An Exact Analysis of the Optical CDMA Noncoherent Receiver

David Brady, Sergio Verdú †
Department of Electrical Engineering
Princeton University,
Princeton, NJ 08544

Abstract

This work studies optical Code Division Multiple Access (CDMA) systems, and presents the exact error expression for the noncoherent, single-user matched-filter receiver based on the electron count in a symbol period. This analysis is valid for arbitrary photomultipliers, adheres fully to the semi-classical model of light, and does not depend on approximations for large user groups or strong received optical fields.

The general error rate expression is specialized to the case of unity gain photodetectors and prime sequences, and the exact minimum probability of error and optimal threshold are compared to those obtained with simplifying assumptions on user transmission coordination or multiple-access-interference (MAI) distribution. We find that the approximation of chip synchronism yields a weak upper bound on the true error rate, and we demonstrate that the approximations of perfect optical-to-electrical conversion and Gaussian MAI yield an optimal hypothesis test whose error rate overestimates the true minimum error rate and underestimates the optimal threshold for moderate and large received optical energies.

Optical CDMA Model

The digital modulation format studied in this paper is optical Direct Sequence Spread Spectrum, i.e., during each symbol interval of duration T, the j^{th} transmitting laser is amplitude- modulated by the product of the data, which takes on values in $\{0,1\}$, and an assigned, signature sequence of relatively short rectangular pulses. This scheme divides the symbol interval into N equal length subintervals, called chips, on which the signature sequence is constant and takes on values in $\{0,1\}$. Further, we define $P_j = P$ as the number of non-zero chips in each signature sequence, $b_{j,n}$ as the transmitted symbol of the j^{th} user in the interval $[nT,(n+1)T]$, and $c_j(t)$ as a periodic replication of the signature sequence of the j^{th} user such that $\{c_j(t),\ t \in [nT,(n+1)T)\}$ is the j^{th} signature sequence for any fixed integer n. Then the transmitted complex scalar field from the j^{th} laser may be expressed as

$$r_j(t) = \sqrt{\frac{s\,N}{T}}c_j(t)b_{jn}e^{j(\nu t + a_j W_j(t) + \theta_j)}, \qquad (1)$$
$$nT \leq \ t - \tau_j \ < (n+1)T$$

where s is proportional to the optical energy per bit of the transmitting laser, ν denotes the optical carrier frequency (assumed to be identical for all users), and θ_j is the phase offset of the j^{th} laser from the first laser. In this expression $W_j(t)$ is a standard Brownian motion, and a_j is related to the j^{th} transmitting laser linewidth, B_j, by $a_j = \sqrt{2\pi B_j}$. The relative delays $\{\tau_j\}$ are defined on $[0,T)$ with reference to the receiver of the first user. With dispersion-free transmission (1) also represents the complex scalar field at the first receiver due to user j.

We shall assume that the symbol rate of each user is the same, the optical fields of the K users add in a noncoherent fashion, and that each single-user receiver acquires the timing of its transmitter's symbol epochs. As there is no cooperation between the users, it is appropriate to model the remaining relative delays, $\{\tau_j\}_{j=2}^{K}$, as independent, identically distributed, random variables that are uniformly distributed on the interval $[0,T]$. It follows that the intensity of the optical field at the receiver of the first user is

$$|r(t)|^2 = \frac{sN}{T}\sum_{j=1}^{K} b_{j,-1}c_j(t-\tau_j)p_t(0,\tau_j) + b_{j,0}c_j(t-\tau_j)p_t(\tau_j,T)$$

Where $p_t(a,b)$ a rectangular pulse of unit height with support $[a,b)$. Due to the modulation shown in (1), the resulting photon point process depends on the data $b_{1,0}$ only on the set $\{t|c_1(t) = 1,\ 0 \leq t < T\}$. A commonly used receiver for this channel is the noncoherent matched-filter, which sums the photon counts in each of the nonzero chip subintervals of the user of interest. Given that the function $c_1(t)$ takes values on $\{0,1\}$, the correlation operation would be easily achieved at extremely low chip rates by an electro-optic modulator, which would allow received light to pass only when $c_1(t) = 1$. A more effective device to achieve the matched-filtering operation at higher chip rates is the fiber optic tap delay line, which uses the finite

† This work was partially supported by the U.S. Army Research Office under Contract DAAL03-87-K-0062 and by the U.S. Office of Naval Research under Contract N0014-87-K-0054

propagation velocity of light to achieve the proper relative delay of two optical signals by passing them through fibers of different lengths. The matched-filter direct-detection receiver has been studied in several experiments [1,2] and will be the CDMA receiver analyzed in this work.

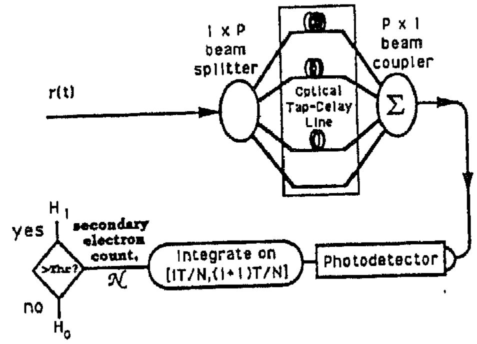

Figure 1. Optical Noncoherent Matched-Filter CDMA Receiver

As shown in Figure 1, the total received optical signal r(t) is coupled to a $1\times P$ beam splitter. Each of the outputs of the splitter are identical copies of the input signal, only attenuated in intensity by P. These signals are input to the tap delay line. The function of the i^{th} tap is to delay the received field so that the optical signal in the i^{th} non-zero chip of the first signature sequence overlaps in time with the last (P^{th}) non-zero chip of the same bit interval in the undelayed signal. Thus, the first tap requires more fiber cable than the second. The tapped signals are noncoherently recombined, and the output optical signal is incident on the photodetector. To decide on the value of $b_{1,0}$, we use the secondary electron count during the last non-zero chip interval of the first signature sequence. For the remainder of this work we denote this secondary electron count by $\mathcal{N}$. We shall employ a common photomultiplier model, in which the intensity of primary electrons is given by $\alpha|r(t)|^2 + \beta$, where α is proportional to the quantum efficiency of the photodetector, and β denotes the rate of primary electrons due to an independent dark current. The n^{th} primary electron yields a random number of secondary (output) electrons g_n, and the collection $\{g_n\}$ is assumed to be mutually independent, identically distributed, and independent of the photon or primary electron point process [3]. The common probability generating function of $\{g_n\}$ is denoted as $G(z) = \sum_{k=0}^{\infty} p_k z^k$. In this case, $\mathcal{N}$ is conditionally compound Poisson given the integrated intensity, which we define as Λ, and the distribution of $\mathcal{N}$ depends only on $G(z)$ and the integrated intensity Λ, given by

$$\Lambda \triangleq \alpha s\, b_{1,0} + d + \frac{\alpha s}{P} \sum_{j=2}^{K} b_{j,-1} R_{j,1}(\tau_j) + b_{j,0} \hat{R}_{j,1}(\tau_j) \quad (2)$$

where $R_{j,1}(\tau)$ and $\hat{R}_{j,1}(\tau)$ are the normalized (partial) cross-correlations

$$R_{j,1}(\tau) \triangleq \frac{N}{T} \int_0^{\tau} c_j(t-\tau) c_1(t)\, dt$$

$$\hat{R}_{j,1}(\tau) \triangleq \frac{N}{T} \int_{\tau}^{T} c_j(t-\tau) c_1(t)\, dt$$

that represent the contributions to the conditional mean Λ due to the j^{th} signature sequence for the duration of $b_{j,-1}$ and $b_{j,0}$, respectively. Also, d represents the portion of the primary electron count mean due to thermoelectrons. In the remainder of this work we set the quantum efficiency of the photodetector to unity, as this effects the distribution of $\mathcal{N}$ only through an attenuation of intensity. Further, we set $x \triangleq \alpha s b_{1,0} = 0$ under hypothesis $\mathcal{H}_0$ and $x = s$ under hypothesis $\mathcal{H}_1$.

Derivation of $\mathcal{P}\Big[\mathcal{N} = n \,\Big|\, x\Big]$

In this section we obtain the general expression for the PMF of the secondary electron count $\mathcal{N}$, at the integrator output for an arbitrary photomultiplier and for synchronous or asynchronous transmission. We will use this result in a later section to compare the error rates under various simplifying approximations to the exact error rate. Also, the form of the general expression will be used in the next section to develop arbitrarily tight, computationally efficient bounds on the cumulative distribution function of $\mathcal{N}$.

