Energy-Distortion Tradeoffs in Multiple-access Channels with Feedback

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***Abstract*—The *energy–distortion function* $E(D)$ for the joint source–channel coding problem in networks is defined and studied. The energy–distortion function $E(D)$ is defined as the minimum energy required to transmit a source to a receiver within the target distortion D, when there is no restriction on the number of channel uses per source sample. For point-to-point channels, $E(D)$ is shown to be equal to the product of the minimum energy per bit $E_{b\min}$ and the rate–distortion function $R(D)$, establishing the optimality of source-channel separation in this setting.**

Then, it is shown that the optimality of separation does not extend to multi-user networks. A scenario with two encoders observing correlated Gaussian sources in which the encoders communicate to the receiver over a Gaussian multiple-access channel (MAC) with perfect channel output feedback is studied. First a lower bound on $E(D)$ is provided and compared against an upper bound achievable by separation. Even though the separation based scheme does not achieve the lower bound in general, its energy requirement is shown to be within a constant gap of $E(D)$ in the low distortion regime, for which the energy requirement grows unbounded. Another upper bound using uncoded transmission based on the well-known Schalkwijk-Kailath (SK) scheme is also considered. Through simulation, it is shown that this scheme outperforms the separation based scheme in various scenarios, thus establishing the sub-optimality of separation in this model of multiple users with correlated sources.

I. Introduction

We study the energy–distortion tradeoff in communicating sources over noisy communication channels, such that the source can be reconstructed at the receiver within a target distortion. The 'energy' refers to the cost per source sample of using the communication channel (see, e.g., [9] and references therein). The more energy that the transmitter has at its disposal, the more information it can transmit to the receiver. This potentially translates to lower distortion at the receiver. Thus, there is a fundamental energy-distortion tradeoff in this setting. We try to capture this tradeoff by defining a fundamental information-theoretic function $E(D)$ which is the minimum energy required per source sample to achieve an average distortion D, for large number of source samples. We note that no restriction has been placed on the number of channel uses per source sample, thus allowing us to maximize the energy efficiency over unlimited bandwidth. Since our model brings together the problems of compression and communication, it is a joint source–channel coding problem.

The model presented here has a number of potential applications. Consider for example, energy-constrained sensor networks, where the fusion center wants to reconstruct the observations taken at the sensor nodes. These observations could be correlated due to the physical proximity of the sensors. Our work quantifies the minimum energy required at the nodes in order to let the fusion center estimate the observations with a certain fidelity.

For a point-to-point memoryless channel and a memoryless source, source–channel separation is widely known to be optimal for a fixed number of channel uses per source sample and under an average power constraint. We show here that this optimality extends to the energy-distortion function as well, that is, the optimal $E(D)$ is achieved by a simple separation scheme that first quantizes the source (at rate $R(D)$) and then transmits the quantized message over the channel using minimum energy per bit $E_{b\min}$ by operating in the wideband regime.

However, as in the case of average power constraints and a fixed number of channel uses per source sample, this optimality of separation does not generalize to multiple sources. In particular, we consider two encoders/transmitters observing Gaussian sources that are correlated. The communication channel from the transmitters to the receiver is assumed to be a multiple-access channel (MAC) with additive white Gaussian noise, in which the perfect causal channel output at the receiver is available to the transmitters. This model is studied in [5] under average power constraints and assuming matching source and channel bandwidths. We are interested in obtaining the minimum energy requirement of reconstructing both sources at the receiver within a target distortion without any restrictions on the ratio of source and channel bandwidths. For the sake of simplicity of analysis we restrict our attention to the case of symmetric sources, energies, channel gains, and target distortions.

