Capacity and Optimal Resource Allocation in the Degraded Gaussian Relay Channel with Multiple Relays

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Abstract

We determine the capacity region of a degraded Gaussian relay channel with multiple relay stages. This is done by building an iterative argument based on the single-relay capacity theorem of Cover and El Gamal. We then give an iterative algorithm for determining the optimal power allocation between the transmitter and the relays. We show that in the case when all noise sources have equal power the results is a coding strategy that delivers SNR growth that is asymptotically linear with the number of relays.

1 Introduction

In their 1979 paper [1] Cover and El Gamal provided a thorough analysis of the single-relay channel, in particular determining the capacity region for the physically degraded version of the channel. Until recently, little has been done to extend these results to channels with multiple relays. However, a renewed interest in ad-hoc networks and network information theory has sparked new research into the relay channel. One set of recent results in this area is in a paper by Cigada and Kumar [2] where they demonstrate an achievable rate region result for a fairly general communication network, of which a degraded relay channel is a special case. The results in [2] are for both the discrete memoryless channel and the additive white Gaussian noise channel. A follow-up paper by Xie and Kumar [7] establishes an asymptotic achievable rate expression for the degraded Gaussian channel with multiple relays which, in general, exceeds the rate in [2].

In this paper, we focus our attention on a Gaussian physically degraded relay channel with multiple relay stages. Figure 1 depicts such a channel with 3 relay stages. We build upon the achievability results of [2] and [7] and use an iterative argument to determine the capacity region of such a channel. Thus, we give an alternative derivation for the achievable rate determined in [7]. With our iterative proof, we demonstrate how the coding strategy can be built recursively on the basis of the back-coding argument utilized in [1]. We then extend our iterative argument to prove the converse to the capacity theorem as well.

Having determined the capacity region of the degraded Gaussian channel with multiple relays we turn our attention to the problem of optimal power allocation between the transmitter and the relays. We provide a general solution to this problem and demonstrate that in the special case when all the noise powers are equal, this strategy achieves...
The value of \( \alpha_1^* \) in equation (3) may be obtained from, \( \alpha_1^* = 1 \). Otherwise, \( \alpha_1^* = \frac{\sqrt{1 + \beta^2 - \beta}}{1 + \beta^2} \).

We can also solve (3) for \( \beta \) under the assumption \( \beta = \sqrt{\alpha_1^*} - \sqrt{1} \).

We next pose the optimum power allocation problem to find the value of \( \beta_k \) such that the optimization problem is subject to \( 1 + \beta_k \leq \frac{P_0}{N_k} \) for some fixed \( P_0 \). Similarly, \( \beta = \sqrt{\alpha_1^*} - \sqrt{1} \), we can remove the same improvement in the "effective SNR" arrived at by (4) from the improvement given by (5).

The expression above is actually an expression in use (3) to express \( \alpha_1^* \) in terms of \( \beta_1 \) or (4) to express is only valid if the resulting optimum is such that \( \alpha_1^* > 0 \). In fact, maximizing \( \beta_1 \) we obtain \( \alpha_1^{\text{opt}} = \frac{\beta_1}{2} \).

Equation (6) becomes important when we consider the problem for multi-relay channels.

3 The multiple relay channel

We now consider the problem of the multiple relay problem previously considered in [3] and [7] where achieving the multiple relay degraded Gaussian channel track that result to prove explicitly the capacity of the channel. It turns out that the achievable rate form the multi-relay channel.

For a specified choice of \( \alpha_{ij} \)'s with \( 0 \leq i \leq j \leq K \) and \( \sum_{j=i}^{K} \alpha_{ij} = 1 \), define

\[ R_k(\alpha) = \sum_{j=i}^{K} \frac{\beta_j^2 \alpha_{ij}}{\sum_{j=i}^{K} \beta_j^2} \]
The value of \( a_1^* \) in equation (2) may be obtained explicitly. As stated, if \( \beta_1 \geq \gamma_1 \) then \( a_1^* = 1 \). Otherwise,

\[
a_1^* = \frac{\sqrt{1 + \gamma_1 - \beta_1 + \sqrt{\beta_1}}}{{(1 + \gamma_1)^2}}
\]  

(3)

We can also solve (3) for \( \beta_1 \) under the assumption that \( a_1^* < 1 \). This results in

\[
\beta_1 = \left(\sqrt{a_1^* \gamma_1} - \sqrt{1 - a_1^*}\right)^2
\]  