In the following, we define M as the upper bound on the set of total cross-correlations $R_{j,k} + \hat{R}_{j,k}$, and as the signature sequences are from $\{0,1\}^N$, these bounds hold for the partial cross-correlations as well. Since the relative delays are uniformly distributed and the chip waveform is rectangular, it is straightforward to show that each cross-correlation is a mixed random variable whose measures have point masses on the integers $\{0, 1, \ldots, M\}$ and continuous portions that are constant between these integers. We shall employ the following notation

$$\mathcal{P}\Big[R_{j,1} = i\Big] = d_j(i) \qquad i \in \{0, 1, \ldots, M\}$$

$$\mathcal{P}\Big[R_{j,1} \in [v, v+dv)\Big] = c_j(i)\ dv \qquad [v, v+dv) \in (i, i+1)$$

and we denote the distribution of $R_{j,1}$ as $\{d_j(0), d_j(1), \ldots, d_j(M), c_j(0), \ldots, c_j(M-1)\}$. Thus the marginal distribution of each cross-correlation is completely specified by 2M parameters. Further, the superscript-T notation will be used to distinguish the distribution of the total cross-correlation $R_{j,1} + \hat{R}_{j,1}$ from that of $R_{j,1}$, and the hat notation will be used for the distribution of $\hat{R}_{j,1}$.

Our approach to finding the PMF of $\mathcal{N}$ is the following: we will derive the z-transform of $\mathcal{N}$ from its conditional compound Poisson nature, and then show that this z-transform has a particularly straightforward and explicit Maclaurin series expansion. The PMF is the collection of coefficients of this series, and may be explicitly represented.

By conditioning on $(x, \{R_{j,1}, \hat{R}_{j,1}\},\ j = 2, \ldots K)$, the count $\mathcal{N}$ has a compound Poisson distribution, whose z-transform is given by

$$\mathrm{E}\left[z^{\mathcal{N}} \,\Big|\, x, \{R_{j,1}, \hat{R}_{j,1}\},\ j = 2, \ldots K\right] = \tag{3}$$
$$e^{(x+d)(G(z)-1)} \times \prod_{j=2}^{K} e^{\frac{s}{P}\{b_{j,-1}R_{j,1}+b_{j,0}\hat{R}_{j,1}\}(G(z)-1)}$$

Due to the mutual independence of the pairs $\{R_{j,1}, \hat{R}_{j,1}\}$ we need to determine only the expectation of each factor in (3), as the j^{th} factor depends only on the random mixture $b_{j,-1}R_{j,1} + b_{j,0}\hat{R}_{j,1}$. It is clear that the random mixture has the same kind of distribution as $R_{j,1}$, and we denote this mixed distribution by $(D_j(0), D_j(1), \ldots, D_j(M), C_j(0), \ldots, C_j(M-1))$. With this notation, the closed form expression of the power series of interest is

$$\mathrm{E}\left[z^{\mathcal{N}} \,\Big|\, x\right] = \mathrm{e}^{(\mathrm{G(z)}-1)(\mathrm{x+d})} \times \tag{4}$$
$$\prod_{j=2}^{K} \left\{ \begin{array}{l} \displaystyle\sum_{q=0}^{M} D_j(q) \exp(q\frac{s}{P}(G(z)-1)) - \\ \displaystyle\frac{P}{s} \frac{\mathrm{e}^{(\mathrm{G(z)}-1)\frac{s}{P}} - 1}{1-\mathrm{G(z)}} \sum_{r=0}^{M-1} C_j(r) \exp(r(G(z)-1)\frac{s}{P}) \end{array} \right\}$$

We are interested in finding $\mathcal{P}\left[\mathcal{N} = n \,\Big|\, x\right]$, which is the coefficient of z^n in the power series of (4) about the origin. This power series is straightforward but unnecessarily general for most signature sequence sets of interest. For example, the number of parameters in the power series is reduced by a factor of $K-1$ by assuming that the marginals of $R_{j,1}, \hat{R}_{j,1}$ and $R_{j,1}+\hat{R}_{j,1}$ are independent of j, i.e., the contribution of user j to the MAI is statistically indistinguishable from the other interferers. We have verified that this is an excellent approximation when the signature sequences come from the prime codes, and will drop the subscript from the distribution of the random mixtures in the sequel. Also, the power series of this expression is concisely written if we define $C(-1) = C(M) = 0$. With these simplifications, (4) becomes

$$\mathrm{E}\left[z^{\mathcal{N}} \,\Big|\, x\right] = \mathrm{e}^{(\mathrm{G(z)}-1)(\mathrm{x+d})} \times \tag{5}$$
$$\left\{ \begin{array}{l} \displaystyle\sum_{q=0}^{M} D(q) \mathrm{e}^{q\frac{s}{P}(\mathrm{G(z)}-1)} \; - \\ \displaystyle\frac{P}{s} \frac{1}{1-G(z)} \sum_{q=0}^{M} [C(q-1) - C(q)] e^{q(G(z)-1)\frac{s}{P}} \end{array} \right\}^{K-1}$$

There are 2M+2 terms inside of the braces. Letting n_q index the number of occurrences of $D(q)$, and m_q the number of occurences of $[C(q-1) - C(q)]$ in a multinomial expansion, we rewrite (5) as

$$\mathrm{E}\left[z^{\mathcal{N}} \,\Big|\, x\right] = \sum \frac{(K-1)!}{\prod_{q=0}^{M} n_q! m_q!} \prod_{q=0}^{M} D(q)^{n_q} \left[\frac{P}{s}\{C(q-1) - C(q)\}\right]^{m_q}$$
$$\times \frac{\exp[(G(z)-1)\{x + d + \frac{s}{P}\sum_{q=0}^{M} q[n_q + m_q]\}]}{(1-G(z))^{\sum_{q=0}^{M} m_q}} \tag{6}$$

where the outer summation is over all the indices such that $\sum_{q=0}^{M} m_q + n_q = K - 1$. We find the PMF of $\mathcal{N}$ in the following way. Suppose that we knew explicitly the coefficients of the following power series

$$\sum_{n=0}^{\infty} \mathcal{R}es(n, \alpha, \beta) z^n \triangleq \frac{e^{\alpha(G(z)-1)}}{(1-G(z))^{\beta}}, \quad \alpha \in \mathbb{R}+ \tag{7}$$

Using (7) in (6) we express the PMF for $\mathcal{N}$ as

$$\mathcal{P}\left[\mathcal{N} = n \,\Big|\, x\right] = \sum \frac{(K-1)!}{\prod_{q=0}^{M} n_q! m_q!} \prod_{q=0}^{M} D_q^{n_q} \left[\frac{P}{s}\{C(q-1) - C(q)\}\right]^{m_q}$$
$$\mathcal{R}es\left(n, \{x + d + \frac{s}{P}\sum_{q=0}^{M} q[n_q + m_q]\}, \sum_{q=0}^{M} m_q\right) \tag{8}$$

All that remains to be determined is an explicit expression for the coefficients $\mathcal{R}es/$ of the power series in (7). In the following we show that $\mathcal{R}es$ may be calculated by a linear recursion on the integers n and β.

The recursion for $\mathcal{R}es$ is most easily seen by substituting the identity

$$\frac{e^{\alpha(G(z)-1)}}{(1-G(z))^{\beta+1}} = G(z)\ \frac{e^{\alpha(G(z)-1)}}{(1-G(z))^{\beta+1}} + \frac{e^{\alpha(G(z)-1)}}{(1-G(z))^{\beta}}$$

where $\beta \in \{0, 1, 2, \ldots\}$, into the definition for $\mathcal{R}es$ (7). This yields

$$(1-p_0)\mathcal{R}es(n+1, \alpha, \beta+1) = \sum_{l=1}^{n+1} p_l \mathcal{R}es(n+1-l, \alpha, \beta+1)$$
$$+ \mathcal{R}es(n+1, \alpha, \beta)\ , \quad n, \beta \in \{0, 1, 2, \ldots\} \tag{9}$$

where $G(z) = \sum_{l=0}^{\infty} p_l z^l$. For most photomultiplier models $p_0 = 0$, which we will assume in the sequel. The initial conditions of this recursion are also easily extracted from the definition of $\mathcal{R}es$,

$$\mathcal{R}es(0, \alpha, \beta) = e^{-\alpha}\ , \quad \beta \in \{0, 1, 2, \ldots\} \tag{10}$$

$$\mathcal{R}es(n, \alpha, 0) = \sum_{k=0}^{n} \frac{\alpha^k}{k!} e^{-\alpha}\ \mathcal{P}\left[\sum_{l=1}^{k} g_l = n\right]\ , \quad n \in \{0, 1, \ldots\}.$$

The linear recursion for $\mathcal{R}es$ on n and β permits fast, efficient computation for any arguments $n, \beta \geq 1$. Note that the second initial condition for this recursion depends on $\mathcal{P}\left[\sum_{l=1}^{k} g_l = n\right]$, which must be known for $n, k \in \{0, 1, 2, \ldots\}$. These probabilities require iterated convolutions of the PMF of the random gain g_l, may be precomputed and stored for small n and k, and may be accurately approximated online for large n, k. We are naturally interested in special cases where $\mathcal{P}\left[\sum_{l=1}^{k} g_l = n\right]$ has an explicit form - it is easy to show that this is the case for random gains that are shifted Poisson-distributed, as well as for the unity gain case.