For the two-user model, we first provide a converse bound by extending the results of [5] to the bandwidth mismatch situation. Then a separation based achievability scheme is analyzed. We also briefly describe an uncoded transmission

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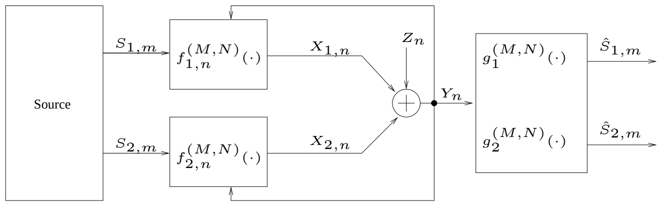


Fig. 1: Bivariate Gaussian source model with perfect channel output feedback.

scheme motivated by the analog transmission schemes employed in [3], [7] and others. While we do not provide a closed form expression for the energy consumption of the uncoded transmission scheme, simulation results suggest that it can outperform the separation based scheme in various situations.

II. System Model

We consider a network with two transmitters and a single receiver, where perfect channel output feedback at the receiver end is provided to the transmitters. The source S_i^M at transmitter i is an M-length random vector of independent and identically distributed (i.i.d.) real-valued Gaussian random variables with zero means and variances $\sigma_{S_i}^2$, i.e., $S_i \sim \mathcal{N}(0, \sigma_{S_i}^2)$ for $i = 1, 2$.

Transmitters encode their observations and transmit them over a MAC. Denoting the input sequence at transmitter i as $X_i^N = (X_{i,1}, ..., X_{i,N})$, and the corresponding channel output vector as $Y^N = (Y_1, ..., Y_N)$, the channel is characterized by

$$Y_n = X_{1,n} + X_{2,n} + Z_n \qquad \text{for } n = 1, \dots, N, \qquad (1)$$

where $Z^N = (Z_1, ..., Z_N)$ is the vector of i.i.d. $\mathcal{N}(0, \sigma_Z^2)$ channel noise variables. See Fig. 1. In this work, we focus on the symmetric scenario in which the source statistics are the same, i.e., $\sigma_{S_1}^2 = \sigma_{S_2}^2 \triangleq \sigma_S^2$. We assume that the sources are correlated, the coefficient of correlation being

$$\rho = \frac{\mathbb{E}[S_1 S_2]}{\sigma_S^2}. \qquad (2)$$

Without loss of generality, ρ can be taken to lie between 0 and 1 since if the sources are negatively correlated, we can replace S_2 by $-S_2$ to get a positive correlation.

We also assume the availability of perfect causal channel output feedback at both of the transmitters. Hence the encoding function at each transmitter can depend on its source realizations as well as the previous channel outputs. Considering block encoding from an M-length source vector to an N-length channel vector, the encoder at transmitter i is described by a sequence of encoding functions $f_{i,n}^{(M,N)} : \mathbb{R}^M \times \mathbb{R}^{n-1} \to \mathbb{R}$ where $X_{i,n} = f_{i,n}^{(M,N)}(S_i^M, Y^{n-1})$, for $i = 1, 2$ and $n = 1, \dots, N$. The decoder is described by the decoding functions $g_i^{(M,N)} : \mathbb{R}^N \to \mathbb{R}^M$ where $\hat{S}_i^M = g_i^{(M,N)}(Y^N)$, for $i = 1, 2$. We define this network as a $(\sigma_S^2, \sigma_Z^2, \rho)$ network.

Definition 2.1: For the given network, we say that an energy–distortion pair (E, D) is *achievable* if there exists a sequence (over M) of encoding functions

$$\{f_{1,n}^{(M,N)}\}_{n=1}^N \text{ and } \{f_{2,n}^{(M,N)}\}_{n=1}^N$$

satisfying the energy constraint

$$\mathbb{E}\left[\sum_{n=1}^{N} X_{i,n}^2\right] \le ME \qquad \text{for } i = 1, 2, \qquad (3)$$

and a sequence of decoding functions

$$g_1^{(M,N)} \text{ and } g_2^{(M,N)}$$

such that the corresponding distortion sequence satisfies

$$\limsup_{M\to\infty} \frac{1}{M} \sum_{m=1}^{M} \mathbb{E}\left[(S_{i,m} - \hat{S}_{i,m})^2\right] \le D, \qquad \text{for } i = 1, 2.$$

