Minimum Energy per Bit for Gaussian Broadcast Channels with Common Message and Cooperating Receivers

Aman Jain  
Department of Electrical Engineering  
Princeton University  
Princeton, NJ 08544, USA  
Email: amanjain@princeton.edu

Sanjeev R. Kulkarni  
Department of Electrical Engineering  
Princeton University  
Princeton, NJ 08544, USA  
Email: kulkarni@princeton.edu

Sergio Verdú  
Department of Electrical Engineering  
Princeton University  
Princeton, NJ 08544, USA  
Email: verdu@princeton.edu

Abstract—This paper considers a three-terminal communication problem with one source node which broadcasts a common message to two destination nodes over a wireless medium. The destination nodes can cooperate over bidirectional wireless links. We study the minimum energy per information bit for this setup when there is no constraint on the available bandwidth. The physical link between any pair of nodes is affected by additive white Gaussian noise and circularly symmetric fading. The channel states are assumed to be not known at the transmitters. We show information-theoretic converse bounds on the minimum possible energy expenditure per information bit. These bounds are then compared against the achievable energy per bit using a decode-and-forward scheme which is shown to be weaker than the converse bounds by a factor of at most two for all cases of channel gains. For many cases of channel gains, decode-and-forward achieves the minimum energy per bit exactly. For the cases where the performance of decode-and-forward does not meet the converse bounds, we propose another scheme based on estimate-and-forward, assuming only phase-fading which is known at the receivers. For the particular cases of symmetric channel gains, we show that estimate-and-forward improves upon decode-and-forward, and is weaker than the converse bounds by at most a factor of 1.74.

I. INTRODUCTION

In communication systems with large available bandwidth and limited energy budget, it might be more appropriate to minimize the energy expenditure rather than maximize the communication rates. In such situations, instead of maximizing the capacity cost function, one should maximize the capacity per unit cost [13]. When 'cost' of a symbol is the energy spent in the transmission of the symbol over the channel, we get the specific formulation of information bits per unit energy. The reciprocal of this quantity – minimum energy per bit is a popular tool for studying the energy efficiency of communication systems.

The minimum energy per bit of point-to-point channels with additive Gaussian noise was studied in wide generality in [12]. Some key results for this setup are known. First, minimizing energy per bit requires vanishing spectral efficiency. Second, the minimum energy requirement can be reduced to checking that the total received energy (per bit) is at least $N_0 \log_2 2$. Third, the minimum energy per bit is not reduced even if channel state information is available at the receivers.

This research was supported in part by the Office of Naval Research under contract numbers W911NF-07-1-0185 and N00014-07-1-0555.

The minimum energy per bit is also known for the Gaussian multiple-access channel, the Gaussian broadcast channel and the Gaussian interference channel [13], [12], [8], [2]. However, the minimum energy per bit is not yet known for the three terminal setting of a Gaussian relay channel, though some progress has been made (see [5], [15] and the references therein).

In this paper, we study multicasting in a three terminal network. We consider a general setup where node 1 is the source node which has to communicate its message to nodes 2 and 3. Nodes 2 and 3 can communicate and cooperate over the wireless medium. This problem has been considered before, from a channel capacity point of view, as a fully cooperative relay broadcast channel (or cooperative broadcast channel, in short) with a common message (see [10], [4], [11], [9], [1] and references therein). Our work differs primarily due to our focus on the minimum energy per bit. This work is also different from [5] due to the fading (not known at the transmitters) and multicasting in our model. For larger networks, scaling laws for the minimum energy per bit for multicasting in random networks were studied in [6], for a similar channel model.

In Section II, we introduce the system model which is the same as in [6]. The physical channel is modeled as being affected by additive white Gaussian noise. We assume the presence of circularly symmetric fast fading. Channel states are assumed to be unknown at the transmitters. In Section III, a converse result on the minimum possible energy consumption per bit is presented. This converse is in terms of the effective network radius which depends only on the set of pairwise channel gains between different nodes in the network (see [6] for a converse based on an alternate definition of effective network radius). In Section IV, the effective network radius is explicitly evaluated for different cases of channel gain values. In almost all cases, either direct transmission or decode-and-forward is shown to achieve the lower bounds. In certain cases, where the destination nodes are close together but separated from the source, neither direct transmission nor decode-and-forward are optimal. However, the energy expenditure of decode-and-forward is shown to be at most twice the lower bound. In Section V, we simplify our setup, considering only the case of symmetric channel gains and presence of only phase-
fading which is known at the receivers. For the symmetric case, an estimate-and-forward based scheme is shown to have an improved performance over decode-and-forward. For an asymptotic case of channel gains, estimate-and-forward performs close to optimal, which is 3 dB better than the performance of decode-and-forward. In fact, the proposed estimate-and-forward scheme is shown to have an energy expenditure which is at most 1.74 away from the lower bound even for asymmetric channel gains. We conclude this paper in Section VI.