Arbitrarily Tight Bounds on $\mathcal{P}\left[\mathcal{N} \leq n \,\middle|\, x\right]$

Computationally efficient bounds must reduce the complexity of (8) in both the multinomial summation and the computation of $\mathcal{R}es$, while controlling the loss of accuracy by a parameter of our selection. In this section we show that by quantizing the random mixtures, we achieve all three objectives.

The complexity of the PMF is due to the smoothing over the joint distribution of the random mixtures - we originally conditioned on these random variables to take advantage of the conditional compound Poisson nature of $\mathcal{N}$. We could have also conditioned according to the conditional mean, Λ, for which $\mathcal{N}$ is also compound Poisson. However, the exact disturbution of the conditional mean Λ is not easily obtained, as it is formed by the convolution of $K-1$ mixed distributions. It is obvious that if the convolved distributions were discrete, say, with $QM+1$ points, then the exact distribution of $\mathcal{N}$ would be straightforward to compute. More importantly, the distribution of Λ would take on $(K-1)QM+1$ points, rather than a number that is exponential in the number of interferers.

But how do we obtain bounds on $\mathcal{P}\left[\mathcal{N} \leq n \,\middle|\, x\right]$ that use a discrete distribution on Λ, and are arbitrarily tight? Suppose we quantize the random mixtures $\{b_{j,-1}R_{j,1} + b_{j,0}\hat{R}_{j,1}\}$ with a $\frac{1}{Q}$ quantization step size, $Q \in \{1, 2, \ldots\}$, and round-up or round-down to form bounds on the random mixtures. That is, we form Λ_l, Λ_u given by

$$\Lambda_l = x + d + \frac{s}{P}\sum_{j=2}^{K} b_{j,-1}\frac{1}{Q}\lfloor QR_{j,1}\rfloor + b_{j,0}\frac{1}{Q}\lfloor Q\hat{R}_{j,1}\rfloor$$

$$\Lambda_u = x + d + \frac{s}{P}\sum_{j=2}^{K} b_{j,-1}\frac{1}{Q}\lceil QR_{j,1}\rceil + b_{j,0}\frac{1}{Q}\lceil Q\hat{R}_{j,1}\rceil$$

where $\lfloor R\rfloor$ ($\lceil R\rceil$) is the greatest (least) integer function of R. Then it is obvious that $\Lambda_l \leq \Lambda \leq \Lambda_u$, but can we use Λ_u, Λ_l to form bounds on the secondary electron count CDF?

A subtle point is raised by considering the form of $\mathcal{N}$

$$\mathcal{N}(\Lambda) = \sum_{p=1}^{\Pi(\Lambda)} g_p$$

where $\Pi(\Lambda)$ is the conditionally Poisson number of primary electrons with conditional mean Λ. Since g_p are non-negative, we have that $\mathcal{N}$ is an increasing function of the primary electron count, Π. It is not clear that (a.s.) bounds on Λ produce similar bounds on $\Pi(\Lambda)$, as $\mathcal{P}\left[\Pi(\Lambda_l) > \Pi(\Lambda) \,\middle|\, x\right] > 0$, and this representation of $\mathcal{N}$ does not guarantee bounds on $\mathcal{P}\left[\mathcal{N} \leq n \,\middle|\, x\right]$. In the lemma below we use a statistically equivalent representation of $\mathcal{N}$ to show that we may achieve bounds on $\mathcal{P}\left[\mathcal{N} \leq n \,\middle|\, x\right]$ by using the distributions of Λ_l, Λ_u.

Lemma. Let $\Pi(\Lambda)$ be a conditional Poisson random variable with mean Λ given Λ, and let $\mathcal{N}(\Lambda) = \sum_{k=1}^{\Pi(\Lambda)} g_k$, where $\{g_k\}$ are independent, identically distributed, non-negative integer-valued random variables. Let $\Lambda' < \Lambda$, a.s. Then

$$\mathcal{P}\Big[\mathcal{N}(\Lambda) \leq n\Big] \leq \mathcal{P}\Big[\mathcal{N}(\Lambda') \leq n\Big] \;,\; n \geq 0.$$

Proof. We recall that $p_k = \mathcal{P}\Big[g_j = k\Big]$, and define $\{\mathcal{M}_k(\Lambda' p_k), \Pi_k(\Lambda p_k)\}$ to be a set of conditionally mutually-independent, Poisson random variables with the indicated means given (Λ, Λ') so that $\Pi(\Lambda) = \sum_{k=1}^{\infty}\Pi_k(\Lambda p_k)$. Under this conditioning, $\mathcal{N}(\Lambda)$ has the same distribution as [4]

$$\mathcal{N}(\Lambda) = \sum_{k=1}^{\infty} k\Pi_k(\Lambda p_k)$$

It is straightforward to show that if $\{X_1, X_2, Y_1, Y_2\}$ are conditionally mutually-independent random variables given Λ, Λ' and

$$\mathcal{P}\left[X_i \leq n \,\middle|\, \Lambda, \Lambda'\right] \leq \mathcal{P}\left[Y_i \leq n \,\middle|\, \Lambda, \Lambda'\right] \;,\; i = 1, 2$$

then the same is true for the sum

$$\mathcal{P}\left[X_1 + X_2 \leq n \,\middle|\, \Lambda, \Lambda'\right] \leq \mathcal{P}\left[Y_1 + Y_2 \leq n \,\middle|\, \Lambda, \Lambda'\right].$$

Since the Poisson CDF is a decreasing function of the mean, we have for $I = 1$

$$\mathcal{P}\left[\sum_{k=1}^{I} k\Pi_k(\Lambda p_k) \leq n \,\middle|\, \Lambda, \Lambda'\right] \leq \mathcal{P}\left[\sum_{k=1}^{I} k\mathcal{M}_k(\Lambda' p_k) \leq n \,\middle|\, \Lambda, \Lambda'\right].$$

The same is true for the unconditioned CDFs by smoothing. The same holds for finite I by induction on the above fact, and for $I \to \infty$ by monotone sequential continuity of the probability measure.$\blacksquare$

Example: Prime Sequences and PIN Photodiodes

A necessary prerequisite to the comparison between error rates of the CDMA matched-filter receiver is the computation of the random mixture distribution $(D(0), \ldots, D(M), C(0), \ldots, C(M-1))$, as seen in (8). These are computed by a knowledge of the signature sequences, as well as the distribution of the relative delay. Since the cross-correlations of prime sequences are bounded above by [5] $M = 2$, we must compute $(D(0), D(1), D(2), C(0), C(1))$ for the chip-synchronous, and asynchronous cases. For the prime sequences from GF(31), we have found that the average distributions for the random mixtures are

$$
\begin{aligned}
& (D(0), D(1), D(2), C(0), C(1)) \\
\textit{chip synchronous} &\Rightarrow (.57, .36, .07, .00, .00) \\
\textit{asynchronous} &\Rightarrow (.44, .22, .01, .24, .09)
\end{aligned}
$$

As noted earlier, we have verified that the MAI for prime sequences is well-modeled by a sum of independent, identically-distributed (IID) random variables in the sense that the mean, variance, and third central moment of the MAI using the IID assumption and the average distribution were identical to the exact MAI moments, while the fourth central moments differed by less than .004% for 29 interferers. Further, these distributions did not differ significantly for the prime sequences from GF(11) and GF(17), and we use these distributions for all calculations.

In Figure 2 we have plotted the minimum error probability of the matched-filter CDMA receiver for the chip-synchronous assumption and for completely asynchronous transmission. We have used the weight 17 and length 289 prime sequences from GF(17), a received optical energy of s=1000 photons per bit, and a dark current contribution of d=50 thermoelectrons per bit. For a bit rate of R bits per second, these numbers correspond to a peak received power of $R \cdot 10^{-7} mW$ and a photodetector dark current of approximately $R \cdot 10^{-8} nA$. From Figure 2 we see that in this particular case the chip synchronous approximation upper-bounds the error rate in the asynchronous case by at least one order of magnitude.