Definition 2.2: We define the *energy–distortion function* for the $(\sigma_S^2, \sigma_Z^2, \rho)$ network as

$$E(D) \triangleq \inf\{E \ge 0 : (E, D) \text{ is achievable}\}. \qquad (4)$$

Note that we do not impose any channel bandwidth constraints, and hence it is possible to transmit as many channel symbols per source observation as needed as long as the total energy constraint is satisfied. Our goal in this paper is to determine $E(D)$ for a $(\sigma_S^2, \sigma_Z^2, \rho)$ network.

III. Single Source Scenario

For completeness, we first treat the single user scenario in which a single Gaussian source S is transmitted over a point-to-point channel with additive white Gaussian noise with variance σ_Z^2. In this situation, separate source–channel coding is optimal and achieves $E(D)$.

To present this result we first define $E_{b\min}$ as the *minimum energy per bit* [9] for the underlying communication channel. We also define $R(D)$ as the rate-distortion function for the given source, that is, the minimum rate (bits per channel use) of encoding S required to achieve an average distortion D.

Lemma 3.1: For the single source scenario, we have

$$E(D) = E_{b\min} R(D), \qquad (5)$$

which can be achieved by separate source and channel coding[1].

For the point-to-point Gaussian channel,

$$E_{b\min} = 2\sigma_Z^2 \log_e 2. \qquad (6)$$

Furthermore, for the Gaussian model considered here, we have

$$R(D) = \frac{1}{2} \log_2^+ \left(\frac{\sigma_S^2}{D}\right), \qquad (7)$$

where $\log^+(x) = \log(x)$ if $x \ge 1$ and 0 otherwise. Therefore, for the single Gaussian source case, we have

$$E(D) = \sigma_Z^2 \log_e^+ \left(\frac{\sigma_S^2}{D}\right). \qquad (8)$$

[1]Proofs of the results given in this paper are omitted due to space limitations.

Remark 3.1: We note that the optimality of source–channel separation is not limited to the Gaussian model considered here. Lemma 3.1 holds for a general discrete memoryless stationary source and channel with any additive distortion measure and any separable cost function respectively. A similar separation result for the remote rate-distortion problem is presented in [2].

Remark 3.2: It is well known that separation is optimal for the single source and point-to-point channel case, and that feedback does not change $E_{b\min}$. Lemma 3.1 can be extended to show that (5) holds even when perfect channel feedback is available at the transmitter.

For the rest of the paper, we study the two source scenario in which the sources are correlated with each other. We provide lower and upper bounds on $E(D)$.

IV. Lower Bound on $E(D)$

In this section, we present two lower bounds on $E(D)$ for the $(\sigma_S^2, \sigma_Z^2, \rho)$ network. The first bound is obtained by assuming perfect cooperation between the encoders. Not only do the sources cooperate to compress their observations, but they also cooperate to transmit their observations over the communication channel. Since this reduces the model to that of a single vector source and point-to-point channel, we can apply Lemma 3.1 with

$$E_{b\min} = \sigma_Z^2 \log_e 2, \tag{9}$$

which is the minimum energy per bit of a Gaussian channel with two transmitter and single receiver antenna. The rate–distortion function under individual distortion constraints is given by [11]

$$R(D) = \begin{cases} \frac{1}{2}\log_2^+\left(\frac{\sigma_S^4(1-\rho^2)}{D^2}\right) & \text{if } 0 \le D \le \sigma_S^2(1-\rho) \\ \frac{1}{2}\log_2^+\left(\frac{\sigma_S^2(1+\rho)}{2D-(1-\rho)\sigma_S^2}\right) & \text{if } \sigma_S^2(1-\rho) < D \le \sigma_S^2. \end{cases} \tag{10}$$