(4)

We next pose the optimum power allocation problem for the single-relay channel. We would like to find the value of \( \beta_1 \) such that the capacity of the channel is maximized subject to \( \beta_1 \leq \tilde{\beta} \) for some fixed \( P_1 \). Since the channel capacity is given by

\[ C = C \left( \frac{P_1}{N_0} \right) = C \left( \frac{P_1(1 + \gamma_1)}{N_0} \right) \]

we can remove the constraint on \( \beta_1 \) by maximizing the improvement in the "effective SNR" delivered by the relay channel. This "effective SNR improvement" is given by

\[
J_1(\alpha, \beta) = \frac{\alpha(1 + \gamma_1)}{1 + \frac{1}{3}}
\]

(5)

The expression above is actually an expression in only one variable since we can either use (3) to express \( \beta_1 \) in terms of \( a_1^* \) or use (4) to express \( a_1^* \) in terms of \( \beta_1 \). Such substitution is only valid if the resulting optimum is such that \( a_{opt} < 1 \); however, this is true for all \( \gamma_1 > 0 \). In fact, maximizing \( J_1 \) we obtain \( a_{opt} = \frac{\gamma_1}{\gamma_1 + N_0} \) and \( \beta_{opt} = \frac{1}{\gamma_1 + N_0} \) where we note that

\[
\beta_{opt} = 1 - a_{opt}
\]

(6)

Equation (6) becomes important when we consider the general resource optimization problem for multiple relays.

3 The multiple relay channel

We now consider the problem of the multiple relay channel. This problem has been previously considered in [3] and [7] where achievable rates were found for a channel of which the multiple relay degraded Gaussian channel is a special case. In this section we extend that result to prove explicitly the capacity of the multiple relay degraded Gaussian channel. It turns out that the achievable rate found in [7] is the capacity of the degraded multi-relay channel.

For a specified choice of \( \alpha_{ij} \)'s with \( \theta \leq i \leq j \leq K \) satisfying

\[
\sum_{j=\theta}^{K} \alpha_{ij} = 1
\]

(7)

and

\[
\sum_{j=\theta}^{K} \alpha_{ij} = \beta_i \quad \forall i \leq \theta \leq K
\]

(8)

define

\[
R_i(\beta) = C \left( \frac{\sum_{j=\theta}^{K} \alpha_{ij} \gamma_j}{\sum_{j=\theta}^{K} N_j} \right)^{\beta_i}
\]

(9)
where we use \( \Delta \) as a shorthand for \( \sum_{i,j} \). We then have the following theorem.

**Theorem 2. Multi-relay capacity**

The capacity of the multi relay degraded Gaussian channel with \( K \) relays is given by

\[
C_K = \sup_{\mathbf{C}(\mathbf{R})} C_K(\mathbf{R})
\]

with \( C_K \) as defined by (9) and (10).

**Proof:** We prove with the achievability and the converse parts of the theorem by induction. In both cases, the single-relay result of (9) as presented in Theorem 1 serves as the initial step in the induction. Indeed, using the notation presented so far, we have for the single-relay channel: \( K = 1 \): \( \alpha_1 = 1 - \gamma_1 \) of Theorem 1 is \( \alpha_K = 1 - \gamma_1 \). Let \( R_d(\gamma_1) = C(\alpha_1, \Delta) = C(\max_i(\Delta_{1,i}), \Delta) \) and \( R_d(\gamma_1) = C(\max_i(\Delta_{1,i}), \Delta) \).

To prove achievability we need to specify our coding strategy. We simply extend the method used in [1]. The resulting coding strategy is similar to the one proposed in [7], although our method for generating it is recursive and builds directly on the coding strategy used in [1]. This is unlike [7], where the coding strategy is specified directly.

Achievability. For the induction step of the proof of achievability, assume that the theorem holds for \( K - 1 \) relays. Consider adding relay \( K \) as the last relay in the chain.