The error rates are ordered in this way due exclusively to the differences of the distributions of the random mixtures shown above. Note that the means of the random mixtures are identical in both cases, while the ordering of the variances coincides with that of the error rates. Thus the MAI has identical means under these distributions, and second moments whose ordering coincides with that of the error rates. It is easy to show that $E[\mathcal{N} \mid x] = \bar{\Lambda}$, and $Var(\mathcal{N}|x) = \bar{\Lambda} - (\bar{\Lambda})^2 + \overline{\Lambda^2}$, which implies that under each hypothesis on x the mean of $\mathcal{N}$ is unchanged by the approximation of chip synchronism, yet the variance of $\mathcal{N}$ given x increases as we proceed from complete asynchronism to chip-synchronism. From the ordering of the minimum error rate curves in Figure 2, we see that an increase in the variance of $\mathcal{N}$ under each hypothesis results in an increased error rate as the conditional means of $\mathcal{N}$ are fixed.

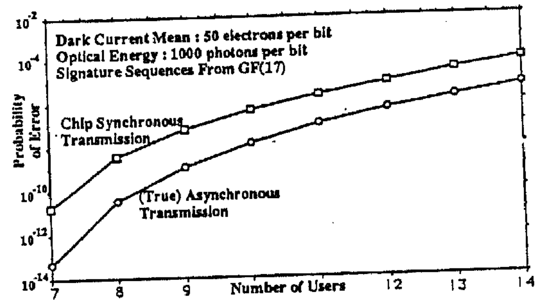

Figure 2. Comparison of the Minimum Error Rates For Complete Asynchronism and Chip Synchronous Approximation

Direct detection systems often require large received optical energies to achieve an acceptable error rate when a PIN photodiode is used, so we are interested in the asymptotic distribution of (a scaled version of) $\mathcal{N}$. The question is more formally worded as: if $\mathcal{N}$ is a conditionally-Poisson random variable with mean Λ given Λ, and $\frac{\Lambda - \bar{\Lambda}}{\sigma_\Lambda}$ tends in distribution to a random variable ϕ as some parameter grows without bound, what does the distribution of $\frac{\mathcal{N} - \bar{\mathcal{N}}}{\sigma_\mathcal{N}}$ tend to? In the simple case when Λ is deterministic, it is well known that the normalized count converges in distribution to a standard Gaussian random variable. Is this the case in general?

The answer was solved independently by Serfozo [6] and Grandell [7] for the special case when $\bar{\Lambda} \to \infty$, and depends on the limit ρ defined as $\lim \sigma_\Lambda^2 / \bar{\Lambda}$. If $\rho = 0$, then the normalized count converges in distribution to a standard Gaussian. If $\rho = \infty$, then the normalized count converges in distribution to ϕ. Finally, if $0 < \rho < \infty$, then the normalized count converges in distribution to an independent mixture of a standard Gaussian and ϕ.

In our case, the parameter is the received signal energy per bit, s, and the condition $\bar{\Lambda} \to \infty$ is satisfied as Λ is proportional to s. It is this fact that also sets ρ to ∞, and we have from the result above that for large signal energies the normalized count converges in distribution to the scaled conditional mean ϕ. This asymptotic result is a weaker form of what is more commonly known as "perfect optical-to-electrical conversion", in which the integrated photocurrent is equal (a.s.) to the integrated optical intensity. It will be seen in the numerical results presented next that the asymptotic statistic is far from being a deterministic signal in Gaussian noise, as the MAI is far from Gaussian even for a moderate number of users.

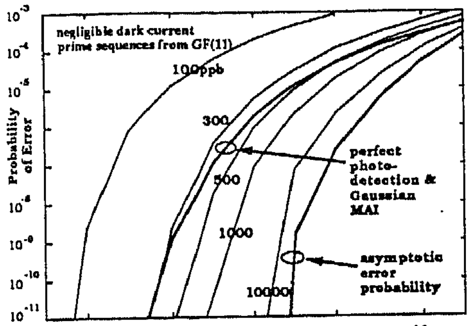

Figure 3. Exact Error Rates vs. Error Rates for Various Approximations

In Figure 3 we have compared the minimum error rates of the CDMA matched-filter receiver based on perfect optical-to-electrical conversion (the high energy limit) to those for the true distribution of $\mathcal{N}$ at various finite optical energies. In this example we have used the prime sequence from GF(11). Also, we have plotted the minimum error rate under the additional assumption of Gaussian-distributed MAI. We note that even for modest received optical energies of 10,000 photons per bit the error rate exceeds that predicted by the asymptotic distribution by at least an order of magnitude. Figure 3 shows that the minimum error rate is a decreasing function of the received optical energy, as expected. Further, we note that a Gaussian assumption on the MAI, together with the perfect optical-to-electrical assumptions is a poor estimate of the true minimum error rate curve, except for user group sizes exceeding, say, 10 users. In particular, this assumption overestimates the error rate for moderate to large incident optical energies.

As a result of the perfect optical-to-electrical-conversion approximation, the boundedness of the MAI leads to an "error-free" condition for sufficiently small numbers of interferers. This occurs since the supports of the conditional distributions of the test statistic are disjoint under these assumptions. Since prime sequences have cross-correlations that are bounded above by 2, the necessary condition for prime sequences is $K-1 < P/2$. This assumption predicts zero error rate for $K \leq 6$ in Figure 3, which indicates that the perfect optical-to-electrical assumption accurately predicts the "error-free condition" only for incident optical energies exceeding 10,000 photons per bit - the error rate for K=6 at this energy is roughly 10^{-14}.

In Figure 4 we have plotted the optimal thresholds, normalized by the signal energy, s, for those error rate curves plotted in Figure 3. As the incident optical energy per bit increases, the normalized optimal threshold increases to unity, which is the curve corresponding to the asymptotically optimal test. Note that the Gaussian MAI, perfect optical-to-electrical approximation predicts a threshold that significantly underestimates the true optimal threshold for those incident optical energies needed to dominate the dark current.

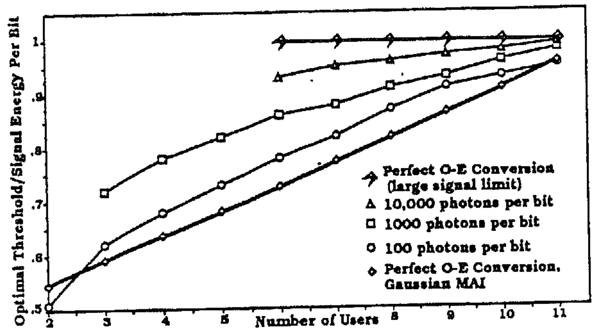

Figure 4. Optimal Thresholds For The Matched-Filter CDMA Receiver

Observe that the asymptotic test yields a more accurate estimate of the optimal threshold for moderate signal energies. Optimal thresholds for large incident optical energies are not plotted for the "error-free" region because they could not be reliably determined due to the vanishing error rate.

References

[1] P.R. Prucnal, M.A. Santoro, T.R. Fan, *"Spread Spectrum Fiber-Optic Local Area Network Using Optical Processing"*, J. Lightwave Tech., Vol. LT-4, No, 5, pp. 547-554, May,1986

[2] J.A. Salehi, C.A. Brackett, *"Fundamental Principles of Fiber Optic Code Division Multiple Access (FO-CDMA)"*, International Conference on Communications, 1987

[3] M.C. Teich, K. Matsuo, B.E.A. Saleh *"Counting Distributions and Error Probabilities for Optical Receivers Incorporating Superlattice Avalanche Photodiodes"*, Electron Devices, ED-33, pp.1475-1488, Oct, 1986

[4] D.L.Snyder, *Random Point Processes,* J. Wiley & Son,1975

[5] A.A. Shaar, P.A.Davies, *"Prime Sequences: Quasi-optimal Sequences For OR Channel Code Division Multiplexing"*, Electronic Letters Vol. 19, No.21,pp. 888-889, 13th, October, 1983

[6] R.F. Serfozo *"Conditional Poisson Processes,"* J. Appl. Prob., 9, pp.288-302 Sept, 1972

[7] J. Grandell, *"Statistical Inference for Double Stochastic Poisson Processes", in Stochastic Point Processes, Ed. P.A.W. Lewis, J. Wiley, New York,* 1972

WORST-CASE PERFORMANCE OF DIRECT-SEQUENCE MODULATED CODES ON CHANNELS WITH UNKNOWN INTERFERENCE

Murad Hizlan and *Brian Hughes*

Department of Electrical and Computer Engineering
The Johns Hopkins University
Baltimore, Maryland 21218

ABSTRACT

We consider a communication channel corrupted by thermal noise and by an unknown and arbitrary interfering signal of bounded power. On this channel, we derive a simple upper bound to the worst-case error probability incurred by a communication system comprised of a binary block code, random interleaving, direct-sequence modulation and a correlation receiver. We compare the performance of this communication system with the performance achievable with optimal random modulation and detection, and also with the performance on a channel corrupted only by additive white Gaussian noise.

