

# Capacity and Optimal Resource Allocation in the Degraded Gaussian Relay Channel with Multiple Relays \*

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## Abstract

We determine the capacity region of a degraded Gaussian relay channel with multiple relay stages. This is done by building an inductive argument based on the single-relay capacity theorem of Cover and El Gamal. We then give an iterative algorithm for determining the optimal power allocation between the transmitter and the relays. We show that in the case when all noise sources have equal power this results in a coding strategy that delivers SNR growth that is asymptotically linear with the number of relays.

## 1 Introduction

In their 1979 paper [1] Cover and El Gamal provided a thorough analysis of the single-relay channel, in particular determining the capacity region for the physically degraded version of the channel. Until recently, little has been done to extend these results to channels with multiple relays. However, a renewed interest in ad-hoc networks and network information theory has sparked new research into the relay channels. One set of recent results in this area is in a paper by Gupta and Kumar [3] where they demonstrate an achievable rate region result for a fairly general communication network, of which a degraded relay channel is a special case. The results in [3] are for both the discrete memoryless channel and the additive white Gaussian noise channel. A follow-up paper by Xie and Kumar [7] establishes an explicit achievable rate expression for the degraded Gaussian channel with multiple relays which, in general, exceeds the rate in [3].

In this paper, we concentrate our attention on a Gaussian physically degraded relay channel with multiple relay stages. Figure 1 depicts such a channel with  $K$  relay stages. We build upon the achievability results of [1] and [3] and use an inductive argument to determine the capacity region of such a channel. Thus, we give an alternative derivation for the achievable rate determined in [7]. With our inductive proof, we demonstrate how the coding strategy can be built recursively on the basis of the bin-coding argument utilized by [1]. We then extend our inductive argument to prove the converse to the capacity theorem as well.

Having determined the capacity region of the degraded Gaussian channel with multiple relays we turn our attention to the problem of optimal power allocation between the transmitter and the relays. We provide a general solution to this problem and demonstrate that in the special case when all the noise powers are equal, this strategy achieves

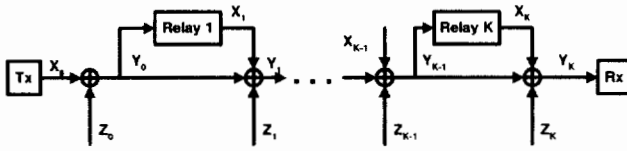


Figure 1: Physically degraded Gaussian relay channel with  $K$  relay stages.

an effective SNR that is asymptotically linear in the number of relays with slope of  $\frac{6}{\pi^2}$ .

We now define the degraded Gaussian channel with multiple relays precisely. As shown in Figure 1, the channel consists of a transmitter, whose output at transmission time  $i$  is  $x_{0,i}$  and  $K$  relays whose outputs at transmission time  $i$  are  $x_{1,i}$  through  $x_{K,i}$ . The input to relay  $k$  at transmission time  $i$  is  $y_{k-1,i}$  with  $y_{k,i}$  received by the receiver. The channel is physically degraded in the sense of [2] since

$$y_{k,i} = x_{k,i} + z_{k,i} + y_{k-1,i} \text{ for } 1 \leq k \leq K, \quad y_{0,i} = x_{0,i} + z_{0,i}$$

At each transmission stage the signal is corrupted by independently generated Gaussian random variables  $Z_0$  through  $Z_K$  with  $Z_k \sim \mathcal{N}(0, N_k)$ . Denote the transmitter power by  $P_0$  and the power of relay  $k$  by  $P_k$ . Let  $\beta_k$  denote the ratio between the relay power and the transmitter power, thus  $\beta_k \stackrel{\text{def}}{=} \frac{P_k}{P_0}$ . To simplify notation we will sometimes use  $\beta_0$  which is always equal to 1. We also define  $\nu_k \stackrel{\text{def}}{=} \frac{N_k}{\sum_{j=0}^{k-1} N_j}$ .

The goal of the paper is to determine the capacity of this channel for any given set of  $P_0, \dots, P_K$  and  $N_0, \dots, N_K$ . Additionally, we seek to determine  $\beta_1, \dots, \beta_K$  such that the capacity is maximized. We are to do this under the constraint that only a finite total amount of power  $P_T$  is available, i.e. under the constraint that  $1 + \sum_{k=1}^K \beta_k \leq \frac{P_T}{P_0}$ .