Because we use it for our inductive step, we reproduce the coding scheme used in [1] to achieve capacity. For \( 0 \leq \gamma_1 \leq 1 \) define \( \alpha_1 = 1 - \gamma_1 \), and let \( X_1 \sim N(0, \alpha_1 I) \) and \( X_2 \sim N(0, \alpha_2 I) \), with \( X_1 \) and \( X_2 \) independent and let \( X_0 = \sum_{i=1}^{K} X_i + X_0 \). Let \( W = \{1, 2, \ldots, 2^K\} \) be the set of messages to be transmitted. Let \( S_2 = \{S_1, S_2, \ldots\} \) be a partition of \( W \) generated in a uniform and random fashion independently from everything else.

We have two random codebooks:

- \( X_i: \mathbb{R} \rightarrow \mathbb{R}^2, x \in W \)
- \( X_i: \mathbb{R} \rightarrow \mathbb{R}^2, x \in S_i \)

Finally, for transmission time \( i, s_{i,j} \) is chosen such that \( s_{i,j} = s_{i,j} \), where \( s_{i,j} \in S_i \). It is shown in [3] that the receiver can decode the message \( s_{i,j} \) at the end of transmission interval \( t \).

Proceeding now to the inductive step, we begin with \( K \) (\( K - 1 \))-relay channel. Fix some appropriate choice of \( \alpha_1, \alpha_2 \). Let \( \mathbf{C}_K(\mathbf{R}) \) be the rate achievable in this channel and with this choice of \( \alpha_1, \alpha_2 \) and assume that this rate is achievable using a codebook such that the output of the transmitter is given by a random variable \( X_K \sim N(0, \Delta_{K} I) \).

Now consider adding another relay at the end of the degradation stage. Because all the relays are decoding all the information, we can think of this as simply adding a new transmitter (indexed \( K \)) and a new receiver (indexed \( K \)). Let \( W = \{1, 2, \ldots, 2^K\} \) be the set of messages to be transmitted. Let \( \mathbf{C}_K(\mathbf{R}_K) \) be a partition of \( W \) generated in a uniform and random fashion independently from everything else.

Define a random codebook \( X_K(\mathbf{X}_K) \) i.i.d. \( \mathcal{N}(0, \Delta) \) we note that \( \alpha_K, \alpha_2, \alpha_1 \sim \mathcal{N}(0, \Delta) \). For transmission time \( t \), \( 1 \leq t \leq K - 1 \), we have it is the property of the power that transmitter \( K \) achieves.

In [1] it was necessary to assume that at the last receiver we successfully decoded messages \( a_k \). However, we assume that at the start of transmission the decoded messages \( a_k \). In particular, we assume that the receiver \( K - 1 \) is successful in decoding messages \( a_k \). Thus, we are able to use the result of Theorem 1 to the result as defined by (9) and (10) is achievable from the receivers 1 through \( K - 1 \). We note that by definition \( \Delta_{K-1} \) is a function of \( \Delta_{K-1} \) and \( \Delta_{K-1} \).

Suppose now that, under the assumption that receivers 1 through \( K - 1 \), it is possible to communicate as defined by (9). We note that because the sum to all the receivers, the rate \( \alpha_K, \alpha_2, \alpha_1, \alpha_2, \alpha_1 \), we can communicate reliably at this stage to all the others of the order of \( \alpha_1, \alpha_2, \alpha_1 \), we obtain the desired capacity.

It remains to show that, assuming that reliability is obtained, it is indeed possible to communicate to done by applying successive interference cancellation argument is provided in [4].

**Converse:** We begin by defining

\[
I_t = I(X_0, \ldots, X_t; X_{t+1} | X_{t+2})
\]

Then from the cut-set bound [2], (Ch. 14) and follows that

\[
C_K \leq \sup_{\mathbb{R}^K} I_t(\cdots, 1) \leq \log(2^{K/2})
\]

Thus we need to show that there exists some \( \alpha_1, \alpha_2, \alpha_1 \) satisfying constraints (7) and (8).

\[
E[ |X_0 + \cdots + X_t; X_{t+1}, \ldots, X_K | X_{t+2}, \ldots, X_K] \leq \log(2^{K/2})
\]

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\[(a) = \min \text{, } R_{\text{ck}}(a) \quad \text{for } a \in \mathbb{C} \]  
\[\text{Min} \sup_{a \in \mathbb{C}} C_k(a) \quad \text{for } a \in \mathbb{C} \]  

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It follows from Theorem 1 that we have the following theorem.