II. SYSTEM MODEL

A. Channel Model

We deal with a discrete-time complex additive Gaussian noise channel with fading. There are three nodes \{1, 2, 3\} in the network, node 1 being the source node and nodes 2 and 3 being the destination nodes. Let node \(i \in \{1, 2, 3\}\) transmit \(x_{i,t} \in \mathbb{C}\) at time \(t\), and let \(y_{j,t} \in \mathbb{C}\) be the received signal at node \(j \in \{1, 2, 3\}\). Then,

\[
y_{j,t} = \sum_{i=1}^{3} h_{ij,t} x_{i,t} + z_{j,t}
\]

(1)

for \(j = 1, 2, 3\), where \(z_{j,t}\) is circularly symmetric complex additive Gaussian noise at the receiver \(j\), distributed according to \(\mathcal{CN}(0, N_0)\). The noise terms are independent for different receivers as well as for different times. The fading between any two distinct nodes \(i\) and \(j\) is modeled by complex-valued circularly symmetric random variables \(h_{ij,t}\) which are i.i.d. for different times. We assume that \(h_{ii,t} = 0\) for all nodes \(i\) and all times \(t\). Also, for all \((i, j) \neq (l, m)\), the pair \(h_{ij,t}\) and \(h_{lm,t}\) is independent for all times \(t\).

Absence of channel state information at a transmitter \(i\) implies that \(x_{i,t}\) is independent of the channel state realization vector \((h_{i1,t}, h_{i2,t}, h_{i3,t})^T\) for all times \(t\). The quantity \(\mathbb{E}[|h_{ij}|^2]\) is referred to as the channel gain between nodes \(i\) and \(j\).

The source node (node 1) has a channel gain of \(g_{12} = \mathbb{E}[|h_{12}|^2]\) and \(g_{13} = \mathbb{E}[|h_{13}|^2]\) to the destination nodes 2 and 3, respectively. Nodes 2 and 3 have gains \(g_{23} = \mathbb{E}[|h_{23}|^2]\) and \(g_{32} = \mathbb{E}[|h_{32}|^2]\) between themselves. (See Fig. 1).

B. Problem Setup

Both destination nodes are assumed to have receiving, processing and transmitting capabilities. Each destination node acts as a relay for the other. Thus, each destination node is provided a relay and a decoding function. The relay function at node \(i \in \{2, 3\}\) decides the channel input symbol \(x_{i,t}\) at time \(t\) based on all the previous \(t-1\) channel outputs at the node. The decoding function at node \(i \in \{2, 3\}\) decodes a suitable message \(\hat{m}_i\) from the message set \(\mathcal{M}\) containing \(M\) messages, once all the \(n\) channel outputs at the node are received. A decoding error is said to have occurred when any of the destination nodes fails to decode the correct message transmitted by the source. The probability of error of the code is defined as

\[
P_e = \frac{1}{M} \sum_{m \in \mathcal{M}} P[\hat{m}_1 \neq m \text{ or } \hat{m}_2 \neq m | m \text{ is the message}]
\]

(2)

Define the expected total energy expenditure (for all nodes) of the code to be

\[
E_{\text{total}} = \frac{3}{M} \sum_{i=1}^{3} \sum_{t=1}^{\infty} \mathbb{E}[|x_{i,t}|^2]
\]

(3)

where the expectation is over the message, noise and fading. The energy per bit of the code is defined as

\[
E_b = \frac{E_{\text{total}}}{\log_2 M}
\]

(4)

An \((n, M, E_{\text{total}}, \epsilon)\) code spans \(n\) channel uses, has \(M\) messages at the source node, expected total energy consumption at most \(E_{\text{total}}\) and probability of error at most \(0 \leq \epsilon < 1\).

Definition 1: [13] Given \(0 \leq \epsilon < 1\), \(E_b \in \mathbb{R}_+\) is an \(\epsilon\)-achievable energy per bit if for every \(\delta > 0\), there exists an \(E_0 \in \mathbb{R}_+\) such that for every \(E_{\text{total}} \geq E_0\) an \((n, M, E_{\text{total}}, \epsilon)\) code can be found such that

\[
\frac{E_{\text{total}}}{\log_2 M} \leq E_b + \delta
\]

(5)

\(E_b\) is an achievable energy per bit if it is \(\epsilon\)-achievable energy per bit for all \(0 < \epsilon < 1\), and the minimum energy per bit \(E_{\text{min}}\) is the infimum of all achievable energy per bit values.

III. CONVERSE

In Theorem 1, we present a converse result which holds irrespective of whether the channel states are known at the receivers.

Theorem 1. A lower bound on the minimum energy per bit for the described model is given by

\[
E_1 = \frac{N_0 \log_e 2}{G}
\]

(6)

where \(G\) is the effective radius of the network, given by

\[
G(g_{12}, g_{13}, g_{23}, g_{32}) = \max_{\alpha_1, \alpha_2, \alpha_3 \geq 0, \alpha_1 + \alpha_2 + \alpha_3 = 1} \min \left\{ (g_{12} + g_{13}) \alpha_1, g_{12} \alpha_1 + g_{32} \alpha_3, g_{13} \alpha_1 + g_{23} \alpha_2 \right\}
\]

(7)

Remark 1: Note that (6)–(7) can be reformulated as the following linear program (8), i.e., the value of \(E_1\) given by
(6)–(7) is the same as the value of $E_1$ given by (8).

$$E_1 \triangleq \min_{\alpha_1, \alpha_2, \alpha_3} \{ \alpha_1 + \alpha_2 + \alpha_3 :$$

$$g_{12} + g_{13} \alpha_1 \geq N_0 \log_e 2$$
$$g_{12} \alpha_1 + g_{23} \alpha_3 \geq N_0 \log_e 2$$
$$g_{13} \alpha_1 + g_{23} \alpha_2 \geq N_0 \log_e 2$$

$$\alpha_1, \alpha_2, \alpha_3 \geq 0 \} \quad (8)$$

Before proving Theorem 1, we first introduce some more notation and state Lemma 1.