## 2 The single relay channel.

We begin by summarizing the result of [1], changing the notation of [1] to that used in this paper.

**Theorem 1** *Single-relay capacity ([1]).*

The capacity  $\mathcal{C}$  of the single-relay degraded Gaussian channel is given by

$$\mathcal{C}_1 = \max_{0 \leq \alpha \leq 1} \min \left\{ C \left( \frac{1 + \beta_1 + 2\sqrt{(1-\alpha)\beta_1} \frac{P_0}{N_0}}{1 + \nu_1} \right), C \left( \alpha \frac{P_0}{N_0} \right) \right\} \quad (1)$$

where  $C(x) \stackrel{\text{def}}{=} \log(1+x)$ .

Moreover, let  $\alpha_1^*$  denote the value of  $\alpha$  which achieves the optimum in (1). Then

$$\mathcal{C}_1 = C \left( \alpha_1^* \frac{P_0}{N_0} \right) = C \left( \alpha_1^* \frac{P_0(1 + \nu_1)}{N_0 + N_1} \right) \quad (2)$$

where  $\alpha_1^* = 1 \Leftrightarrow \beta_1 \geq \nu_1$ .

□

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The value of  $\alpha_1^*$  in equation (2) may be obtained explicitly. As stated, if  $\beta_1 \geq \nu_1$  then  $\alpha_1^* = 1$ . Otherwise,

$$\alpha_1^* = \frac{(\sqrt{1 + \nu_1 - \beta_1} + \sqrt{\nu_1 \beta_1})^2}{(1 + \nu_1)^2} \quad (3)$$

We can also solve (3) for  $\beta_1$  under the assumption that  $\alpha_1^* < 1$ . This results in

$$\beta_1 = \left( \sqrt{\alpha_1^* \nu_1} - \sqrt{1 - \alpha_1^*} \right)^2 \quad (4)$$

We next pose the optimum power allocation problem for the single-relay channel. We would like to find the value of  $\beta_1$  such that the capacity of the channel is maximized subject to  $1 + \beta_1 \leq \frac{P_T}{P_0}$  for some fixed  $P_T$ . Since the channel capacity is given by  $C\left(\alpha_1^* \frac{P_0}{N_0}\right) = C\left(\alpha_1^*(1 + \nu_1) \frac{P_0}{N_1 + N_2}\right)$  we can remove the constraint on  $\beta_1$  by maximizing the improvement in the “effective SNR” delivered by the relay channel. This “effective SNR improvement” is given by

$$J_1(\alpha, \beta) = \frac{\alpha(1 + \nu_1)}{1 + \beta} \quad (5)$$

The expression above is actually an expression in only one variable since we can either use (3) to express  $\alpha_1^*$  in terms of  $\beta_1$  or (4) to express  $\beta_1$  in terms of  $\alpha_1^*$ . Such substitution is only valid if the resulting optimum is such that  $\alpha_{1,opt} < 1$ , however, this is true for all  $\nu_1 > 0$ . In fact, maximizing  $J_1$  we obtain  $\alpha_{1,opt} = \frac{4}{4 + \nu_1}$  and  $\beta_{1,opt} = \frac{\nu_1}{4 + \nu_1}$  where we note that

$$\beta_{1,opt} = 1 - \alpha_{1,opt} \quad (6)$$

Equation (6) becomes important when we consider the general resource optimization problem for multiple relays.

### 3 The multiple relay channel

We now consider the problem of the multiple relay channel. This problem has been previously considered in [3] and [7] where achievable rates were found for a channel of which the multiple relay degraded Gaussian channel is a special case. In this section we extend that result to prove explicitly the capacity of the multiple relay degraded Gaussian channel. It turns out that the achievable rate found in [7] is the capacity of the degraded multi-relay channel.