By degraded Gaussian channel with K relays is given

\[C_k = \sup_{a \in \mathbb{C}} C_k(a) \]  

For transmission time \(i\), \(a_n\) is chosen so that \(a_n = j\) where \(\gamma_{n,i} \in S_{n,i}\). Now for \(0 \leq k \leq K - 1\) define \(X_k = \sqrt{B - \gamma_{n,k}}X_n + \sqrt{\gamma_{n,K}}X_k\) where \(\gamma_{n,K}\) is the power of the transmitter \(K\) allocated to the newly added receiver \(i\).

In [3] it was necessary to assume that at the start of transmission time \(i\) the relay \(K\) has successfully decoded messages \(w_1, \ldots, w_{i-1}\). Extending this assumption, we assume that at the start of transmission time \(i\), the receiver \(K\) has successfully decoded messages \(w_1, \ldots, w_{i-1}\). In particular, at transmission time \(i\), all receivers up to and including receiver \(K - 1\) know \(w_{i-1}\). Thus, receiver \(K - 1\) can successfully remove the contribution from \(X_n\) to the received signal. Thus the rate \(C_{\text{ck}}(a)\) as defined by (9) and (10) is achievable from the point of view of communicating to receivers \(1\) through \(K - 1\). We note that by adding zero-zero \(a_n\), we reduce \(C_k(a)\) since \(\gamma_{n,k} = 0\) is reduced from \(1\) to \(a_n = 0\).

Suppose now that under the assumption that reliable communication is achieved to receivers \(1\) through \(K - 1\), it is possible to communicate to receiver \(K\) at a rate \(R_k(a)\) as defined by (9). We note that because the same information is being communicated to all the receivers, the rate \(\min C_k(a), R_k(a)\) is achievable since we can communicate reliably at this rate to all the receivers. Finally, taking a supremum over the choice of \(\gamma_{n,K}\), we obtain the desired capacity rate.

It remains to show that, assuming that reliable communication to all other receivers is achieved, it is indeed possible to communicate to receiver \(K\) at a rate \(R_k(a)\). This is done by applying successive interference cancellation similarly to [3] and [7]. The detailed argument is provided in [4].

After \(B\) transmission instances, we can therefore achieve a rate of \(R_k(a)\) which approaches \(R_k(a)\) as \(B \rightarrow \infty\). This shows that under the assumption that reliable communication to receivers \(1\) through \(K - 1\) is achieved, we can achieve reliable communication to receiver \(K\) at a rate \(R_k(a)\) and this completes the proof of the achievability part of the theorem.

Converse: We begin by defining

\[T_k = I(X_1, \ldots, X_k; Y_{k+1}, \ldots, Y_K) \]  

Then from the cut-set bound ([2], Ch. 14) and the degradations of the channel it follows that

\[C_k \leq \sup_{a \in \mathbb{C}} T_k \]  

To prove the converse, we show that \(I_k \leq R_k\). We have

\[I_k = \frac{1}{2} \text{Var}(X_1 + \cdots + X_k; X_{k+1}, \ldots, Y_K) \]  

Thus we need to show that there exists some choice of real values \(\{a_n\}\) defined for \(0 \leq i \leq j \leq K\) and satisfying constraints (7) and (8) such that

\[\text{Var}(X_1 + \cdots + X_k; X_{k+1}, \ldots, Y_K) \leq \frac{B}{2} \sum_{i=1}^{K} \gamma_{n,i} \]  

Define a random codebook \(X_k(y_{k+1}, \ldots, y_K) ; y_{k+1}, \ldots, y_K \in \{1, 2, \ldots, 2^{B(y_K)}\}\) where we note that \(\gamma_{n,k} = 0\). For transmission time \(i\), \(a_n\) is chosen so that \(a_n = j\) where \(\gamma_{n,i} \in S_{n,i}\). Now for \(0 \leq k \leq K - 1\) define \(X_k = \sqrt{B - \gamma_{n,k}}X_n + \sqrt{\gamma_{n,K}}X_k\) where \(\gamma_{n,K}\) is the power of the transmitter \(K\) allocated to the newly added receiver \(i\).