Therefore, from Lemma 3.1, the energy–distortion tradeoff can be lower bounded as in Theorem 4.1

Theorem 4.1: In a $(\sigma_S^2, \sigma_Z^2, \rho)$ network, $E(D)$ is lower bounded by $E_{LB1}(D)$ where

$$E_{LB1}(D) = \begin{cases} \frac{\sigma_Z^2}{4}\log_e^+\left(\frac{\sigma_S^4(1-\rho^2)}{D^2}\right) \\ \qquad \text{if } 0 \le D \le \sigma_S^2(1-\rho) \\ \frac{\sigma_Z^2}{4}\log_e^+\left(\frac{\sigma_S^2(1+\rho)}{2D-(1-\rho)\sigma_S^2}\right) \\ \qquad \text{if } \sigma_S^2(1-\rho) < D \le \sigma_S^2. \end{cases} \tag{11}$$

Next, we present another lower bound following the arguments in [4, Section 3]. The main difference from the point-to-point case considered above is that now we consider separate channel encoding at the transmitters, which allows us to improve the lower bound by limiting beamforming gains.

We define the conditional rate-distortion functions $R_{S_1|S_2}(D)$ and $R_{S_2|S_1}(D)$ as the minimum rate required to achieve a distortion of D (for one source) when the other source is available at both the encoder and the decoder. It can be shown that

$$R_{S_1|S_2}(D) = R_{S_2|S_1}(D) = \frac{1}{2}\log_2^+\left(\frac{\sigma_S^2(1-\rho^2)}{D}\right). \tag{12}$$

Theorem 4.2: In a $(\sigma_S^2, \sigma_Z^2, \rho)$ network, $E(D)$ is lower bounded by $E_{LB2}(D)$ as given in (13).

Remark 4.1: It can be shown that $E_{LB1}(D) \le E_{LB2}(D)$ always. Therefore, for the rest of the paper, we restrict our attention to $E_{LB2}(D)$ as far as lower bounds are concerned.

V. Separate Source and Channel Coding

Almost all practical communication systems operate based on separate source and channel coding. Apart from the practical motivation due to the modularity it provides, separate source and channel coding is also motivated by its theoretical optimality in the point-to-point scenario. In this section, we outline a separation based scheme and analyze its energy–distortion tradeoff.

The basic idea of the scheme is for the encoders to first quantize their sources at identical rates (see [6] or [10]) and then transmit the quantization information over the MAC with feedback (see [7]).

The rate-distortion function $R_{\text{sep}}(D)$ to achieve symmetric distortion D for each source is given as [6], [10]

$$R_{\text{sep}}(D) = \max\left\{\frac{1}{2}\log_2^+\left(\frac{\sigma_S^2(1-\rho^2)}{2D}\left(1+\sqrt{1+\frac{4D\rho^2}{\sigma_S^2(1-\rho^2)^2}}\right)\right), \frac{1}{4}\log_2^+\left(\frac{\sigma_S^4(1-\rho^2)}{2D^2}\left(1+\sqrt{1+\frac{4D^2\rho^2}{\sigma_S^4(1-\rho^2)^2}}\right)\right)\right\}. \tag{14}$$

Each transmitter now transmits $R_{\text{sep}}(D)$ bits to the receiver in a separate, orthogonal band, i.e., without interfering with the other transmitter. Note that the feedback signal is not used in this scheme. This is due to the fact that, despite feedback enlarges the capacity region of a MAC under average power constraints at the users, it does not improve the minimum energy per bit which is achieved by orthogonal transmissions. We thus get the achievable energy–distortion tradeoff given by $E_{\text{sep}}(D)$ in Theorem 5.1.