For a specified choice of  $\alpha_{i,j}$ 's with  $0 \leq i \leq j \leq K$  satisfying

$$\sum_{j=0}^K \alpha_{0,j} = 1 \quad (7)$$

and

$$\sum_{j=i}^K \alpha_{i,j} = \beta_i \quad \forall 1 \leq i \leq K \quad (8)$$

define

$$\mathcal{R}_k(\bar{\alpha}) = C \left( \frac{P_0 \sum_{j=0}^k \left( \sum_{i=0}^j \sqrt{\alpha_{i,j}} \right)^2}{\sum_{j=0}^k N_j} \right) \quad (9)$$

and

$$\mathcal{C}_K(\bar{\alpha}) = \min_{0 \leq k \leq K} \mathcal{R}_k(\bar{\alpha}) \quad (10)$$

where we use  $\bar{\alpha}$  as a shorthand for  $\{\alpha_{i,j}\}_{0 \leq i \leq j \leq K}$ . We then have the following theorem

**Theorem 2** *Multi-relay capacity*

The capacity of the multiple relay degraded Gaussian channel with  $K$  relays is given by

$$\mathcal{C}_K = \sup_{\{\alpha_{i,j}\}} \mathcal{C}_K(\bar{\alpha}) \quad (11)$$

with  $\mathcal{C}_K$  as defined by (9) and (10).

**Proof:** We prove both the achievability and the converse parts of the theorem by induction. In both cases, the single-relay result of [1] as presented in Theorem 1 serves as the initial step in the induction. Indeed, using the notation presented above, we have for the single-relay channel:  $K = 1$ ;  $\alpha_1$  of Theorem 1 is  $\alpha_{0,0}$ ;  $1 - \alpha_1$  of Theorem 1 is  $\alpha_{0,1}$ ;  $\alpha_{1,1} = \beta_1$ ;

$$\mathcal{R}_0(\bar{\alpha}) = C\left(\alpha_1 \frac{P_0}{N_0}\right); \mathcal{R}_1(\bar{\alpha}) = C\left(\frac{1 + \beta_1 + 2\sqrt{(1 - \alpha_1)\beta_1} \frac{P_0}{N_0}}{1 + \nu_1}\right) = C\left(\frac{(\sqrt{\alpha_1})^2 + (\sqrt{1 - \alpha_1} + \sqrt{\beta_1})^2 \frac{P_0}{N_0}}{1 + \nu_1}\right).$$

To prove achievability we need to specify our coding strategy. We simply extend the method used in [1]. The resulting coding strategy is similar to the one proposed in [7], although our method for generating it is recursive and builds directly on the coding strategy used in [1]. This is unlike [7], where the coding strategy is specified directly.

*Achievability:* For the induction step of the proof of achievability, assume that the theorem holds for  $K - 1$  relays. Consider adding relay  $K$  as the last relay in the chain.

Because we use it for our inductive step, we reproduce the coding scheme used in [1] to achieve capacity. For  $0 \leq \alpha_{0,0} \leq 1$  define  $\alpha_{0,1} = 1 - \alpha_{0,0}$  and let  $X_1 \sim \mathcal{N}(0, \alpha_{1,1}P)$  and  $\hat{X}_0 \sim \mathcal{N}(0, \alpha_{0,0}P)$  with  $\hat{X}_0, X_1$  independent and let  $X_0 = \sqrt{\frac{\alpha_{0,1}}{\alpha_{1,1}}}X_1 + \hat{X}_0$ .

Let  $\mathcal{W} = \{1, 2, \dots, 2^{nR}\}$  be the set of messages to be transmitted. Let  $\mathcal{S}_1 = \{S_{1,1}, S_{1,2}, \dots\}$  be a partition of  $\mathcal{W}$  generated in a uniform and random fashion independently from everything else.

We have two random codebooks:

- $\hat{X}_0(w)$  i.i.d  $\sim \mathcal{N}_n(0, \alpha_{0,0}P)$ ,  $w \in \mathcal{W}$
- $X_1(s_1)$  i.i.d  $\sim \mathcal{N}_n(0, \alpha_{1,1}P)$ ,  $s_1 \in \mathcal{S}_1$

Finally, for transmission time  $i$ ,  $s_{1,i}$  is chosen so that  $s_{1,i} \in \mathcal{S}_{1,j}$  where  $w_{i-1} \in \mathcal{S}_{1,j}$ . It is shown in [1] that the receiver can decode the message  $w_{i-1}$  at the end of transmission interval  $i$ .