1. INTRODUCTION

Direct-sequence modulation and other forms of spread-spectrum modulation have long been used as a means of achieving robust communication when the statistics of the interference are at least partially unknown. Typical examples of such situations include channels corrupted by multiple-access interference or jamming.

There has been a tendency in the communications literature to assume highly optimistic models for this unknown interference. Bounds on the performance of direct-sequence systems have been found for several simple canonical forms of interference, such as pulsed, memoryless Gaussian noise, tones, etc [e.g. 1-4]. However, in some applications (e.g. jamming), it is far from obvious that the interference can be accurately represented by one of these simple models.

In this paper, we consider a communication system comprised of a binary block code, pseudo-random interleaving, direct-sequence modulation and a correlation receiver. Our aim is to develop bounds to the error probability of this communication system on a channel corrupted by thermal noise and by an unknown interfering signal of bounded power. This model encompasses a far broader class of interfering signals than has previously been considered.

Our main results can be summarized as follows. We derive a simple upper bound to the worst-case error probability incurred by the communication system described above when used on a channel corrupted by additive white Gaussian noise (AWGN) and by an unknown arbitrary signal of bounded power. This bound is exponentially tight as the blocklength of the code becomes large. We compare the performance of this system on this channel to that on a channel corrupted only by additive white Gaussian noise with the same total interference power.

The remainder of this paper is organized as follows. In Section 2, we define the channel and system models precisely. We obtain a two-stage upper bound on the worst-case error probability of direct-sequence modulated, interleaved codes in Section 3. In Section 4, we conclude with comparisons of the system performance with the asymptotically optimal system, and with the performance on a pure AWGN channel.

2. SYSTEM MODEL AND DEFINITIONS

The channel model adopted in this paper is illustrated in Figure 1. Every T seconds, an integer message $m \in \{1, \ldots, M\}$ is encoded into a real waveform $x(t)$ and sent to the receiver. In transmission, $x(t)$ is corrupted by two independent, additive noise processes so that

$$y(t) \equiv x(t) + n(t) + s(t), \qquad 0 \le t < T, \quad (2.1)$$

is received. Here, $n(t)$ represents thermal noise and is modeled by a white Gaussian noise process with (one-sided) power spectral density N_0 Watts/Hz. The signal $s(t)$ models interference from sources with partially or completely unknown statistics such as jammers and non-cooperative multiple-access transmitters. We place no restrictions on this interfering signal except that $s(t)$ is independent of m and $n(t)$ and that its time-averaged power is bounded

$$\frac{1}{T}\int_0^T s^2(t)\,dt \le P_J \qquad \text{almost surely (a.s.)}. \quad (2.2)$$

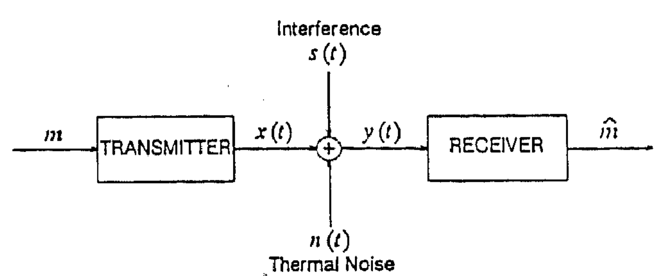

Figure 1: A channel with unknown interference.

A complete version of this paper is available form the authors. This work was supported by the National Science Foundation under grant no. NSF-NCR-8804257.

The transmitted signal $x(t)$ is formed from m by using a direct-sequence modulated, interleaved, binary block code with block length N. First, the interleaved code waveform, $x^{(m)}(t)$ is formed:

$$x^{(m)}(t) = \sum_{i=1}^{N} x_{j_i}^{(m)} Q(t-(i-1)T_c), \quad 0 \le t < T, \quad (2.3)$$

where $\mathbf{x}^{(m)} = (x_1^{(m)}, \ldots, x_N^{(m)})'$ is the binary $\{\pm\sqrt{E}\}$ codeword that represents the message m; $Q(t)=1$ in the interval $[0, T_c)$, and vanishes outside; and $T_c \equiv T/N$. The index sequence $\{j_1, j_2, \ldots, j_N\}$ in (2.3) corresponds to the interleaving of bits in the codeword, and is a pseudorandom permutation of the sequence $\{1, 2, \ldots, N\}$. We model this as a random permutation with a probability of $(N!)^{-1}$ for each distinct ordering. Note that we assume that only the symbols *within* a codeword are interleaved. The more general situation in which symbols from $L > 1$ *successive* codewords are interleaved can be treated in a similar way by taking $N' = LN$. The bounds obtained later then apply to the probability that any of L successive messages is received incorrectly.

Next, the transmitted signal $x(t)$ is obtained from $x^{(m)}(t)$ by direct-sequence modulation:

$$x(t) \equiv \sqrt{2/T_c}\cos(\omega_c t)\, c(t)\, x^{(m)}(t), \quad 0 \le t < T, \quad (2.4)$$

where $c(t)$ is the *direct-sequence carrier* and ω_c is the *carrier frequency* with $\omega_c >> T_c^{-1}$. The direct-sequence carrier is defined by

$$c(t) \equiv \sum_{i=1}^{N} \alpha_i \psi(t-(i-1)T_c), \quad 0 \le t < T. \quad (2.5)$$

Here, the *chip waveform*, $\psi(t)$, is a low-pass waveform that vanishes outside the interval $[0, T_c)$, and satisfies

$$\frac{1}{T_c}\int_0^{T_c} \psi^2(t)\, dt = 1. \quad (2.6)$$

The *signature sequence* $\{\alpha_i\}$ is a pseudorandom sequence of ± 1's, which we model as an independent, identically distributed (i.i.d.) sequence of symmetric Bernoulli random variables with $\Pr\{\alpha_i = +1\} = \Pr\{\alpha_i = -1\} = 1/2$.

The definition of direct-sequence in (2.5) seems to require that the modulator use only *one chip per coded bit*, in sharp contrast to the typical case where $L' > 1$ chips/bit is used. Here, the general case is easily modeled by conceptually forming a new concatenated code using our original code for the outer code and a repeat-L' code for the inner code. Stated another way: an (N, k, d) block code used with L' chips/bit spreading is equivalent to an $(L'N, k, L'd)$ concatenated block code with one chip/bit.

The signal detection is performed by a standard correlation receiver comprised of M coherent filters, each one matched to one of the M possible transmitted signals. The decision is the index of the largest correlation receiver output

$$\hat{m} \equiv \arg \max_{1 \le i \le M} U_i, \quad (2.7)$$

where

$$U_i \equiv (2/T)^{1/2} \int_0^T y(t) \cos(\omega_c t)\, c(t)\, x^{(i)}(t)\, dt. \quad (2.8)$$

Before proceeding further, it is convenient to recast the waveform system described above in an equivalent vector form. It is easily seen that the N signals

$$\phi_i(t) = \sqrt{2}\cos(\omega_c t)\psi(t-(i-1)T_c), \quad 1 \le i \le N, \quad (2.9)$$

are an orthonormal basis for $x^{(1)}(t), \ldots, x^{(M)}(t)$. By projecting $x(t)$ and $y(t)$ onto this basis, we can easily show that the waveform system in Figure 1 is equivalent to the vector system illustrated in Figure 2. Here, the received vector is

$$\mathbf{y} \equiv \mathbf{x} + \mathbf{s} + \mathbf{n}, \quad (2.10)$$

where $\mathbf{n} = (n_1, \ldots, n_N)'$ is an N-tuple of i.i.d. $N(0, N_0/2)$ random variables and $\mathbf{s}$ is an unknown and arbitrary interfering vector. The power constraint on $s(t)$ becomes

$$\frac{1}{N}\sum_{i=1}^{N} s_i^2 \le c \quad (2.11)$$

for the vector channel, where $c \equiv P_J T/N$.

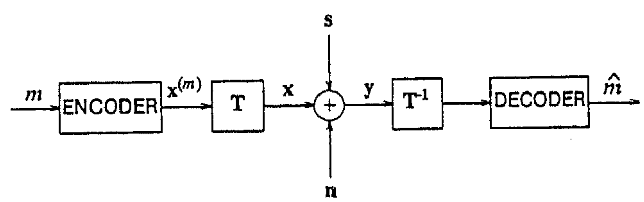

Figure 2: Equivalent Vector Channel.