$$E_{LB2}(D) = \begin{cases} \min_{0\le\hat\rho\le 1}\max\left\{\frac{\sigma_Z^2}{(1-\hat\rho^2)}\log_e^+\left(\frac{\sigma_S^2(1-\rho^2)}{D}\right), \frac{\sigma_Z^2}{2(1+\hat\rho)}\log_e^+\left(\frac{\sigma_S^4(1-\rho^2)}{D^2}\right)\right\} & \text{if } 0 \le D \le \sigma_S^2(1-\rho) \\ \min_{0\le\hat\rho\le 1}\max\left\{\frac{\sigma_Z^2}{(1-\hat\rho^2)}\log_e^+\left(\frac{\sigma_S^2(1-\rho^2)}{D}\right), \frac{\sigma_Z^2}{2(1+\hat\rho)}\log_e^+\left(\frac{\sigma_S^2(1+\rho)}{2D-(1-\rho)\sigma_S^2}\right)\right\} & \text{if } \sigma_S^2(1-\rho) < D \le \sigma_S^2 \end{cases} \tag{13}$$

Theorem 5.1: In a $(\sigma_S^2, \sigma_Z^2, \rho)$ network, $E(D)$ is upper bounded by $E_{\text{sep}}(D)$, where

$$E_{\text{sep}}(D) = \max\left\{\sigma_Z^2 \log_e^+\left(\frac{\sigma_S^2(1-\rho^2)}{2D}\left(1+\sqrt{1+\frac{4D\rho^2}{\sigma_S^2(1-\rho^2)^2}}\right)\right), \frac{\sigma_Z^2}{2}\log_e^+\left(\frac{\sigma_S^4(1-\rho^2)}{2D^2}\left(1+\sqrt{1+\frac{4D^2\rho^2}{\sigma_S^4(1-\rho^2)^2}}\right)\right)\right\}. \quad (15)$$

Remark 5.1: For $\rho = 0$, it can be easily checked that the lower bound $E_{LB2}(D)$ and upper bound $E_{\text{sep}}(D)$ match. For $\rho = 1$, $E_{\text{sep}}(D)$ is twice the lower bound $E_{LB2}(D)$. This is expected since the transmissions at the encoders have a correlation of zero which implies that there are no beamforming gains.

Remark 5.2: While separation is not optimal in general, it has a finite gap with the lower bound even as $E(D)$ goes to infinity in the low distortion regime. Thus, we have the following result in Proposition 5.2.

Proposition 5.2: In a $(\sigma_S^2, \sigma_Z^2, \rho)$ network, the following holds:

$$\lim_{D\to 0} E_{\text{sep}}(D) - E_{LB2}(D) = -\frac{\sigma_Z^2}{2}\log_e(1-\rho^2) \quad (16)$$

Note that

$$\lim_{D\to 0} E_{\text{sep}}(D) = \lim_{D\to 0} E_{LB2}(D) = \infty. \quad (17)$$

Based on Proposition 5.2, separation can be a viable alternative for real-world applications when the source correlation is low. Note that the gap between $E_{\text{sep}}(D)$ and $E_{LB2}(D)$ diverges when $\rho = 1$.

VI. Uncoded Transmission

While separation is optimal for point-to-point systems and for a MAC with independent sources, this is not the case when the sources are correlated. In this section, we describe an achievability scheme with uncoded transmission based on the Schalkwijk–Kailath (SK) scheme [3].

In the SK scheme for a point-to-point channel, in each step the transmitter transmits a scaled version of the estimation error at the receiver in an uncoded manner. It is shown in [3] that this simple scheme achieves the channel capacity without employing block codes, hence without delay. This optimality, without incurring any delay, can also be extended to joint source–channel coding in the single source and point-to-point channel setting for a fixed bandwidth ratio [8]. So, the SK scheme extends the optimality of uncoded transmission to the bandwidth mismatch case. Not surprisingly, the SK scheme achieves the energy–distortion tradeoff (8) for single Gaussian source as well.