Proceeding now to the inductive step, we begin with a  $(K - 1)$ -relay channel. Fix some appropriate choice of  $\alpha_{i,j}$ 's. Let  $\mathcal{C}_{K-1}(\bar{\alpha})$  be the rate achievable in this channel and with this choice of  $\alpha_{i,j}$ 's and assume that this rate is achievable using a codebook such that the output of transmitter  $k$  is given by a random variable  $\tilde{X}_k \sim \mathcal{N}(0, \beta_k P_0)$

Now consider adding another relay at the end of the degradation stage. Because all the relays are decoding all the information, we can think of this as simply adding a new transmitter (indexed  $K$ ) and a new receiver (indexed  $K$ ). Let  $\mathcal{W} = \{1, 2, \dots, 2^{nR}\}$  be the set of messages to be transmitted. Let  $\mathcal{S}_K = \{S_{K,1}, S_{K,2}, \dots, S_{K,2^{nR,K}}\}$  be a partition of  $\mathcal{W}$  generated in a uniform and random fashion independently from everything else.

Define a random codebook  $\mathbf{X}_K(s_K)$  i.i.d.  $\sim \mathcal{N}_n(0, \alpha_{K,K} P_0)$ ,  $s_K \in \{1, \dots, 2^{nR_K}\}$ , where we note that  $\alpha_{K,K} \equiv \beta_K$ . For transmission time  $i$ ,  $s_{K,i}$  is chosen so that  $s_{K,i} = j$  where  $w_{i-K} \in S_{K,j}$ . Now for  $0 \leq k \leq K-1$  define  $X_k = \sqrt{1 - \alpha_{k,K}} \tilde{X}_k + \sqrt{\frac{\alpha_{k,K}}{\alpha_{K,K}}} X_K$ , where  $\alpha_{k,K}$  is the proportion of the power that transmitter  $k$  allocated to the newly added receiver  $K$ .

In [1] it was necessary to assume that at the start of transmission time  $i$  the relay(receiver) has successfully decoded messages  $w_1, \dots, w_{i-1}$ . Extending this assumption, we assume that at the start of transmission time  $i$ , the receiver  $k$  has successfully decoded messages  $w_1, \dots, w_{i-k}$ . In particular, at transmission time  $i$ , all receivers up to and including receiver  $K-1$  know  $w_{i-K}$ . Thus, receivers 1 through  $K-1$  can successfully remove the contribution from  $X_K$  to the received signal. Thus the rate  $\mathcal{C}_{K-1}(\bar{\alpha})$  as defined by (9) and (10) is achievable from the point of view of communicating to receivers 1 through  $K-1$ . We note that by adding non-zero  $a_{i,K}$ 's we reduced  $\mathcal{C}_{K-1}(\bar{\alpha})$  since  $\sum_{j=1}^{K-1} \alpha_{i,j}$  is reduced from  $\beta_i$  to  $\beta_i - \alpha_{i,K}$ .

Suppose now that, under the assumption that reliable communication is achieved to receivers 1 through  $K-1$ , it is possible to communicate to receiver  $K$  at a rate  $\mathcal{R}_K(\bar{\alpha})$  as defined by (9). We note that because the same information is being communicated to all the receivers, the rate  $\min(\mathcal{C}_{K-1}(\bar{\alpha}), \mathcal{R}_K(\bar{\alpha})) = \min_{1 \leq k \leq K} \mathcal{R}_k(\bar{\alpha})$  is achievable since we can communicate reliably at this rate to all the receivers. Finally, taking a supremum over the choices of  $\alpha_{i,j}$ 's we obtain the desired capacity rate.

It remains to show that, assuming that reliable communication to all other receivers is attained, it is indeed possible to communicate to receiver  $K$  at a rate  $\mathcal{R}_K(\bar{\alpha})$ . This is done by applying successive interference cancellation similarly to [3] and [7]. The detailed argument is provided in [4].

After  $B$  transmission instances, we can therefore achieve a rate of  $\mathcal{R}_K(\bar{\alpha}) \frac{B-K}{B}$  which approaches  $\mathcal{R}_K(\bar{\alpha})$  as  $B \rightarrow \infty$ . This shows that under the assumption that reliable communication to receivers 0 through  $K-1$  is achieved, we can achieve reliable communication to receiver  $K$  at a rate  $\mathcal{R}_K$ ; and this completes the proof of the achievability part of the theorem.