The encoder in Figure 2 is an (N, M, d) binary block encoder $\mathbf{e}$, defined by

$$\mathbf{e}: \{1, \ldots, M\} \rightarrow \{+\sqrt{E}, -\sqrt{E}\}^N, \quad (2.12)$$

The *random modulator* $\mathbf{T} \equiv \mathbf{T}_1\mathbf{P}$ is composed of a random diagonal matrix $\mathbf{T}_1$, representing direct-sequence modulation, and a random permutation matrix $\mathbf{P}$, representing random interleaving. The matrix $\mathbf{T}_1$ is defined by

$$\mathbf{T}_1 = \begin{bmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_N \end{bmatrix} \quad (2.13)$$

with $\{\alpha_i\}$ as the random signature sequence defined earlier. The random permutation matrix $\mathbf{P} = \{\delta_{ij_k}\}$ is an $N \times N$ random matrix with a single "1" in each row and column, and "0"s elsewhere such that $\mathbf{P}(1, 2, \ldots, N)'$ is uniformly distributed over all permutations of $(1, 2, \ldots, N)'$. Therefore, the transmitted signal on the vector channel for a message m is given by

$$\mathbf{x} \equiv \mathbf{Te}(m) = (\alpha_1 x_{j_1}^{(m)}, \ldots, \alpha_N x_{j_N}^{(m)}), \quad (2.14)$$

where the index sequence $\{j_1, \ldots, j_N\}$ represents a random permutation of the sequence $\{1, \ldots, N\}$ as acted upon by **P**. Note that we can also represent an uncoded direct-sequence spread-spectrum system by using a repetition code for the encoder e.

Finally, the decoder in Figure 2 is a correlation detector, $d : \mathbb{R}^N \rightarrow \{1, \ldots, M\}$ defined by

$$\hat{m} \equiv \arg \max_{1 \le i \le M} (\mathrm{T}'\mathrm{y})'\mathrm{e}(i) = \arg \max_{1 \le i \le M} \mathrm{y}'\mathrm{Te}(i), \tag{2.15}$$

where ties are resolved randomly.

As stated earlier, our main objective is to investigate the worst-case performance of the system described above. Here, we consider the worst-case *average error probability*

$$\varepsilon(\mathrm{e}, \sigma_n, \sigma_s) \equiv \sup_{\mathrm{s}} \Pr\{\hat{m} \neq m\} \tag{2.16}$$

where the supremum is over all s that satisfy (2.11) and,

$$\sigma_n^2 \equiv 2E/N_0, \qquad \sigma_s^2 \equiv E/c \tag{2.17}$$

are *signal-to-thermal-noise* and *signal-to-interference* ratios, respectively. We emphasize, however, that our methods can be applied to many other error measures, e.g. maximum message error, bit-error-rate, etc. A simple upper bound to $\varepsilon(\mathrm{e}, \sigma_n, \sigma_s)$ can be found by applying the union bound [5, pp. 58] to (2.16):

$$\varepsilon(\mathrm{e}, \sigma_n, \sigma_s) \le \sup_{\mathrm{s}} \frac{1}{M} \sum_{i \neq j} \Pr\{i \rightarrow j \mid \mathrm{s}\} \le \frac{1}{M} \sum_{i \neq j} \sup_{\mathrm{s}} \Pr\{i \rightarrow j \mid \mathrm{s}\} \tag{2.18}$$

where

$$\Pr\{i \rightarrow j \mid \mathrm{s}\} \equiv \Pr\{\mathrm{y}'\mathrm{Te}(i) \le \mathrm{y}'\mathrm{Te}(j)\} \tag{2.19}$$

for $\mathrm{y} = \mathrm{Te}(i) + \mathrm{n} + \mathrm{s}$. Clearly, (2.19) depends only upon σ_n, σ_s, s, and the Hamming distance d between $\mathrm{e}(i)$ and $\mathrm{e}(j)$. Accordingly, we define

$$\varepsilon_2(p, \sigma_n, \sigma_s, \mathrm{s}) \equiv \Pr\{i \rightarrow j \mid \mathrm{s}\}, \tag{2.20}$$

$$\varepsilon_2(p, \sigma_n, \sigma_s) \equiv \sup_{\mathrm{s}} \varepsilon_2(\mathrm{e}, \sigma_n, \sigma_s, \mathrm{s}), \tag{2.21}$$

where $p \equiv d/N$ and the supremum is over all s that satisfy (2.11).

3. AN UPPER BOUND TO THE WORST-CASE ERROR PROBABILITY

Our aim in this section is to find an upper bound to $\varepsilon_2(p, \sigma_n, \sigma_s)$, defined in (2.21). We begin by expressing the pairwise error probability, $\varepsilon_2(p, \sigma_n, \sigma_s, \mathrm{s})$, in a simpler form. Let k and l be two messages such that $\mathrm{e}(k)$ and $\mathrm{e}(l)$ are Hamming distance d apart. The pairwise error probability is given by

$$\begin{aligned}
\varepsilon_2(p, \sigma_n, \sigma_s, \mathrm{s}) &\equiv \Pr\{k \rightarrow l \mid \mathrm{s}\} \\
&= \Pr\{\mathrm{y}'\mathrm{Te}(k) \le \mathrm{y}'\mathrm{Te}(l)\} \\
&= \Pr\left\{ \sum_{i=1}^{N} (\alpha_i x_{j_i}^{(k)} + s_i + n_i)\, \alpha_i\, (x_{j_i}^{(k)} - x_{j_i}^{(l)}) \le 0 \right\} \\
&= \Pr\left\{ \sum_{i=1}^{N} (s_i + n_i)\, \alpha_i (x_{j_i}^{(l)} - x_{j_i}^{(k)}) \ge \sum_{i=1}^{N} \alpha_i^2 x_{j_i}^{(k)} (x_{j_i}^{(k)} - x_{j_i}^{(l)}) \right\}.
\end{aligned} \tag{3.1}$$

Since the Hamming distance between the codewords $\mathrm{x}^{(k)}$ and $\mathrm{x}^{(l)}$ is d, the sum on the right side of the inequality in (3.1) is given by

$$\sum_{i=1}^{N} \alpha_i^2 x_{j_i}^{(k)} (x_{j_i}^{(k)} - x_{j_i}^{(l)}) = 2dE, \tag{3.2}$$

and the term $\alpha_i (x_{j_i}^{(l)} - x_{j_i}^{(k)})$ has the same distribution as $2\alpha_i J_i \sqrt{E}$ for $1 \le i \le N$, where $\mathrm{J} \equiv (J_1, \ldots, J_N)'$ is uniformly distributed over all binary N-vectors of weight d. Thus,

$$\begin{aligned}
\varepsilon_2(p, \sigma_n, \sigma_s, \mathrm{s}) &= \Pr\left\{ \sum_{i=1}^{N} \alpha_i J_i (s_i + n_i) \ge d\sqrt{E} \right\} \\
&= \Pr\left\{ \sum_{i=1}^{N} \alpha_i J_i s_i + \eta \ge d\sqrt{E} \right\},
\end{aligned} \tag{3.3}$$

where $\eta \sim N(0, dN_0/2)$.

Next we place an upper bound to the pairwise error probability by applying the Chernoff upper bound [6, pp. 97] on equation (3.3):

$$\begin{aligned}
&\varepsilon_2(p, \sigma_n, \sigma_s, \mathrm{s}) \\
&\le \mathop{\mathrm{E}}_{\alpha, \mathrm{J}}\left\{ \exp(\lambda \sum_{i=1}^{N} \alpha_i J_i s_i) \right\} \cdot \mathop{\mathrm{E}}_{\eta}\left\{ \exp(\lambda\eta) \right\} \cdot \exp(-\lambda d\sqrt{E}) \\
&= \mathop{\mathrm{E}}_{\mathrm{J}}\left\{ \mathop{\mathrm{E}}_{\alpha}\left\{ \exp(\lambda \sum_{i=1}^{N} \alpha_i J_i s_i) \mid \mathrm{J} \right\} \right\} \cdot \exp(\lambda^2 d N_o/4 - \lambda d\sqrt{E}) \\
&= \Phi(d) \cdot \exp(\lambda^2 d N_o/4 - \lambda d\sqrt{E}), \qquad \lambda \ge 0,
\end{aligned} \tag{3.4}$$

where we have defined

$$\Phi(k) \equiv \mathop{\mathrm{E}}_{\mathrm{J}(k)}\left\{ \prod_{i=1}^{N} \cosh(\lambda s_i J_i(k)) \right\}, \quad 0 \le k \le N, \tag{3.5}$$

for $\mathrm{J}(k) \equiv (J_1(k), \ldots, J_N(k))'$ uniformly distributed over all binary N-vectors of weight k. The factor $\Phi(d)$ is generally difficult to reduce further. Our aim here is to develop a simple upper bound for $\Phi(d)$ in terms of a similar expectation for Bernoulli random variables. Therefore define

$$\Psi(q) \equiv \mathop{\mathrm{E}}_{\mathrm{B}(q)}\left\{ \prod_{i=1}^{N} \cosh(\lambda s_i B_i(q)) \right\}, \quad 0 \le q \le 1, \tag{3.6}$$