Let $u \in U \triangleq \{\{1\}, \{1, 2\}, \{1, 3\}\}$ be a subset of nodes. We use the notation $x_u \in \mathbb{C}^3$ to denote the transmissions at the nodes in the subset $u$, i.e., the $i$th element of $x_u$ is given by

$$(x_u)_i = \begin{cases} x_i & \text{if } i \in u \\ 0 & \text{if } i \notin u \end{cases}$$

Similarly, we use notation $y_u \in \mathbb{C}^3$ to denote the observation vector at the set of nodes $u$. Without subscripts, $x$ and $y$ denote the transmissions and observations at all the nodes. Let $H$ be the random matrix formed by the fading coefficients $h_{ij}$, for $i, j = 1, 2, 3$. The $(i, j)^{th}$ entry of the random fading matrix $H_u$ is given by

$$(h_{u})_{ij} = \begin{cases} h_{ij} & \text{if } j \in u \\ 0 & \text{otherwise} \end{cases}$$

Dropping the time indices, we can rewrite (1) to express the observation vector at the set $u^c = \{1, 2, 3\} \setminus u$ as

$$y_{u^c} = H_{u^c}^T x + z_{u^c} \quad (9)$$

where $z_{u^c}$ is the noise vector denoting noise only at the set of nodes $u^c$, i.e., $(z_{u^c})_i = 0$ if $i \notin u^c$. For different possibilities of $u \in U$, (9) denotes the channel seen by the nodes in the set $u^c$.

Lemma 1. The multicasting minimum energy per bit is lower bounded by

$$E_2 \triangleq \inf_{p_1, p_2, p_3 \geq 0} \max_{p_k; \|x_k\| \leq p_k \text{ for } i = 1, 2, 3} \frac{N_0 \log_e 2}{\sum_{i=1}^{3} p_i} \frac{\sum_{i \in u} \sum_{j \in u^c} \mathbb{E}[|h_{ij}|^2]}{\sum_{i \in u} \sum_{j \in u^c} \mathbb{E}[|h_{ij}|^2]} \quad (10)$$

Proof: The proof is omitted due to space constraints. However, it is along the lines of the proof of [7, Lemma 1] (see also, [6, Lemma 1]). Crucial to the proof is the fact that the set $U$ is a collection of all cut-sets separating the source and a destination in the given network.

Proof of Theorem 1: We first need to apply (10) to our channel model. Fix the constraints on average power per channel use: $p_1, p_2, p_3 \geq 0$ for transmissions from nodes 1, 2, and 3 respectively, such that $\sum_{i=1}^{3} p_i > 0$ and also fix a cut $u \in U$. For a given probability distribution of $x$ (independent of $H$), we can bound the mutual information in (10) by

$$I(x_u; H_{u^c}^T x + z_{u^c}|x_{u^c}, H)$$

$$= I(x_u; H_{u^c}^T x + z_{u^c}|x_{u^c}, H)$$

$$\leq \mathbb{E} \left[ \log_2 \det \left( I + \frac{1}{N_0} \text{cov}(H_{u^c}^T x_{u^c}|x_{u^c}, H) \right) \right]$$

$$\leq \frac{\log_2 e}{N_0} \mathbb{E} \left[ \text{tr} \left( \text{cov}(H_{u^c}^T x_{u^c}|x_{u^c}, H) \right) \right]$$

$$= \frac{\log_2 e}{N_0} \sum_{i \in u} \sum_{j \in u^c} \mathbb{E}[|h_{ij}|^2]$$

$$\leq \frac{\log_2 e}{N_0} \sum_{i \in u} \sum_{j \in u^c} \mathbb{E}[|h_{ij}|^2]$$

where (11) is an upper bound on the mutual information based on the maximum received power under AWGN; (12) is due to Hadamard’s inequality and the fact that $\log(1+x) \leq x$ for any $x \geq 0$; (13) is from the expansion of the trace term in (12); and, (14) is obtained by maximizing the right hand side of (13) among all $x$ independent of $h_{ij} = (h_{1j}, h_{2j}, h_{3j})$ such that $\mathbb{E}[|x_i|^2] \leq p_i$, taking into account that the channel coefficients are independent with zero mean.

Substituting (14) in (10), we get

$$E_2 \geq \inf_{p_1, p_2, p_3 \geq 0} \max_{p_k; \|x_k\| \leq p_k \text{ for } i = 1, 2, 3} \frac{N_0 \log_e 2}{\sum_{i=1}^{3} p_i} \frac{\sum_{i \in u} \sum_{j \in u^c} \mathbb{E}[|h_{ij}|^2]}{\sum_{i \in u} \sum_{j \in u^c} \mathbb{E}[|h_{ij}|^2]} \quad (15)$$

$$= \min_{\alpha_1, \alpha_2, \alpha_3 \geq 0} \frac{N_0 \log_e 2}{\sum_{i \in u} \sum_{j \in u^c} \mathbb{E}[|h_{ij}|^2]} \quad (16)$$

where (16) is obtained from (15) by replacing $p_i/\sum p_i$ with $\alpha_i$. The statement of Theorem 1 is now immediate from the fact $U \triangleq \{\{1\}, \{1, 2\}, \{1, 3\}\}$.