In multi-user scenarios the advantage of uncoded transmission has been shown previously. In Gaussian sensor networks, when the source and channel bandwidths match, it is known that uncoded transmission is optimal [1]. In [2], we show that an SK based uncoded transmission scheme has better energy–distortion performance than the separation based scheme in certain cases. In fact, the channel coding part of the separation based transmission in Section V inherently uses uncoded transmission in applying the SK scheme [7]. Here, we basically use the transmission scheme of [7] with correlated sources.

The basic idea is similar to the SK scheme for a point-to-point channel. At each step, each transmitter calculates the ‘error’ or the difference between the estimate at the receiver and the actual source realization at the transmitter. These errors are then scaled and transmitted simultaneously by both transmitters over the MAC. The power of these transmissions are taken to be fixed and very small (approaching zero). Based on the received signals, the receiver updates its estimates for both the sources. The scheme is terminated once the target distortions are achieved at the receiver. In the next section, we present simulation results for the energy–distortion tradeoff achieved by this uncoded transmission scheme, denoted by $E_{\text{uncoded}}(D)$. It is observed that the performance of this uncoded transmission scheme outperforms the separation based scheme, proving the suboptimality of separate source-channel coding in this model.

Remark 6.1: We note that the achievability part (channel coding) of the separation scheme uses a similar uncoded scheme at its core. The main difference between the scheme here andn the separation based scheme is that now we eliminate the ‘quantization’ step and deal directly with the source realizations at the transmitters.

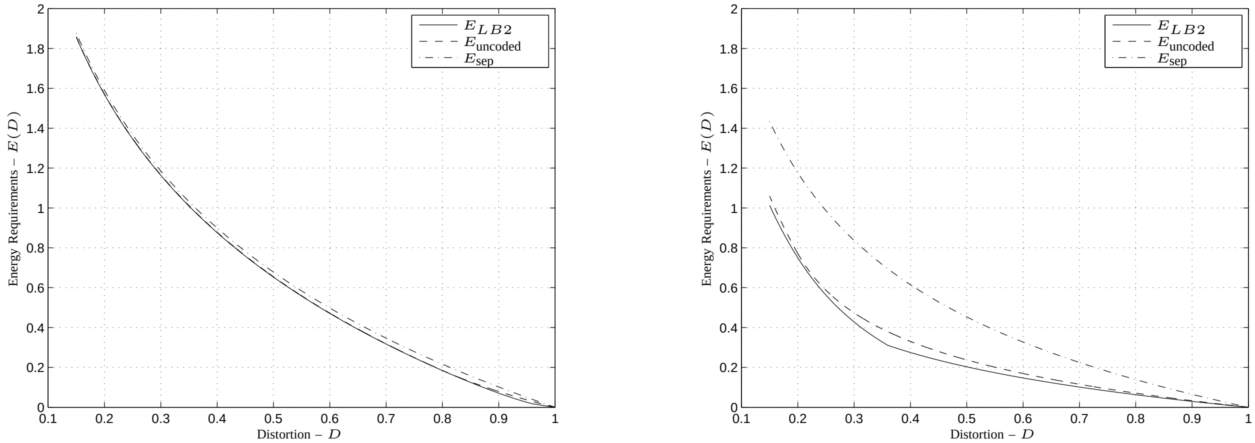
VII. Numerical Results and Discussions

In Figs. 2a and 2b, we plot the lower and upper bounds on $E(D)$ for $\sigma_S^2 = \sigma_Z^2 = 1$ and ρ equal to 0.2 and 0.8 respectively. We observe in both cases that the uncoded transmission scheme performs better than separation. For $\rho = 0.2$, all the bounds are close to each other. However, the gap between the performance of the uncoded transmission scheme and the lower bound is almost indistinguishable except at higher values of distortion. The curves are more separated when $\rho = 0.8$. In this case, the uncoded transmission scheme has a clear advantage over the separation based scheme.