In [3] it was necessary to assume that at the start of transmission time \(i\) the relay \(K\) has successfully decoded messages \(w_1, \ldots, w_{i-1}\). Extending this assumption, we assume that at the start of transmission time \(i\), the receiver \(K\) has successfully decoded messages \(w_1, \ldots, w_{i-1}\). In particular, at transmission time \(i\), all receivers up to and including receiver \(K - 1\) know \(w_{i-1}\). Thus, receiver \(K - 1\) can successfully remove the contribution from \(X_n\) to the received signal. Thus the rate \(C_{\text{ck}}(a)\) as defined by (9) and (10) is achievable from the point of view of communicating to receivers \(1\) through \(K - 1\). We note that by adding zero-zero \(a_n\), we reduce \(C_k(a)\) since \(\gamma_{n,k} = 0\) is reduced from \(1\) to \(a_n = 0\).

Suppose now that under the assumption that reliable communication is achieved to receivers \(1\) through \(K - 1\), it is possible to communicate to receiver \(K\) at a rate \(R_k(a)\) as defined by (9). We note that because the same information is being communicated to all the receivers, the rate \(\min C_k(a), R_k(a)\) is achievable since we can communicate reliably at this rate to all the receivers. Finally, taking a supremum over the choice of \(\gamma_{n,K}\), we obtain the desired capacity rate.

It remains to show that, assuming that reliable communication to all other receivers is achieved, it is indeed possible to communicate to receiver \(K\) at a rate \(R_k(a)\). This is done by applying successive interference cancellation similarly to [3] and [7]. The detailed argument is provided in [4].

After \(B\) transmission instances, we can therefore achieve a rate of \(R_k(a)\) which approaches \(R_k(a)\) as \(B \rightarrow \infty\). This shows that under the assumption that reliable communication to receivers \(1\) through \(K - 1\) is achieved, we can achieve reliable communication to receiver \(K\) at a rate \(R_k(a)\) and this completes the proof of the achievability part of the theorem.
Define $a_{i\lambda} \Delta \frac{1}{P_i} \mathbb{E}(X_i | X_{\lambda})$ (16) 

The rest of the a's will be defined implicitly based on our induction argument: at this point we only note that from (7) and (9) these must satisfy the constraint

$$
\sum_{\lambda \leq \kappa} a_{\lambda} = \frac{1}{P_0} \mathbb{E}(X_0 | X_{\lambda}) \quad 0 \leq \kappa < K - 1
$$

(17)

We also note that in the case of a single-relay channel the definition in (16) reduces to the definition of (II-a) in the proof of the converse part of Theorem 1 in [1]; and the left-hand side of (17) reduces to $a_0$ in the proof of the converse part of Theorem 1 in [1]. Thus, we can use the single-relay proof of [1] as the initial step in an induction proof of the overall theorem.

For the inductive step, let us assume that the converse holds for a $(K-1)$-relay channel. Thus, we assume that for every choice of the transmitter output distribution and noise powers there exists a choice of real values $(\lambda_0, \ldots, \lambda_{K-1})$ for $0 \leq i \leq j \leq K-1$ satisfying (16) and satisfying (17) for any choice of real values $\eta_j, 1 \leq j \leq K-1$ such that

$$
\sum_{j=0}^{K-1} \left( \sum_{i \leq j} \sqrt{\lambda_i} \right)^2 
$$

holds for all $0 \leq \kappa < K - 1$.

Now consider a $K$-relay channel. Define, as we did throughout, $A$ to be the power of relay $i$, normalized to the transmitter power $P_0$.

First we show that (15) must hold for $0 \leq \kappa < K - 1$. Our argument is constructed as follows. We fix $(a_{i\lambda})$'s for all $0 \leq i < K$. Conditioned on such a choice (i.e. conditioned on $X_0$) we construct an equivalent $K-1$ relay channel with the following properties:

- The set of $(a_{i\lambda})_{0 \leq \lambda \leq K-1}$ satisfying (7) and (9) in the $(K-1)$-relay channel we construct is in one-to-one correspondence with the set of $(a_{i\lambda})_{0 \leq \lambda \leq K-1}$ satisfying (17) in the original channel (conditioned on $X_0$).

- Equation (18) holds for $0 \leq \kappa \leq K-1$ for the $(K-1)$-relay channel we constructed and only if equation (15) holds for $0 \leq \kappa \leq K-1$ for the original $K$-relay channel.

Having shown this, we conclude that (15) must hold for $0 \leq \kappa < K - 1$, for otherwise we violate the induction hypothesis. The details of the proof are provided in [4].