where $\mathrm{B}(q) \equiv (B_1(q), \ldots, B_N(q))'$ is an N-vector of i.i.d. Bernoulli random variables with $\Pr\{B_i(q) = 1\} = q$, $\Pr\{B_i(q) = 0\} = 1-q$ for $1 \le i \le N$. Note that $\Psi(q)$ can be evaluated as

$$\Psi(q) = \prod_{i=1}^{N} \mathop{\mathrm{E}}_{B_i(q)}\left\{ \cosh(\lambda s_i B_i(q)) \right\}$$

$$= \prod_{i=1}^{N} [\, q\cosh(\lambda s_i) + 1 - q \,], \tag{3.7}$$

and can be also expressed in terms of $\Phi(k)$ by conditioning on the weight of $\mathbf{B}(q)$:

$$\Psi(q) = \underset{\Sigma}{\mathbf{E}}\left\{ \underset{\mathbf{B}(q)}{\mathbf{E}} \left\{ \prod_{i=1}^{N} \cosh(\lambda s_i B_i(q)) \;\middle|\; \Sigma = k \right\}\right\}$$

$$= \sum_{k=0}^{N} \underset{\mathbf{J}(k)}{\mathbf{E}} \left\{ \prod_{i=1}^{N} \cosh(\lambda s_i J_i(k)) \right\} \binom{N}{k} q^k (1-q)^{N-k}$$

$$= \sum_{k=0}^{N} \Phi(k) \binom{N}{k} q^k (1-q)^{N-k}, \tag{3.8}$$

where $\Sigma \equiv \sum_{i=1}^{N} B_i(q)$.

In [7], we prove that $\Phi(k)$ is increasing in k. Hence, we have the inequality

$$\sum_{k=d}^{d'} \Phi(d) \binom{N}{k} p^k (1-p)^{N-k} \le \sum_{k=d}^{d'} \Phi(k) \binom{N}{k} p^k (1-p)^{N-k},$$

and consequently,

$$\Phi(d) \le \frac{\sum_{k=0}^{N} f(k)\Phi(k) \binom{N}{k} p^k (1-p)^{N-k}}{\sum_{k=d}^{d'} \binom{N}{k} p^k (1-p)^{N-k}}, \tag{3.9}$$

for all $d \le d' \le N$, where

$$f(k) \equiv \begin{cases} 1 & d \le k \le d', \\ 0 & \text{elsewhere}. \end{cases} \tag{3.10}$$

We can further bound above the right side of (3.9) by invoking the fact that $f(k) \le \exp(-\rho(k-d'))$ for all $\rho \ge 0$, to get

$$\Phi(d) \le \frac{\sum_{k=0}^{N} \exp(-\rho(k-d'))\,\Phi(k) \binom{N}{k} p^k (1-p)^{N-k}}{\sum_{k=d}^{d'} \binom{N}{k} p^k (1-p)^{N-k}}$$

$$= \frac{\exp(\rho(d'-d) + ND(p,q))\,\Psi(q)}{\sum_{k=d}^{d'} \binom{N}{k} p^k (1-p)^{N-k}}, \tag{3.11}$$

for all $\rho \ge 0$ and $d \le d' \le N$, where

$$q \equiv \frac{p e^{-\rho}}{p e^{-\rho} + 1 - p}, \qquad 0 < q \le p, \tag{3.12}$$

and $D(p, q)$ is the *binary discrimination function* [8, pp. 110]

$$D(p, q) \equiv p \ln\left(\frac{p}{q}\right) + (1-p)\ln\left(\frac{1-p}{1-q}\right). \tag{3.13}$$

We now use the definition in (3.5), its upper bound in (3.11), and equation (3.7) to further bound the inequality in (3.4). The resulting bound on the pairwise error probability is then given by

$$\varepsilon_2(p, \sigma_n, \sigma_s, \mathbf{s}) \tag{3.14}$$
$$\le \frac{\exp\left[\rho(d'-d) + ND(p,q) + \lambda^2 d N_0/4 - \lambda d\sqrt{E}\right] \cdot \Psi(q)}{\sum_{k=d}^{d'} \binom{N}{k} p^k (1-p)^{N-k}}$$

for all $\lambda \ge 0$, $\rho \ge 0$, and $d \le d' \le N$, where q and $D(p, q)$ are defined as in (3.12) and (3.13), respectively.

We now return to the main objective of this section, which is to develop an upper bound for the worst-case error probability between pairs of codewords, $\varepsilon_2(p, \sigma_n, \sigma_s)$, defined in (2.21). By maximizing the upper bound presented in (3.14) over all $\mathbf{s}$ subject to the power constraint stated in (2.11), we can place an upper bound on $\varepsilon_2(p, \sigma_n, \sigma_s)$. The only term in (3.14) which is a function of $\mathbf{s}$ is the product term, $\Psi(q)$. Hence, the maximization problem at hand may be stated as

$$\max_{\mathbf{s}} \prod_{i=1}^{N} [\, q\cosh(\lambda s_i) + 1 - q \,] \quad \text{subject to} \quad \frac{1}{N}\sum_{i=1}^{N} s_i^2 \le c, \tag{3.15}$$

or equivalently as

$$\max_{\mathbf{s}} \frac{1}{N}\sum_{i=1}^{N} h(x_i) \quad \text{subject to} \quad \frac{1}{N}\sum_{i=1}^{N} x_i \le c\lambda^2, \tag{3.16}$$

where

$$h(x) = \ln[\, q\cosh(\sqrt{x}) + 1 - q \,], \quad x \ge 0. \tag{3.17}$$

Let $\hat{h}(x)$ be the *concave envelope* of $h(x)$:

$$\hat{h}(x) \equiv \inf\{\, g(x) \ge h(x) \mid g(x) \text{ concave} \,\}; \tag{3.18}$$

in other words, the minimum concave function which is greater than or equal to $h(x)$ for all x. In [7] we show that $h(x)$ is an increasing function which is concave for all $x > 0$ if $q \ge 1/3$, and is first convex, then concave if $q < 1/3$. For the non-trivial case of $q < 1/3$, the behavior of $h(x)$ and $\hat{h}(x)$ is depicted in Figure 3.

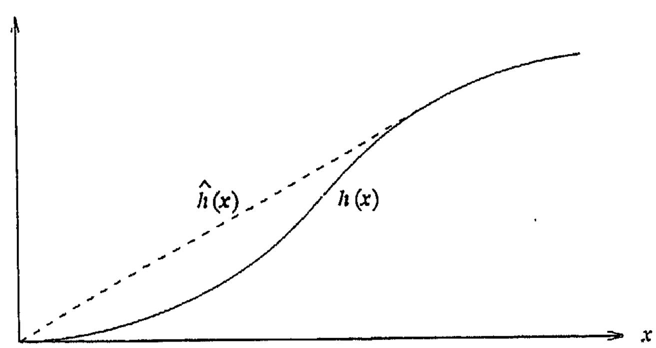

Figure 3: The behavior of $h(x)$ and $\hat{h}(x)$.

Since $h(x)$ is increasing, concave and $\hat{h}(x) \ge h(x)$, it follows that

$$\frac{1}{N}\sum_{i=1}^{N} h(x_i) \le \frac{1}{N}\sum_{i=1}^{N}\hat{h}(x_i)$$
$$\le \hat{h}\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)$$
$$\le \hat{h}(c\lambda^2). \tag{3.19}$$

Hence, we have

$$\max_{s} \prod_{i=1}^{N} [\,q\cosh(\lambda s_i) + 1 - q\,] \le \exp\left\{ N\hat{h}(c\lambda^2)\right\}$$
$$\text{for } \frac{1}{N}\sum_{i=1}^{N} s_i^2 \le c. \tag{3.20}$$

We remark here that the upper bound in (3.19) can be achieved as N becomes large.

In the trivial case of $h(x)$ concave for all $x > 0$, we have $\hat{h}(x) = h(x)$, $x \ge 0$. Otherwise, as seen in Figure 3, $\hat{h}(x)$ is either equal to $h(x)$, or may be obtained as a two-point interpolation of $h(x)$. Observe that in either case, we may express $\hat{h}(c\lambda^2)$ in terms of $h(\cdot)$ as

$$\hat{h}(c\lambda^2) = \max_{0<\alpha\le 1}\left[(1-\alpha^2)h(0) + \alpha^2 h(c\lambda^2/\alpha^2)\right]$$
$$= \max_{0<\alpha\le 1} \alpha^2 h(c\lambda^2/\alpha^2). \tag{3.21}$$

It is interesting to observe that the worst-case interference for the upper bound takes on the form of on-off pulsing, with duty factor given by the maximizing value of α^2.