*Converse:* We begin by defining

$$\mathcal{I}_k = I(X_0, \dots, X_k; Y_k | X_{k+1}, \dots, X_K) \quad (12)$$

Then from the cut-set bound ([2], Ch. 14) and the degradedness of the channel it follows that

$$\mathbf{C}_K \leq \sup_{p(x_0, \dots, x_K)} \min \mathcal{I}_k \quad (13)$$

To prove the converse, we show that  $\mathcal{I}_k \leq \mathcal{R}_k$ . We have

$$\mathcal{I}_k = \frac{1}{2} C \left( \frac{\mathbb{E} \text{Var}(X_0 + \dots + X_k | X_{k+1}, \dots, X_K)}{\sum_{j=0}^k N_j} \right) \quad (14)$$

Thus we need to show that there exists some choice of real values  $\{\alpha_{i,j}\}$  defined for  $0 \leq i \leq j \leq K$  and satisfying constraints (7) and (8) such that

$$\mathbb{E} \text{Var}(X_0 + \dots + X_k | X_{k+1}, \dots, X_K) \leq P_0 \sum_{j=0}^k \left( \sum_{i=0}^j \sqrt{\alpha_{i,j}} \right)^2 \quad (15)$$

Define

$$\alpha_{i,K} \stackrel{\text{def}}{=} \frac{1}{P_0} \mathbb{E} \mathbb{E}^2(X_i | X_K) \quad (16)$$

The rest of the  $\alpha$ 's will be defined implicitly based on our induction argument; at this point we only note that from (7) and (8) these must satisfy the constraint

$$\sum_{j=i}^{K-1} \alpha_{i,j} = \frac{1}{P_0} \left( \mathbb{E} X_i^2 - \mathbb{E} \mathbb{E}^2(X_i | X_K) \right) \quad 0 \leq i \leq K-1 \quad (17)$$

We also note that in the case of a single-relay channel the definition in (16) reduces to the definition of  $(1 - \alpha)$  in the proof of the converse part of Theorem 1 in [1]; and the left-hand side of (17) reduces to  $\alpha$  in the proof of the converse part of Theorem 1 in [1]. Thus, we can use the single-relay proof of [1] as the initial step in an induction proof of the overall theorem.

For the inductive step, let us assume that the converse holds for a  $(K-1)$  relay channel. Thus, we assume that for every choice of the transmitter output distribution and noise powers there exists a choice of real values  $\{\tilde{\alpha}_{i,j}\}$  for  $0 \leq i \leq j \leq K-1$  satisfying (16) and satisfying (17) for any choice of real values  $\beta_j$ ,  $1 \leq j \leq K-1$  such that

$$\mathbb{E} \text{Var}(X_0 + \dots + X_k | X_{k+1}, \dots, X_{K-1}) \leq P_0 \sum_{j=0}^k \left( \sum_{i=0}^j \sqrt{\tilde{\alpha}_{i,j}} \right)^2 \quad (18)$$

holds for all  $0 \leq k \leq K-1$ .

Now consider a  $K$ -relay channel. Define, as we did throughout,  $\beta_i$  to be the power of relay  $i$ , normalized to the transmitter power  $P_0$ .

First we show that (15) must hold for  $0 \leq k \leq K-1$ . Our argument is constructed as follows. We fix  $\{\alpha_{i,K}\}$ 's for all  $0 \leq i \leq K$ . Conditioned on such a choice (i.e. conditioned on  $X_K$ ) we construct an equivalent  $K-1$  relay channel with the following two properties:

- The set of  $\{\tilde{\alpha}_{i,j}\}_{0 \leq i \leq j \leq K-1}$  satisfying (7) and (8) in the  $(K-1)$ -relay channel we construct is in one-to-one correspondence with the set of  $\{\tilde{\alpha}_{i,j}\}_{0 \leq i \leq j \leq K-1}$  satisfying (17) in the original channel (conditioned on  $X_K$ ).
- Equation (18) holds for  $0 \leq k \leq K-1$  for the  $(K-1)$ -relay channel we constructed if and only if equation (15) holds for  $0 \leq k \leq K-1$  for the original  $K$ -relay channel.

Having shown this, we conclude that (15) must hold for  $0 \leq k \leq K-1$ , for otherwise we violate the induction hypothesis. The details of the proof are provided in [4] □

We have now expressed the capacity of the multiple-relay degraded Gaussian channel as a max-min problem. Given a general distribution of power between the transmitter and the relays, the optimum in (11) may be very difficult to determine. A general closed-form expression would certainly become too unruly for more than a few relays. However, as we shall see in the next section, the problem of maximizing the power allocation between relays under a total power constraint lends itself to a simple iterative solution.