IV. PERFORMANCE OF DECODE-AND-FORWARD

A. Decode-and-Forward

To achieve minimum energy per bit in Gaussian point-to-point channels, we require vanishing spectral efficiency. Likewise, in this work, we assume availability of arbitrarily large bandwidth. In particular, we assign each transmitter its own wide frequency band that is orthogonal to the bands of other transmitters. Thus, each transmission is a wideband broadcast not affected by the interference from other transmissions. All receiver nodes listen to transmissions over all the bands. Various wideband communication schemes can be constructed which let a receiver decode the message reliably as soon as the accumulated energy per bit at the receiver exceeds $N_0 \log_e 2$ [12]. Therefore, without defining the coding scheme explicitly, it is understood that the decoder will be able to decode the message once the total receiver energy per bit exceeds $N_0 \log_e 2$ at its terminal.

In the wideband scenario, it was shown in [12] that the knowledge of the channel states at the receiver does not reduce the minimum energy per bit requirements. Therefore, the minimum energy per bit expended at the transmitter depends only on the noise spectral density and the channel
gains, as long as the transmissions are independent of the channel states. On the other hand, knowledge of the channel states at the receivers is crucial for estimate-and-forward, as will be discussed in Section V.

Direct transmission (or broadcast) from node 1 to nodes 2 and 3 is taken to be a special case of decode-and-forward where the destination nodes completely decode the message but do not forward it to each other.

B. Evaluation of network radius and comparison with decode-and-forward

Next, we use Theorem 1 to identify different regimes of operation (in terms of the channel gains) by explicitly evaluating the effective network radius $G$ for each regime. Also, within each regime, performance of decode-and-forward (cf. [3, Theorem 7]) or direct transmission is compared with the lower bounds. We only provide the final results here, relegating the details to Appendix A.

To determine $E_b$ according to Theorem 1, the analysis is divided into following multiple cases of channel gains. For each case, we assume, without loss of generality,

$$g_{13} \leq g_{12} \quad (17)$$

1) $g_{12} \leq g_{13} + g_{32}$

The three subcases are:

a) $g_{23} \leq g_{13}$: In this case, $G = g_{13}$. Hence,

$$\frac{E_b}{N_0 \min} \geq \frac{\log_e 2}{g_{13}} \quad (18)$$

The minimum energy per bit in (18) is achievable by a broadcast with enough energy for node 3 to be able to decode the message. This energy is also enough for node 2 to be able to decode the message.

b) $g_{32} g_{13} \leq g_{32} g_{23} \leq g_{12} g_{23} + g_{32} g_{13}$: In this case, $G = (g_{12} g_{23})/(g_{12} + g_{23} - g_{13})$. Hence,

$$\frac{E_b}{N_0 \min} \geq \left( \frac{1}{g_{23}} + \frac{1}{g_{12}} - \frac{g_{13}}{g_{12} g_{23}} \right) \log_e 2 \quad (19)$$

This minimum energy per bit is achieved by decode-and-forward where the message is first decoded by node 2 which then forwards it to node 3. Node 3 is able to decode the message based on the initial transmission by the source node (with transmission energy per bit $E_b = N_0 \log_e 2/g_{12} + \epsilon$, for any $\epsilon > 0$) and the subsequent transmission by node 2 (with $E_b = N_0 \log_e 2 (1 - (g_{13}/g_{12})) /g_{23} + \epsilon$, for any $\epsilon > 0$).

c) $g_{12} g_{23} + g_{32} g_{13} \leq g_{32} g_{23}$: This case differs from a) and b) in that decode-and-forward does not achieve the minimum energy per bit lower bound. Gains are now in the regime where it is better for nodes to cooperatively decode their messages rather than one by one. For this case,

$$G = \frac{g_{12} + g_{13}}{1 + \frac{g_{12}}{g_{23}} + \frac{g_{13}}{g_{32}}} \quad (20)$$

Hence,

$$\frac{E_b}{N_0 \min} \geq \left( \frac{1 + \frac{g_{12}}{g_{23}} + \frac{g_{13}}{g_{32}}}{g_{12} + g_{13}} \right) \log_e 2 \quad (21)$$

A few remarks: First, note that as $g_{23}, g_{32} \to \infty$, $G \to g_{12} + g_{13}$. This is the regime where the links between the two destination nodes are so strong that the cooperation entails negligible energy. In this case,

$$\frac{E_b}{N_0 \min} \geq \frac{\log_e 2}{g_{12} + g_{13}} \quad (22)$$

which is also the minimum energy per bit of a point-to-point channel with two antennas (nodes 2 and 3) at the destination.

Second, in general, decode-and-forward does not achieve the lower bound in (21). However, decode-and-forward by node 2 achieves

$$\frac{E_b}{N_0_{DF}} = \left( \frac{1}{g_{23}} + \frac{1}{g_{12}} - \frac{g_{13}}{g_{12} g_{23}} \right) \log_e 2 \quad (23)$$

The ratio of the upper and lower bound is thus,

$$\frac{E_b}{E_{0 \min}} \leq \frac{g_{12} + g_{13}}{1 + \frac{g_{12}}{g_{23}} + \frac{g_{13}}{g_{23}} g_{12}} \leq \frac{g_{12} + g_{13}}{1 + \frac{g_{13}}{g_{23}}} \leq \left( 1 + \frac{g_{13}}{g_{12}} \right) \leq 2 \quad (24)$$

since $g_{13} \leq g_{12}$ by (17). The ratio of 2 (i.e., 3 dB difference) is approached in the case where $g_{12} = g_{13} = a$, $g_{23} = g_{32} = b$ and $\frac{a}{b} \to 0$.

