{N}} tr\{\sum_{j=k_o}^{N-2} \theta_jK^T_{Lj}W_{j+1}(Q)K_{Lj}\}, \tag{3.21}$$

and

$$\Sigma_{oL} = \arg\max_{\Sigma_o\in\mathcal{S}} tr\{\Sigma_oW_{k_o}(Q)\}, \tag{3.22}$$

where

$$K_{Lk} = [A_k\Sigma^L_{k/k-1}C^T_k + \Psi_{Lk}][C_k\Sigma^L_{k/k-1}C^T_k + \theta_{Lk}]^{-1}, \tag{3.23}$$

$$\begin{aligned}\Sigma^L_{k+1/k} &= A_k\Sigma^L_{k/k-1}A^T_k + \Xi_{Lk} \\ &\quad - [A_k\Sigma^L_{k/k-1}C^T_k+\Psi_{Lk}][C_k\Sigma^L_{k/k-1}C^T_k+\theta_{Lk}]^{-1}\cdot \\ &\quad [A_k\Sigma^L_{k/k-1}C^T_k+\Psi_{Lk}]^T,\end{aligned} \tag{3.24}$$

and

$$\Sigma^L_{k_o/k_o-1} = \Sigma_{oL}. \tag{3.25}$$

Notice that while the optimal observer for fixed covariances is independent of the error weighting matrix Q, this is not the case here since the least favorable covariances may depend on Q. Moreover, the least favorable covariances (from $k = k_o$ to N-1) depend on the final time N. Note also the recursive character of these equations, which makes them attractive for an iterative solution of the minimax Kalman gain.

C. Uncertainty in Means and Covariances

Now that we have studied the problem for uncertainty only in the covariance matrices, we drop the assumption that the means are known and consider the more general minimax problem. The penalty function is derived from (3.1), (3.6);

$$L_p(h,y,\Lambda) = \sum_{k=k_o}^{N-1} (tr\{Q_k \Sigma^a_{k/k-1}\} + (y-y_L)^T (IK)^T \Phi_k^T Q_k \Phi_k (IK)(y-y_L)). \quad (3.26)$$

It is not possible to apply Lemmas 1 and 3 to this penalty function because it is not concave in the uncertainties (in fact, it is convex). However, we can put the uncertainty set as the Cartesian product $\mathcal{X} \times \mathcal{P} \times \mathcal{N} \times \mathcal{J} \times \mathcal{M}$, and the part of the penalty function due to the error variance can be decomposed in the same way as in Theorems 2 and 3 in order to apply Lemma 2 (note that no further decomposition of the penalty due to the means is possible). Unfortunately, except for trivial uncertainty classes for the means, (2.13) has no nonrandomized solution in this case. The reason for this can be seen intuitively by considering that the performance of a filter built for a given set of means can only be deteriorated when different means are truly present, and hence no saddle-point solution to the filtering game exists. Since we require a deterministic solution for the means in the observers and regulators, alternatively we deal with the soft minimax [6] solution to the problem. According to this approach, useful to model frequent situations in which there is not "total" uncertainty, operating points that are closer to a given nominal are more likely to occur than those in the uncertainty class more distant from it. This further knowledge can be taken advantage of by adding to the penalty function an additional term accounting for the distance between the operating point and the nominal. Applying this philosophy to the uncertainty in the means, we arrive at a penalty function

$$L'_p(h,y,\Lambda) = L_p(h,y,\Lambda) - (y-y_N)^T D (y-y_N) \quad (3.27)$$

where D is a positive semidefinite matrix, and $y_N$ is the vector of nominal means. Then, the following result holds [6].

Theorem 4. Suppose we have for all possible covariances

$$\sum_{k=k_o}^{N-1} (IK)^T \Phi_k^T Q_k \Phi_k (IK) - D \le 0 , \quad (3.28)$$

then the Kalman filter given by (3.19-25) and with nominal means is robust for the game $(\mathcal{M} \times \mathcal{C}, \mathcal{K}, L'_p)$.