## 4 Optimum power allocation for multiple relays

In this section we consider the problem of optimum power allocation between relays in the multiple relay degraded channel. We start with a  $K$ -relay degraded Gaussian channel.

We wish to determine  $\beta_1, \dots, \beta_K$  subject to  $1 + \beta_1 + \dots + \beta_K \leq \frac{P_T}{P_0}$  such that the capacity of the channel as given by (11) is maximized. We begin with the following theorem.

**Theorem 3** Consider a  $K$ -relay degraded Gaussian channel with the power allocated among the relays in such a way that the capacity of the overall channel is optimized subject to  $1 + \beta_1 + \dots + \beta_K \leq \frac{P_T}{P_0}$ . Then the optimum in (11) is achieved by setting

$$\alpha_{0,j} = \alpha_{1,j} = \dots = \alpha_{j,j} \stackrel{\text{def}}{=} \alpha_j.$$

We note that if  $\alpha_{0,j} = \alpha_{1,j} = \dots = \alpha_{j,j} \stackrel{\text{def}}{=} \alpha_j$ , then  $\beta_k = \sum_{j=k}^K \alpha_j$  and, in particular,  $\sum_{j=0}^K \alpha_j = \beta_0 \equiv 1$ . Thus, if Theorem 3 holds, in order to find the optimum power allocation scheme for a  $K$ -relay channel, it suffices to determine the values of  $\alpha_0, \dots, \alpha_K$  subject to  $\sum_{j=0}^K \alpha_j = 1$ . The values of  $\beta_1, \dots, \beta_K$  are then determined directly by  $\beta_k = \sum_{j=k}^K \alpha_j$ . Additionally, as long as  $\alpha_k > 0 \forall 0 \leq k \leq K$ ,  $\beta_k > \beta_{k+1}$  - and we shall see that this holds as long as  $N_k > 0 \forall k$ .

**Proof Sketch:** The proof of Theorem 3 is an application of the following principle. Let  $\{x_m\}$  be a collection of  $M$  non-negative variables. Then the solution to the problem

$$\min \sum x_m^2 \text{ subject to } \sum x_m \geq K_1$$

or the solution to its dual

$$\sum x_m \text{ subject to } \sum x_m^2 \leq K_2$$

is to set  $x_m = x \forall m$  in such a way that the constrained is satisfied with equality.

The theorem then follows by induction with the initial step provided in Section 2 where we showed that if the power is allocated optimally between the relay and the transmitter, then we have  $\beta_1 = 1 - \alpha_1 \Rightarrow \alpha_{1,1} = \alpha_{0,1}$ . The details are provided in [4]  $\square$

To derive an explicit expression for the capacity of a  $K$ -relay channel as given by (9)-(11) we need to find a set  $\{\alpha_{i,j}\}$  such that  $\mathcal{R}_0 = \mathcal{R}_1 = \dots = \mathcal{R}_K$ . While the problem is in general very difficult, under the assumption of optimum power allocation we can use Theorem 3 to simplify it significantly. We begin by re-writing (9) under the assumption that  $\alpha_{i,j} = \alpha_j$ . This results in

$$\mathcal{R}_k = C \left( P_0 \frac{\sum_{j=0}^k (j+1)^2 \alpha_j}{\sum_{j=0}^k N_j} \right) \quad (19)$$

Setting  $\mathcal{R}_k = \mathcal{R}_{k-1}$  we get

$$\frac{\sum_{j=0}^k (j+1)^2 \alpha_j}{\sum_{j=0}^k N_j} = \frac{\sum_{j=0}^{k-1} (j+1)^2 \alpha_j}{\sum_{j=0}^{k-1} N_j} \quad (20)$$

which yields

$$\frac{(k+1)^2 \alpha_k}{\sum_{j=0}^k N_j} = \sum_{j=0}^{k-1} (j+1)^2 \alpha_j \left( \frac{1}{\sum_{j=0}^{k-1} N_j} - \frac{1}{\sum_{j=0}^k N_j} \right) \quad (21)$$

$$= \sum_{j=0}^{k-1} (j+1)^2 \alpha_j \frac{N_k}{\left( \sum_{j=0}^{k-1} N_j \right) \left( \sum_{j=0}^k N_j \right)} \quad (22)$$