2) $g_{13} + g_{32} \leq g_{12}$

The two subcases of this case show the same behavior as the first two subcases of the previous case.

a) $g_{23} \leq g_{13}$: In this case, $G = g_{13}$ which implies that

$$\frac{E_b}{N_0 \min} \geq \frac{\log_e 2}{g_{13}} \quad (25)$$

The minimum energy per bit is achieved by a broadcast with enough energy to reach both nodes 2 and 3.

b) $g_{13} \leq g_{23}$: In this case, $G = (g_{12} g_{23})/(g_{12} + g_{23} - g_{13})$. Hence,

$$\frac{E_b}{N_0 \min} \geq \left( \frac{1}{g_{23}} + \frac{1}{g_{12}} - \frac{g_{13}}{g_{12} g_{23}} \right) \log_e 2 \quad (26)$$

This minimum energy per bit is achieved by decode-and-forward from node 2 to node 3.

Remark 2: The case of simple broadcast channels (where the nodes do not cooperate) with a common message, falls within the case (2a) above, where it is shown that it is sufficient to broadcast the message to the weaker receiver.

V. THE SYMMETRIC CHANNEL GAINS CASE: PERFORMANCE OF ESTIMATE-AND-FORWARD

In this section, we describe an estimate-and-forward scheme for the particular case of symmetric gains in the situation where decode-and-forward fails to meet the lower
bound on the minimum energy per bit. The main idea here is for node 2 to quantize and forward its observations to node 3 which is then able to decode the message. The final step consists of node 3 transmitting the decoded message with just enough power so that node 2 too can decode the message. For the Gaussian relay channels, a very similar scheme, without the last step, is discussed in [5] as the ‘side information lower bound’.

Let
\[ g_{12} = g_{13} \triangleq a \]  
and,
\[ g_{23} = g_{32} \triangleq b \]  
Therefore, we have reduced the general setup to a symmetric one, as shown in Fig. 2. Since the channel coefficients are known at the receivers, perfect phase correction takes place at all the receivers. We further simplify the model by assuming only phase-fading (fixed amplitudes of fading coefficients) which is known at the receivers. This is justified if the transmissions are spread over a sufficiently large number of independent time or frequency slots, thus mitigating amplitude fading by maximal ratio combining of received signals over the various slots. Note that lack of channel state information at the transmitters precludes any possibility of beamforming. Due to perfect phase correction at the receivers, we only deal with real valued transmissions and additive noise in this section.

First, the converse. Recall from Section IV that decode-and-forward fails to provide tight bounds when \( b/a \geq 2 \). In this case, the lower bound on the minimum energy per bit is given by
\[ E_{b_{\min}} \geq E_{b_{lower}} \triangleq N_0 \log_e 2 \left( \frac{1}{2a} + \frac{1}{b} \right) \]  
(29)
On the other hand, the minimum energy per bit achieved by decode-and-forward is
\[ E_{b_{DF}} = \frac{N_0 \log_e 2}{a} \]  
(30)
We now describe an estimate-and-forward based scheme which has lower energy expenditure than (30). Let the transmission by the source be a zero mean Gaussian random variable \( x \), such that \( \mathbb{E}[x^2] = p \). This is received at node \( i \) as \( y_i = \sqrt{a} x + z_i \) for \( i = 2, 3 \), where \( z_i \sim N_0(0, N_0/2) \). Next, suppose that node 2 transmits an estimate \( u \) based on \( y_2 \) to node 3. Define
\[ u \triangleq y_2 + \tilde{z} \]  
(31)
where \( \tilde{z} \sim N_0(0, d) \) is a random variable independent of both \( z_1 \) and \( z_2 \), for \( d > 0 \). The variable \( u \) acts as the “quantization” variable which quantizes and carries the information about \( y_2 \) [3]. Another way to think of it is in terms of the Wyner-Ziv side information problem [14], where node 3 tries to reconstruct \( y_2 \) from \( u \) using \( y_3 \) as side information.

Note that the variable \( u \) is Gaussian with zero mean and variance \( ap + (N_0/2) + d \). Also,
\[ I(y_2; u) - I(y_3; u) = \frac{1}{2} \log \left( 1 + N_0 \frac{1}{a p + \frac{2p}{d}} \right) \triangleq R(d) \]  
(32)
Therefore, by [14, Theorem 4], \( u \) can be encoded at a rate \( R(d) \) such that \( \mathbb{E}[(y_2 - u)^2] \leq d \). From [12], we know that we can carry \( R(d) \) bits over a channel of gain \( b \) with a total energy expenditure of \( R(d) \) times \( N_0 \log_e 2/b \) which is same as
\[ \frac{N_0 \log_e 2}{2b} \log \left( 1 + N_0 \frac{1}{a p + \frac{2p}{d}} \right) \]  
(33)
Therefore, by the end of the transmission, node 3 has the estimates \( u \) and \( y_3 \), which implies that a communication rate of
\[ \frac{1}{2} \log \left( 1 + \frac{ap}{N_0} \right) \]  
(34)
to node 3 is possible. Once node 3 successfully decodes the message, it transmits \( x \) at power
\[ p_2 \triangleq \frac{a}{b} \frac{p}{1 + \frac{2p}{N_0}} \]  
(35)
so that the rate given in (34) is also possible at node 2.

Finally, adding up the energy expenditures at all the nodes, we get the minimal total energy per bit of this communication scheme as
\[ E_{b_E F} = (N_0 \log_e 2) \times \inf_{p > 0, d > 0} \frac{p + \frac{1}{b} \log_e \left( 1 + \frac{1 + \frac{ap}{d}}{p} \right) + \frac{a}{b} \frac{p}{1 + \frac{2p}{N_0}}}{\log_e \left( 1 + \frac{1 + \frac{ap}{d}}{p} \right)} \]  
(36)
where, for simplification, we have replaced \( p/(N_0/2) \) with \( p \) and \( d/(N_0/2) \) with \( d \).