While we do not have a general closed form expression for the performance of the uncoded transmission scheme, it is optimal when ρ is 0 or 1. Furthermore, when $\rho = 0$, the separation based scheme is also optimal though it has exactly twice the energy consumption of the lower bound when $\rho = 1$. This also implies that the energy consumption of the uncoded transmission scheme is half as much as that of the separation based scheme when $\rho = 1$.

VIII. Conclusions

We have studied the fundamental energy–distortion tradeoff function $E(D)$ in networks with one or two Gaussian sources and a single receiver when there is no constraint on the available channel bandwidth per source sample. Furthermore, perfect channel output has been assumed to be available at the transmitters causally. Using separation, $E(D)$ has been



(a) $\rho = 0.2$

(b) $\rho = 0.8$

Fig. 2: $E(D)$ bounds for a $(1, 1, \rho)$ network, for $\rho = 0.2$ and $\rho = 0.8$.

established for the single source, point-to-point channel case. For the case of two sources, we have first provided a lower bound on $E(D)$. This lower bound represents the absolute minimum energy (in Joules) that is required to reconstruct the sources within target distortion at the receiver, regardless of the communication/reconstruction strategies or the bandwidth used in the system. The lower bound is tight when the sources are independent. Besides the lower bound, we have also studied two different upper bounds based on separation and uncoded transmission, respectively. Simulation results suggest that uncoded transmission can beat the separation based scheme in many situations, proving the suboptimality of separation in this model. This also illustrates that uncoded transmission might be attractive in multi-user systems from an energy efficiency perspective, extending a similar observation in [5] to the wideband regime.

References

[1] M. Gastpar, "Uncoded transmission is exactly optimal for a simple Gaussian "sensor" network," *IEEE Trans. Inf. Theory*, vol. 54, no. 11, pp. 5247–5251, Nov. 2008.

[2] A. Jain, D. Gündüz, S. Kulkarni, H. V. Poor, and S. Verdú, "Energy efficient lossy transmission over sensor networks with feedback," *Thirty-Fifth IEEE International Conference on Acoustics, Speech, and Signal Processing (ICASSP 2010)*, submitted for publication.

[3] T. Kailath, "An application of Shannon's rate-distortion theory to analog communication over feedback channels," in *Proc. Princeton Symp. Sys. Science*, Princeton, NJ, USA, Mar. 1967.

[4] A. Lapidoth and S. Tinguely, "Sending a bivariate Gaussian source over a Gaussian MAC with feedback," in *Proc. International Symposium on Information Theory (ISIT)*, Nice, France, Jun. 2007.

[5] ——, "Sending a bivariate Gaussian source over a Gaussian MAC with feedback," Mar. 2009. [Online]. Available: http://arxiv.org/abs/0903.3487

[6] Y. Oohama, "Gaussian multiterminal source coding," *IEEE Trans. Inf. Theory*, vol. 43, no. 6, pp. 1912–1923, Nov. 1997.

[7] L. H. Ozarow, "The capacity of the white Gaussian multiple access channel with feedback," *IEEE Trans. Inf. Theory*, vol. 30, no. 4, pp. 623–628, Jul. 1984.

[8] J. P. M. Schalkwijk and L. I. Bluestein, "Transmission of analog waveforms through channels with feedback," *IEEE Trans. Inf. Theory*, vol. 13, pp. 617–619, Oct. 1967.

[9] S. Verdú, "On channel capacity per unit cost," *IEEE Trans. Inf. Theory*, vol. 36, no. 5, pp. 1019–1030, Sep. 1990.

[10] A. B. Wagner, S. Tavildar, and P. Viswanath, "Rate region of the quadratic Gaussian two-encoder source-coding problem," *IEEE Trans. Inf. Theory*, vol. 54, no. 5, pp. 1938–1961, May 2008.

[11] J.-J. Xiao and Z.-Q. Luo, "Compression of correlated Gaussian sources under individual distortion criteria," in *Proc. Forty-Third Annual Allerton Conference on Communication, Control, and Computing*, Monticello, IL, USA, Sep. 2005.