## IV. THE ROBUST REGULATOR PROBLEM

A. Formulation

The optimal regulator for linear quadratic optimal control is the solution to

$$\arg\min_{H \in \mathcal{K}} J(H,y,\Lambda) , \quad (4.1)$$

where

$$J(H,y,\Lambda) = E[x_N^T F x_N | Z_{N-1}] + \sum_{k=k_o}^{N-1} E[x_k^T Q_{1k} x_k + U_k^T Q_{2k} U_k | Z_{k-1}] . \quad (4.2)$$

Here $x_k$ is the state of the system (2.1-2) with zero-mean process noise, $\mathcal{K}$ is the set of linear filters with input $U_{k-1}$ and $Z_{k-1}$ and output $U_k$, and where F and $Q_{1k}$ are positive semidefinite and $Q_{2k}$ is positive definite. The optimal filter when the noise covariances are fixed is given by [8]

$$U_k = -G_k \hat{x}_{k/k-1}, \quad (4.3)$$

where

$$G_k = [Q_{2k} + B_k^T S_{k+1} B_k]^{-1} B_k^T S_{k+1} A_k , \quad (4.4)$$

$$S_k = A_k^T S_{k+1} A_k + Q_{1k} - A_k^T S_{k+1} B_k [Q_{2k} + B_k^T S_{k+1} B_k]^{-1} B_k^T S_{k+1} A_k \quad (4.5)$$

$$S_N = F , \quad (4.6)$$

and $\hat{x}_{k/k-1}$ the optimal predictor estimate given by (3.2),(3.3),(3.12),(3.13). This is the consequence of the separation principle of stochastic control which enunciates the optimality of the feedback of the state estimate with the same gain as that in the known state case. Since the feedback gain G does not depend on the statistics of the noise, it should be expected that when these are unknown the optimal value of G is unchanged, and thus the control and state estimation problems can still be solved separately. We will prove this statement rigorously and will show that in general the state estimates are not given by the minimax observer of Section III. These phenomena were demonstrated for the situation of steady-state, continuous time control by Looze, et al. [2].

In the following sections, we will use the following identity [11].

Lemma 5. If the process noise has zero mean, then

$$E[x_N^T F x_N + \sum_{k=k_o}^{N-1} x_k^T Q_{1k} x_k + U_k^T Q_{2k} U_k] = m_o^T S_{k_o} m_o + tr\{S_{k_o} \Sigma_o\} + \sum_{k=k_o}^{N-1} tr\{S_{k+1} \Xi_k\} + \sum_{k=k_o}^{N-1} E[(U_k + G_k x_k)^T [B_k^T S_{k+1} B_k + Q_{2k}](U_k + G_k x_k)] \quad (4.7)$$

where G and S are given by (4.4-6).

B. Uncertainty in Covariances

As in the case of regulator design, we consid consider first the case in which the means are known and the covariances are possibly unknown. Let us define the positive semidefinite matrix

$$N_k \triangleq A_k^T S_{k+1} B_k [Q_{2k} + B_k^T S_{k+1} B_k]^{-1} B_k^T S_{k+1} A_k . \quad (4.8)$$

Then the penalty function can be expressed (by Lemma 5 and (4.3) and taking into account that the means are known) as

$$J(H,\Lambda) = m_o^T S_{k_o} m_o + \mathrm{tr}\{S_{k_o}\Sigma_o\} + \sum_{k=k_o}^{N-1} \mathrm{tr}\{S_{k+1}\Xi_k\} + \sum_{k=k_o}^{N-1} \mathrm{tr}\{N_k \Sigma^a_{k/k-1}\} \qquad (4.9)$$

Then we have the following result which is analogous to Theorem 1 (see [6]).

Theorem 6. The regulator consisting of the feedback for complete state information of the state estimates produced by the Kalman filter for the least favorable collection of covariances for linear quadratic optimal control is robust for the game $(\mathcal{C},\mathcal{K},J)$.

Note that, as the following theorem states, the least favorable set of covariances for this problem may be different from that for state estimation, hence our previous assertion that the Kalman filter for the minimax regulator is not generally a minimax observer. Using the same kind of manipulations that led to (3.18), the penalty function in this case can be expressed as

$$J(H,\Lambda) = m_o^T S_{k_o} m_o + \mathrm{tr}\{S_{k_o}\Sigma_o + S_N \Xi_{N-1}\} + \mathrm{tr}\{\Sigma_o W_{k_o}(N) + \sum_{j=k_o}^{N-2} (\Xi_j S_{j+1} + M_j W_{j+1}(N))\}. \qquad (4.10)$$

Lemma 3 can be applied to this expression to yield the following result.