Using (29), the ratio of the upper and lower bounds is given by
\[ \frac{E_{b_E F}}{E_{b_{lower}}} = \left( \frac{1}{2a} + \frac{1}{b} \right)^{-1} \times \inf_{p > 0, d > 0} \frac{p + \frac{1}{b} \log_e \left( 1 + \frac{1 + \frac{ap}{d}}{p} \right) + \frac{a}{b} \frac{p}{1 + \frac{2p}{N_0}}}{\log_e \left( 1 + \frac{1 + \frac{ap}{d}}{p} \right)} \]  
(37)
It can be checked that the ratio in (37) does not depend on the absolute values of \( a \) and \( b \), rather only on their ratio, since replacing \( a, b \) with \( \lambda a, \lambda b \) respectively and \( p \) with \( p/\lambda \), for \( \lambda > 0 \), does not change (37). Therefore, it is enough to set \( a = 1 \) and vary \( b \) over a large range (\( \geq 2 \)) to understand the
behavior of the ratio (37). Also notice from (37) that estimate-and-forward never performs worse than decode-and-forward due to the fact that as \( d \to \infty \) and \( p \to 0 \), estimate-and-forward effectively reduces to decode-and-forward. Define

\[
f(p, d, b) \triangleq \left(\frac{1}{2} + \frac{1}{b}\right)^{-1} p + \frac{1}{b} \log_e \left(1 + \frac{1 + \frac{p}{1+d}}{d}\right) + \frac{1-p}{b(1+d)}
\]

which is just (37) without the optimization over \( p \) and \( d \). For \( 2 \leq b \leq 13.3 \), set \( d = \infty \) and \( p \to 0 \), so that \( f(p, d, b) \leq 1.739 \). For \( b > 13.3 \), set \( p = 0.2912 \) and \( d = 0.8497 \). Checking that \( f(0.2912, 0.8497, 13.3) = 1.737 \) and that \( \frac{df}{dp}(0.2912, 0.8497, b) < 0 \) for \( b \geq 13.3 \), we immediately establish that \( \inf_{p>0,d>0} f(p, d, b) \leq 1.739 \) for all \( b \geq 2 \). Numerical calculation of the ratio in (37) is plotted in Fig. 3, for \( a = 1 \) and \( b \geq 2 \). Note that the y-axis in Fig. 3 is used to denote both the ratio of upper and lower bounds and the normalized energy per bit values (i.e., \( E_b/(N_0 \log_e 2) \)). Also note that the proposed estimate-and-forward scheme appears to beat decode-and-forward only when \( b/a \) is greater than about 13.

Though we have proposed an estimate-and-forward scheme only for the symmetric case, it is easy to extend the analysis to the asymmetric case. This gives

\[
E_{b,\text{EF}} = N_0 \log_e 2 \min \left\{ \begin{array}{c} p + \frac{1}{g_{12}} \log_e \left(1 + \frac{1 + \frac{p}{1+d}}{d}\right) + \frac{p}{g_{12}} \left(1 + \frac{g_{12}d}{g_{12}d + 1}\right) \\ \inf_{p>0,d>0} \end{array} \right. 
\]

\[
E_{\text{lower}} = N_0 \log_e 2 \min \left\{ \begin{array}{c} p + \frac{1}{g_{13}} \log_e \left(1 + \frac{1 + \frac{p}{1+d}}{d}\right) + \frac{p}{g_{13}} \left(1 + \frac{g_{13}d}{g_{13}d + 1}\right) \\ \inf_{p>0,d>0} \end{array} \right. 
\]

Simulation results suggest that the ratio of \( E_{b,\text{EF}}/E_{\text{lower}} \) is less than 1.74 even for asymmetric cases.

VI. Conclusion

In this work, we studied the maximum energy efficiency in a three terminal Gaussian channel with two cooperating destinations and a common message. The minimum energy per bit was established for various conditions on channel gains. In all these cases, the minimum energy per bit was achieved by suitable decode-and-forward. In the situation where the channel gains between the destination nodes are high and the channel gains from the source node to the destination nodes are low, decode-and-forward is not optimal. It is clear that some other kind of cooperation between the destination nodes is required that involves the destinations to share their ‘noisy’ estimates without decoding the message. Prompted by this observation, we have proposed an estimate-and-forward scheme. For the case of symmetric channel gains and in the presence of phase-fading only, estimate-and-forward is shown to reduce the worst case gap between the upper and lower bound from a factor of 2 (for decode-and-forward) to a factor of less than 1.74. Based on simulation results, we expect the factor of 1.74 to hold even for the case of asymmetric gains.

Complete characterization of the minimum energy per bit for the given setup is still an open problem.