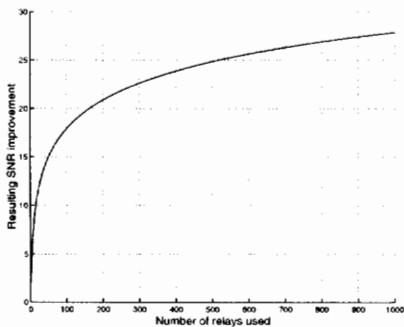


Figure 2 SNR improvement (dB)

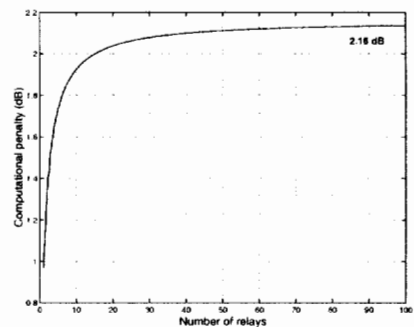


Figure 3 Distributed computation penalty (dB)

Thus,

$$\alpha_k = \frac{1}{(k+1)^2} \nu_k \left( \sum_{j=0}^{k-1} (j+1)^2 \alpha_j \right) \quad (23)$$

where

$$\nu_k \stackrel{\text{def}}{=} \frac{N_k}{\sum_{j=0}^{k-1} N_j} \quad (24)$$

and one can easily check that if we set  $K = 1$  we obtain the solution obtained in Section 2 by direct solution of the single relay optimization problem. Additionally, we note that if  $N_k > 0$ ,  $\nu_k > 0$  and  $\alpha_k > 0$ .

Equation (23) provides us with an iterative approach to generate the sequence  $\alpha_0, \alpha_1, \dots$  for any  $K$ . For each  $K$ , the starting point,  $\alpha_0$  needs to be chosen so that  $\sum_{j=0}^K \alpha_j = 1$ . However, in practice, we can always start with  $\alpha_0 = 1$ , thus producing the same (infinite) sequence for any  $K$ . Then, for each  $K$ , we can re-normalize  $\alpha_0, \dots, \alpha_K$  to satisfy the constraint.

By choosing  $\alpha_j$ 's in this way, we make sure that  $\mathcal{R}_0 = \mathcal{R}_1 = \dots = \mathcal{R}_K$ . Thus, the capacity of the  $K$ -relay channel can then be written simply as

$$C_K = \mathcal{R}_0 = C \left( \alpha_0 \frac{P_0}{N_0} \right) = C \left( \alpha_0 \prod_{k=1}^K (1 + \nu_k) \frac{P_0}{\sum_{k=0}^K N_k} \right) \quad (25)$$

where we can now define

$$J(K, N_0, \dots, N_K) \stackrel{\text{def}}{=} \alpha_0 \prod_{k=1}^K (1 + \nu_k) \quad (26)$$

to be the maximum SNR improvement provided by having some fixed amount of power  $P_0$  at the transmitter and  $K$  relays. This quantity depends on the total number of relays available and the noise power at each of the noise stages. For a given sequence  $\{N_k\}$  and a given  $K$  it can be computed iteratively using (23). Figure 2 shows the improvement achievable with  $N_k = N \forall k$ . The SNR improvement in dB is plotted in Figure 2 for up to 1000 relays.

We observe from Figure 2 that as we increase the number of relays we get less and less improvement out of the additional relays. For example, we gain approximately 18 dB from the first 100 relays and only about another 10 dB from the next 900. A natural question to ask is whether the improvement which we can attain increases without bound.

In the case of equal strength noises the answer turns out to be yes. In this case the SNR improvement offered by adding the  $K^{\text{th}}$  relay stage satisfies

$$\lim_{K \rightarrow \infty} J(K, N, \dots, N) - J(K-1, N, \dots, N) = \frac{6}{\pi^2} \approx 0.6079 \quad (27)$$

The proof of this statement is provided in [4].

Thus, as the number of relays grows, the SNR improvement, in dB, grows asymptotically as  $\log 0.6079K = \log K - \log 0.6079 = \log K - 2.16\text{dB}$ . We define this loss of slope in the SNR growth as the distributed computation penalty and this is plotted in Figure 3 for 1 through 100 relays.