We may now replace the product term in (3.14) by its upper bound from (3.20) to express the upper bound on the worst-case error probability between pairs of codewords, in terms of a maximization over a single variable, α. First, however, observe that the upper bound on $\varepsilon_2(p, \sigma_n, \sigma_s, \mathbf{s})$ holds for all $\lambda \ge 0$, $\rho \ge 0$, and $d \le d' \le N$, thus, we may tighten the bound by optimizing over these variables. Since α and λ are optimization variables, we choose to redefine $\sqrt{c}\,\lambda/\alpha$ simply as λ, and replace all the occurrences. With this definition, and the equations in (3.14), (3.20) and (3.21) we obtain the main result of this section:

$$\varepsilon_2(p, \sigma_n, \sigma_s) \le \max_{\alpha}\min_{d',\lambda,q} \frac{\exp\left(\rho(d'-d) - NE_L(p, \sigma_n, \sigma_s, \alpha, \lambda, q)\right)}{\sum_{k=d}^{d'}\binom{N}{k} p^k (1-p)^{N-k}}, \tag{3.22}$$

where $0 < \alpha \le 1$, $d \le d' \le N$, $\lambda \ge 0$, $\rho \ge 0$, and

$$E_L(p, \sigma_n, \sigma_s, \alpha, \lambda, q) \equiv \alpha\lambda p\,\sigma_s - D(p, q) - \frac{p}{2}\left(\frac{\alpha\lambda\sigma_s}{\sigma_n}\right)^2 - \alpha^2 h(\lambda^2). \tag{3.23}$$

Here, q, $D(p, q)$ and $h(x)$ are as previously defined in (3.12), (3.13), and (3.17), respectively.

We have also obtained a lower bound to the worst-case error probability between pairs of codewords, and observed that the upper and lower bounds have the same asymptotic exponent when optimized over their respective parameters. Hence, the upper bound in (3.22) is asymptotically tight, and the asymptotic error exponent is given by

$$E_{\text{ds}}(p, \sigma_n, \sigma_s) = \min_{\alpha}\max_{\lambda, q} E_L(p, \sigma_n, \sigma_s, \alpha, \lambda, q). \tag{3.24}$$

As a final remark, we have shown that the optimizing λ and q in (3.24) are uniquely defined by the solution of the first order necessary conditions.

4. EXAMPLES AND DISCUSSION

In the previous section, we have obtained a simple, asymptotically tight upper bound to the worst-case error probability between pairs of codewords. When combined with the union bound (2.18), this result provides an upper bound on the worst-case average error probability. In this section, we present some examples.

Figure 4 shows a plot of the upper bound (labeled UAI) in (2.18) against the *bit-energy-to-interference-power ratio* $E/2c\log_2 M$ for an orthogonal code with $N = M = 512$, one chip/bit direct-sequence modulation, and $N_0 = 0$ (no thermal noise). Also shown is the union-Chernoff bound for the same communication system used on a channel corrupted only by white Gaussian noise of the same power (labeled G). We observe that there is at most 1 dB difference between these curves.

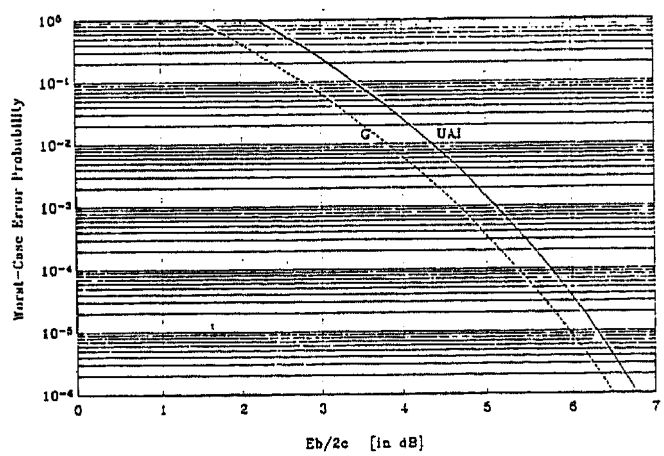

Figure 4: Orthogonal code performance.

In an earlier paper [9], we derived, for a fixed encoder, a random modem and detector that asymptotically minimize the worst-case error probability suffered on the channel (2.1) as the blocklength of the encoder becomes large. The feasible class of modems and detectors considered included the direct-sequence system described in this paper. Thus, by comparing the performance of these two systems, we can determine how the performance of the direct-sequence system described in Section 2 deviates from that of the optimally robust random modem and detector.

In [9], we showed that the asymptotic error exponent corresponding to the optimal random modem and detector on channel (2.1) reduces to

$$E_{\text{opt}}(p, \sigma_s, \infty) = -\frac{1}{2}\ln[1 - p\sigma_s^2] \qquad (4.1)$$

in the absence of thermal noise ($N_0 = 0$). In Figure 5, E_{opt} and the direct-sequence exponent (3.24) are plotted against the *signal-to-interference-power-per-dimension*, $SNR = p\sigma_s^2$, for several values of the Hamming-distance-to-blocklength ratio, $p = 0.1, 0.3, 0.5$. Also shown, labeled G, is the exponent for the pure Gaussian noise channel with the same power. We observe that the asymptotic performance of low minimum distance codes ($p = 0.1$) is worse on the channel (2.1) than on the white Gaussian noise channel with equal power. However, the opposite is true for large minimum distance codes. In particular, we see that as N is increased for the orthogonal code in Figure 4, the bound on unknown arbitrary interference UAI will eventually lie below the Gaussian curve G.

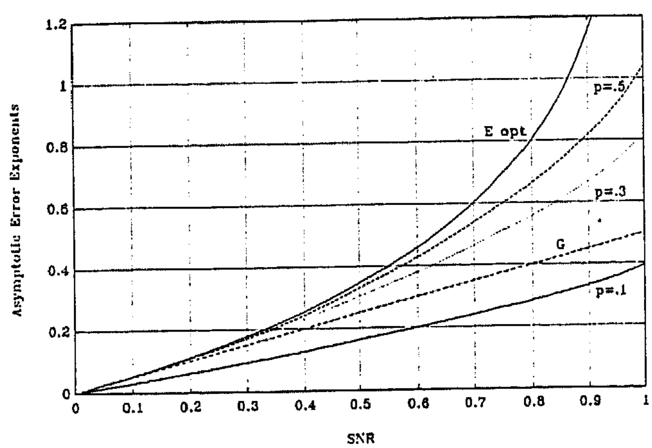

Figure 5: Comparison of error exponents.

Moreover, the exponents for orthogonal codes with direct-sequence modulation ($p = 0.5$) and the same codes with optimal random modulation differ by at most 0.7 dB and are asymptotically the same at low signal-to-noise ratios ($SNR \to 0$). Hence, the performance of direct-sequence modulation, random interleaving and a correlation receiver is close to the optimum performance achievable by any random modem and detector for orthogonal codes. For smaller minimum distance codes, however, it appears that a direct-sequence modem and a correlation receiver are markedly suboptimal.

Finally, we note that E_{opt} and the direct-sequence curves in Figure 5 become unbounded for $SNR > 1$. This corresponds to a zero-error region which is present only when there is no thermal noise. When thermal noise is introduced, all of the exponents are bounded and converge rapidly to G as the relative power of the thermal noise to the unknown interference power is increased.

REFERENCES

[1] R. Dou and L. B. Milstein, "Error probabilities and approximations for DS spread-spectrum with multi tone or multiple access interference," *IEEE Trans. Comm.*, vol. COM-32, no. 5, pp. 493-502, May 1984.

[2] R. A. Iltis and L. B. Milstein, "Performance analysis of narrow band interference rejection techniques in DS spread-spectrum systems," *IEEE Trans. Comm.*, vol. COM-32, no. 11, pp. 1169-1177, Nov. 1984.

[3] E. Masry and L. B. Milstein, "Performance of DS spread-spectrum receiver employing interference suppression filters under a worst-case jammimg condition," *IEEE Trans. Comm.*, vol. COM-34, no. 1, pp. 13-21, Jan. 1986.

[4] Y. Wang and L. B. Milstein, "Rejection of multiple narrow-band interference in both BPSK and QPSK DS spread-spectrum systems," *IEEE Trans. Comm.*, vol. COM-36, no. 2, pp. 195-204, Feb. 1988.

[5] A. J. Viterbi and J. K. Omura, *Principles of Digital Communication and Coding.* New York: McGraw-Hill, 1979.

[6] J. M. Wozencraft and I. M. Jacobs, *Principles of Communication Engineering.* New York: Wiley, 1965.

[7] M. Hizlan and B. Hughes, "Worst-case performance of direct-sequence modulated codes on channels with unknown interference," in preparation.

[8] R. E. Blahut, *Principles and Practice of Information Theory.* Massachusetts: Addison-Wesley, 1987.

[9] B. Hughes and M. Hizlan, "An asymptotically optimal random modulator and detector for robust communication," in preparation.