APPENDIX A

EVALUATION OF EFFECTIVE NETWORK RADIUS IN THEOREM 1

We begin by reducing the number of optimization variables in (7) to get

\[
G(g_{12}, g_{13}, g_{23}, g_{32}) = \max_{\alpha_1, \alpha_2 \geq 0} \min_{\alpha_1 + \alpha_2 \leq 1} \{I_1, I_2, I_3\} \tag{40}
\]

where,

\[
I_1(\alpha_1, \alpha_2) \triangleq (g_{12} + g_{13})\alpha_1 \tag{41}
\]

\[
I_2(\alpha_1, \alpha_2) \triangleq g_{12}\alpha_1 + g_{32}(1 - \alpha_1 - \alpha_2) \tag{42}
\]

\[
I_3(\alpha_1, \alpha_2) \triangleq g_{13}\alpha_1 + g_{23}\alpha_2 \tag{43}
\]

Note that,

\[
I_1 \leq I_2 \implies \alpha_1 \leq -\frac{g_{32}}{g_{13} + g_{32}}(1 - \alpha_2) \tag{44}
\]

\[
I_2 \leq I_3 \implies (g_{12} - g_{32} - g_{13})\alpha_1 + g_{32} \leq (g_{23} + g_{32})\alpha_2 \tag{45}
\]

\[
I_3 \leq I_1 \implies \alpha_2 \leq \frac{g_{12}}{g_{23}}\alpha_1 \tag{46}
\]

Next, we evaluate \( G(g_{12}, g_{13}, g_{23}, g_{32}) \) for different cases of channel gains, in the following manner. For every case of channel gains, we first find out the set of values \( (\alpha_1, \alpha_2) \) for which each of the terms \( I_1, I_2 \) and \( I_3 \) are minimized. Let \( (\alpha_1, \alpha_2)_1 \) be the value amongst all the values of \( (\alpha_1, \alpha_2) \) for which \( I_1 \leq I_2 \) and \( I_1 \leq I_3 \), and thus, for which \( I_1 \) is maximized. Similarly, \( (\alpha_1, \alpha_2)_2 \) and \( (\alpha_1, \alpha_2)_3 \) corresponding to \( I_2 \) and \( I_3 \) respectively, are determined. Then, for the given case of channel gains \( g_{12}, g_{13}, g_{23} \) and \( g_{32} \), we have

\[
G(g_{12}, g_{13}, g_{23}, g_{32}) = \max\{I_1((\alpha_1, \alpha_2)_1), I_2((\alpha_1, \alpha_2)_2), I_3((\alpha_1, \alpha_2)_3)\} \tag{47}
\]
Now, let us calculate $G$ for the different cases of channel gains as following –

1) $g_{12} \geq g_{13}$ and $g_{13} + g_{32} \geq g_{12}$

Note that $(\alpha_1, \alpha_2)$ can only take values in the triangle $EOD$ shown in Fig. 4, which also shows the regions for which either $I_1$, $I_2$ or $I_3$ are the minimum of all three terms. In particular from (44) and (46), for triangle $EOA$, $I_1 \leq I_2, I_3$. From (45) and (44), for triangle $EAB$, $I_2 \leq I_1, I_3$. And from (45) and (46), for quadrilateral $OABD$, $I_3 \leq I_1, I_2$. The point

$$A \triangleq \left( \frac{g_{23} g_{32}}{g_{23} g_{32} + g_{13} g_{23} + g_{12} g_{32}}, \frac{g_{12} g_{23}}{g_{23} g_{32} + g_{13} g_{23} + g_{12} g_{32}} \right)$$

is common to all three regions.

A few remarks about the points $F$, $J$, and $H$. The point $F \triangleq (g_{32}/(g_{32} + g_{13}), 0)$ always lies between $O$ and $D$, where $O \triangleq (0, 0)$ and $D \triangleq (1, 0)$. Similarly, the point $J \triangleq (0, g_{32}/(g_{32} + g_{23}))$ always lies between $O$ and $E$ where $E \triangleq (0, 1)$. Due to the defining conditions of this case, the point $H \triangleq (g_{32}/(g_{13} + g_{32} - g_{12}), 0)$ always lies beyond $D$. Also, the points $B$ and $C$ are given by $B \triangleq (g_{23}/(g_{12} + g_{23} - g_{13}), (g_{12} - g_{13})/(g_{12} + g_{23} - g_{13}))$ and $C \triangleq (g_{23}/(g_{12} + g_{23}), g_{12}/(g_{12} + g_{23}))$.

Next, note that in region $EOA$, maximizing $I_1 = (g_{12} + g_{13})\alpha_1$ implies maximizing $\alpha_1$, which in turn implies operating at point $A$, i.e.,

$$I_1((\alpha_1, \alpha_2)_1) = \frac{(g_{12} + g_{13}) g_{23} g_{32}}{g_{23} g_{32} + g_{13} g_{23} + g_{12} g_{32}}$$

In the region $OABD$, we have to maximize

$$I_3((\alpha_1, \alpha_2)_3) = (g_{13}, g_{23}) \cdot (\alpha_1, \alpha_2)_3$$

Since $(g_{13}, g_{23})$ lies in the first quadrant only, the maximum value of the inner product on the right hand side of (50) is maximized at either $A$, $B$ or $D$.

Thus, for $g_{13} \geq g_{23}$, the optimal operating point is $(\alpha_1, \alpha_2)_3 = D$, which implies

$$I_3((\alpha_1, \alpha_2)_3) = g_{13}$$

(51)

For $g_{13} \leq g_{23}$ and $g_{32} g_{23} \leq g_{12} g_{23} + g_{32} g_{13}$, $(\alpha_1, \alpha_2)_3 = B$, thus

$$I_3((\alpha_1, \alpha_2)_3) = \frac{g_{12} g_{23}}{g_{12} + g_{23} - g_{13}}$$

(52)

And, for $g_{12} g_{23} + g_{32} g_{13} \leq g_{23} g_{32}$, $(\alpha_1, \alpha_2)_3 = A$, thus

$$I_3((\alpha_1, \alpha_2)_3) = \frac{(g_{12} + g_{13}) g_{23} g_{32}}{g_{23} g_{32} + g_{13} g_{23} + g_{12} g_{32}}$$

(53)

In the region $EAB$, we have to maximize

$$I_2((\alpha_1, \alpha_2)_2) = (g_{12} - g_{32}, -g_{32}) \cdot (\alpha_1, \alpha_2)_2 + g_{32}$$

(54)

where, $(\alpha_1, \alpha_2)_2$ can only be one of the points $E$, $A$ and $B$ depending upon the channel gains. Note that the vector $(g_{12} - g_{32}, -g_{32})$ lies only in third and fourth quadrants.