## 5 Wideband performance in the Gaussian degraded relay channel

In a recent paper [5], Verdú demonstrated that second-order effects must be taken into account when wideband performance of communication systems is considered. Using these results it was shown [6] that in the wideband regime TDMA is strictly sub-optimal for both the multi-access Gaussian channel and the broadcast Gaussian channel. This is so despite that fact that it does achieve the optimal  $\frac{E_b}{N_{0\min}}$  defined as

$$\frac{E_b}{N_{0\min}} = \lim_{\text{SNR} \rightarrow 0} \frac{\text{SNR}}{\mathcal{C}(\text{SNR})} = \frac{\log_e 2}{\mathcal{C}'(0)} \quad (28)$$

where  $\mathcal{C}$  is the capacity of the channel as a function of SNR.

However, in the case of the degraded Gaussian relay channel, time division multiplexing (TDM) does not even achieve the same  $\frac{E_b}{N_{0\min}}$  as the information theoretically optimal communication scheme. We illustrate this using the single-relay channel. Define  $Q \stackrel{\text{def}}{=} \frac{P_t}{N_0}$  to be the signal-to-noise ratio between the transmitter and the first relay stage. While  $Q$  may not always be the best measure of the signal-to-noise ratio, it is sufficient for our purposes since any reasonable measure of SNR should go to 0 as  $Q$  goes to 0.

Then for the single-relay channel, the capacity as a function of  $Q$  is given by (2) as

$$\mathcal{C}_1 = \mathcal{C}(\alpha_1^* Q) \quad (29)$$

Thus,  $\lim_{Q \rightarrow 0} \mathcal{C}(Q) = \alpha^*$ , resulting in

$$\frac{E_b}{N_{0\min}} = \frac{\log_e 2}{\alpha^*} \quad (30)$$

Turning now to TDM, we note that we can alternate between two modes of communication: the transmitter can transmit directly to the receiver or the transmitter can transmit to the relay which then relays the message to the receiver. In the second instance, the transmitter and relay transmissions have to be time-multiplexed as well. Let  $\mathcal{C}_d(Q)$  be the capacity function for direct transmitter-to-receiver communication and  $\mathcal{C}_r(Q)$  be the capacity function for communication using the relay. Then the TDM capacity function is given by

$$\mathcal{C}_{TDM}(Q) = \max_{0 \leq \alpha \leq 1} [\alpha \mathcal{C}_d(Q) + (1 - \alpha) \mathcal{C}_r(Q)] = \max[\mathcal{C}_d(Q), \mathcal{C}_r(Q)] \quad (31)$$

Thus, the optimal strategy is to either always use a relay or to never use it. Which of the two options is to be chosen in a specific scenario depends on the values of  $\beta_1$  and  $\nu_1$ . In order to specify this more precisely we need to examine the functions  $\mathcal{C}_d$  and  $\mathcal{C}_r$  themselves.

$\mathcal{C}_d$  is the capacity function for point-to-point communication without a relay, thus it is given by

$$\mathcal{C}_d(Q) = C\left(Q\frac{1}{1+\nu_1}\right) \quad (32)$$

If we are using the relay to communicate, then we must time-share the channel between the transmitter and the relay. Thus, the capacity function in this case is given by

$$\mathcal{C}_r(Q) = \max_{0 \leq \alpha \leq 1} \min[\alpha C(Q), (1-\alpha)C(\beta Q)] \quad (33)$$

The minimax point of (33) is achieved by setting the two arguments of the minimum equal which results in

$$\mathcal{C}_r(Q) = \frac{C(Q)C(\beta Q)}{C(Q) + C(\beta Q)} = \frac{1}{\frac{1}{C(Q)} + \frac{1}{C(\beta Q)}} \quad (34)$$

Since both  $\mathcal{C}_d$  and  $\mathcal{C}_r$  are monotonically increasing and concave, (28) and (31) give us

$$\frac{E_b^{TDM}}{N_{0\min}} = \frac{\log_e 2}{\max[\dot{\mathcal{C}}_d(0), \dot{\mathcal{C}}_r(0)]} = \left(1 + \nu_1 \min\left(1, \frac{1}{\beta_1}\right)\right) \log_e 2 \quad (35)$$

Equation (35) demonstrates that in the wideband regime the optimal TDM strategy is to use the relay if and only if it is capable of transmitting at a higher power than the transmitter. This strategy is clearly worse than the optimal communication strategy and [4] shows that  $\alpha^* > \frac{1}{1+\nu_1 \min(1, \frac{1}{\beta_1})}$  for all  $0 < \nu_1 < \infty$ .

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