Theorem 7. Under the same assumption and definitions as in Theorem 2, $(\Xi_L, \Psi_L, \Theta_L, \Sigma_{oL})$ is the least favorable collection of covariances for linear quadratic optimal control if and only if (3.23-25) and the following are satisfied:

$$\Xi_L = \arg\max_{\Xi \in \mathcal{X}} \mathrm{tr}\{\Xi_{N-1}F + \sum_{j=k_o}^{N-2} \Xi_j (S_{j+1} + W_{j+1}(N))\}, \qquad (4.11)$$

$$\Psi_L = \arg\min_{\Psi \in \mathcal{P}} \mathrm{tr}\{\sum_{j=k_o}^{N-2} \Psi_j K_{Lj}^T W_{j+1}(N)\}, \qquad (4.12)$$

$$\Theta_L = \arg\max_{\Theta \in \mathcal{N}} \mathrm{tr}\{\sum_{j=k_o}^{N-2} \Theta_j K_{Lj}^T W_{j+1}(N) K_{Lj}\}, \qquad (4.13)$$

and

$$\Sigma_{oL} = \arg\max_{\Sigma_o \in \mathcal{S}} \mathrm{tr}\{\Sigma_o (S_{k_o} + W_{k_o}(N))\}. \qquad (4.14)$$

C. Uncertainty in Means and Covariances

We now consider the case, analogous to that treated in Subsection IIIc, in which the means of the initial state and of the observation noise are also uncertain. In this situation an appropriate penalty function is, using soft minimax,

$$J(h,y,\Lambda) = \mathrm{tr}\{S_{k_o}\Sigma_o\} + \sum_{k=k_o}^{N-1} \mathrm{tr}\{S_{k+1}\Xi_k\} + \sum_{k=k_o}^{N-1} \mathrm{tr}\{N_k \Sigma^a_{k/k-1}\}$$
$$+ (V-V_L)^T (JK)^T \sum_{k=k_o}^{N-1} \Phi_k^T N_k \Phi_k (JK)(V-V_L) + V^T (IS) V$$
$$- (V-V_N)^T D (V-V_N), \qquad (4.15)$$

where $(JK) = \mathrm{diag}\{I_n, -K_{k_o}, \ldots, -K_{N-2}\}$; $(IS) = \mathrm{diag}\{S_{k_o}, 0_{(N-1-k_o)r}\}$, $V = \mathrm{col}\{m_o, \bar{v}_{k_o}, \ldots, \bar{v}_{N-2}\}$ (analogously $V_L$ and $V_N$) then, using Lemma 2, we have the following result.

Theorem 8. The regulator consisting of the feedback for complete state information of the state estimates produced by the Kalman filter given by (3.23-25) and (4.11-14) and with means $V_R$ that solve

$$V_R = \min_{V_L \in \mathbb{R}^{n+r(N-1-k_o)}} (V^* - V_L)^T (JK)^T \sum_{k=k_o}^{N-1} \Phi_k^T N_k \Phi_k (JK)(V^* - V_L) \qquad (4.16)$$

where

$$V^* = \max_{V \in \mathcal{M}} (V-V_R)^T (JK)^T \sum_{k=k_o}^{N-1} \Phi_k^T N_k \Phi_k (JK)(V-V_R)$$
$$+ V^T (IS) V - (V-V_N) D (V-V_N) \qquad (4.17)$$

is robust for the game $(\mathcal{M} \times \mathcal{C},\mathcal{K},J)$.

## V. CONCLUSIONS

The application of the general formulation of minimax robust filtering has allowed us to present minimax theorems for state estimation and for quadratic control under general classes of uncertainties in the second-order statistics of the linear stochastic system. These results represent a generalization of earlier works [1],[2] devoted to the steady-state case for invariant continuous-time systems with uncorrelated noises. The minimax theorems state that the minimax filter is the optimal for the least favorable collection of noise covariances; this saddle-point property implies a very attractive feature, namely, that when the actual covariances differ from the least favorable ones, the performance of the robust filter is upgraded. Sets of necessary and sufficient conditions are given for the least favorability of the four types of covariance matrices involved. These conditions lend themselves to a recursive solution, and are applied successfully to deviation classes of practical interest. It is worthwhile to underscore the fact that under the same type of uncertainty the worst case covariances do not necessarily coincide for the types of filters treated here and thus the observer used for the minimax regulator is not, in general, a minimax predictor state estimator. Given a particular expected deviation behavior, the tradeoff between the decrease of performance (with respect to the nominal filter) in the nominal model and the improvement of the worst case, should be assessed for the practical application of the minimax filters presented. The soft minimax approach has been used to deal with unknown means, and if the covariance uncertainty classes can also be modeled using nominals that are more likely to occur, then a corresponding soft minimax solution is straightforward from the previous results.

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