For $g_{32} g_{23} \geq g_{12} g_{23} + g_{32} g_{13}$, $(\alpha_1, \alpha_2)_2 = B$, thus

$$I_2((\alpha_1, \alpha_2)_2) = \frac{g_{12} g_{23}}{g_{12} + g_{23} - g_{13}}$$

(55)

And, for $g_{12} g_{23} + g_{32} g_{13} \leq g_{23} g_{32}$, $(\alpha_1, \alpha_2)_3 = A$, thus

$$I_2((\alpha_1, \alpha_2)_2) = \frac{(g_{12} + g_{13}) g_{23} g_{32}}{g_{23} g_{32} + g_{13} g_{23} + g_{12} g_{32}}$$

(56)

The conditions for operating at point $E$ are never satisfied.

Before moving further, we note the following implications:

$$g_{13} \leq \frac{g_{12} g_{23}}{g_{12} + g_{23} - g_{13}}$$

(57)

and,

$$g_{12} g_{23} \leq \frac{(g_{12} + g_{13}) g_{23} g_{12}}{g_{23} g_{32} + g_{13} g_{23} + g_{12} g_{32}}$$

(58)

Finally, for each instance of channel gains within this case, we need to maximize $G$ over the maximal values within the three regions. Putting together our analysis, the following sub-cases of channel gains emerge.

a) $g_{23} \leq g_{13}$

For this case,

$$G(g_{12}, g_{13}, g_{23}, g_{32}) = \max \left\{ \frac{(g_{12} + g_{13}) g_{23} g_{12}}{g_{23} g_{32} + g_{13} g_{23} + g_{12} g_{32}}, \frac{g_{12} g_{23}}{g_{12} + g_{23} - g_{13}} \right\}$$

(59)

From (57) and (58),

$$G = g_{13}$$

(60)
2) \( g_{13} \leq g_{23} \leq \frac{g_{12} g_{23}}{g_{12}} g_{23} + g_{13} \)

For this case,

\[
G(g_{12}, g_{13}, g_{23}, g_{32}) = \max \left\{ \frac{(g_{12} + g_{13})g_{23}g_{32}}{g_{23}g_{32} + g_{13}g_{23} + g_{12}g_{32}}, \frac{g_{12}g_{23}}{g_{12} + g_{23} - g_{13}} \right\}
\]

From (58),

\[
G = \frac{g_{12}g_{23}}{g_{12} + g_{23} - g_{13}} \quad (62)
\]

For this case, maximal value within all regions is the same.

\[
G = \frac{(g_{12} + g_{13})g_{23}g_{32}}{g_{23}g_{32} + g_{13}g_{23} + g_{12}g_{32}} \quad (63)
\]

2) \( g_{12} \geq g_{13} \) and \( g_{13} + g_{32} \leq g_{12} \)

The analysis for this case is almost the same as before. Different regions where \( I_1, I_2 \) and \( I_3 \) are minimum are same as before. The only difference from the solution region of the previous case is that the point \( H \) now lies to the left of origin. For this case, in region \( EOA \), where term \( I_1 \) is the least, operating at point \( A \) maximizes \( I_1 \). Therefore, \( I_1(\alpha_1, \alpha_2) \) is as given in (49).

In region \( OABD \), if \( g_{13} \geq g_{23} \), the optimal operating point is at \( D \), giving (51); if \( g_{13} \leq g_{23} \), the optimal operating point is \( B \) giving (52). Note that since \( g_{12} \geq g_{32} \), the condition \( g_{32}g_{23} \leq g_{12}g_{23} + g_{32}g_{13} \) always hold, so we never operate at point \( A \). As before, in region \( EAB \), we only operate at point \( B \) giving us (55). We don’t operate at point \( A \) since the condition \( g_{32}g_{23} \leq g_{12}g_{23} + g_{32}g_{13} \) always holds.

Therefore, the following subcases of the channel gains emerge.

a) \( g_{23} \leq g_{13} \)

For this case,

\[
G(g_{12}, g_{13}, g_{23}, g_{32}) = \max \left\{ \frac{(g_{12} + g_{13})g_{23}g_{32}}{g_{23}g_{32} + g_{13}g_{23} + g_{12}g_{32}}, g_{13} \right\}
\]

From (57) and (58),

\[
G = g_{13} \quad (65)
\]

b) \( g_{13} \leq g_{23} \)

For this case,

\[
G(g_{12}, g_{13}, g_{23}, g_{32}) = \max \left\{ \frac{g_{12}g_{23}}{g_{12} + g_{23} - g_{13}}, \frac{(g_{12} + g_{13})g_{23}g_{32}}{g_{23}g_{32} + g_{13}g_{23} + g_{12}g_{32}} \right\}
\]

From (58),

\[
G = \frac{g_{12}g_{23}}{g_{12} + g_{23} - g_{13}} \quad (67)
\]

REFERENCES