We have now expressed the capacity of the multiple-relay degraded Gaussian channel as a max-min problem. Given a general distribution of power between the transmitter and the relays, the optimum in (11) may be very difficult to determine. A general closed-form expression would certainly become too unwieldy for more than a few relays. However, as we shall see in the next section, the problem of maximizing the power allocation between relays under a total power constraint lends itself to a simple iterative solution.

4 Optimum power allocation for multiple relays

In this section we consider the problem of optimum power allocation between relays in the multiple relay degraded channel. We start with a $K$-relay degraded Gaussian channel.
\[
\frac{1}{P_0} \mathbb{E} \left[ \mathbb{E}(Y_0 | X_0) \right] \tag{16}
\]

implies, based on our induction argument, at this point the moment we satisfy the constraint

\[
(1 - \epsilon) \mathbb{E}(X_k | X_i) \tag{17}
\]

single-relay channel, the definition of \(S = \mathbb{E}(X_k | X_i)\) reduces to that of the converse part of Theorem 3 in [1], and the remission of the proof of the converse part of Theorem 2 in [1], of our best guess as the initial step in an induction proof of

assume that the converse holds for a \((K-1)\) relay.

every choice of the transmitter output distribution \(P_r\) and real values \(\{\alpha_j\}\) for \(0 \leq j \leq K-1\) satisfying \(\sum_j \alpha_j = 1\) such that

\[
X_{\alpha_1}, \ldots, X_{\alpha_{K-1}} \leq P_0 \sum_{j=0}^{K-1} \sqrt{\beta_j} \tag{18}
\]

Defining, as we did throughout, \(\beta_j\) to be the power of the \(j\)th relay power \(P_{0j}\) for \(0 \leq j \leq K-1\). Our argument is constrained to be \(\leq K\). Conditioned on such a choice (i.e., conditioned \(K\)-relay channel with the following two properties: satisfying (7) and (8) in the \((K-1)\)-relay channel we can proceed with the set \(\{\alpha_j\}_{0,0,1} \) satisfying conditions on \(X_i\)).

\[
\sum_{j=0}^{K-1} \alpha_j \leq K - 1 \tag{15}
\]

subject to \(\sum_j \alpha_j = 1\), for the original \(K\)-relay channel. (15) must hold for \(0 \leq k \leq K-1\), for otherwise the details of the proof are provided in [4].

We wish to determine \(\alpha_1, \ldots, \alpha_K\) subject to \(1 = \beta_1 + \cdots + \beta_K \leq P_0\) such that the capacity of the channel as given by (11) is maximized. We begin with the following theorem.

**Theorem 3** Consider a \(K\)-relay degraded Gaussian channel with the power allocated among the relays in such a way that the capacity of the overall channel is optimized subject to \(1 = \beta_1 + \cdots + \beta_K \leq P_0\). Then the optimum in (11) is achieved by setting \(\alpha_0 = \alpha_1 = \cdots = \alpha_K\) to \(\frac{1}{K}\),

\[
\sum_{j=0}^{K-1} \alpha_j \leq K - 1 \tag{19}
\]

We note that if \(\alpha_k = 0\), then \(\beta_k = \sum_{j=k}^{K} \alpha_j\); and in particular, \(\alpha_k = 0\) if \(k = 1, \ldots, K\). Thus, if Theorem 3 holds, in order to find the optimum power allocation scheme for a \(K\)-relay channel, it suffices to determine the values of \(\alpha_0, \ldots, \alpha_K\) subject to \(\sum_{j=0}^{K} \alpha_j = 1\). The values of \(\beta_0, \ldots, \beta_K\) are then determined directly by \(\beta_k = \sum_{j=k}^{K} \alpha_j\). Additionally, as long as \(\alpha_0 > 0\) for \(0 \leq k \leq K\), \(\beta_k > 0\), and we shall see that this holds so long as \(N_{y} > 0\).

**Proof Sketch:** The proof of Theorem 3 is an application of the following principle. Let \(\{x_k\}_{0,k,K+1}\) be a collection of \(M\) non-negative variables. Then the solution to the problem

\[
\min \sum_{k=0}^{K} x_k \text{ subject to } x_k \geq 1
\]

or the solution to its dual

\[
\max \sum_{k=0}^{K} x_k \text{ subject to } x_k \leq K
\]

is to set \(x_k = \frac{1}{k}\) for each \(k\) such that the constraint is satisfied with equality.

The theorem then follows by induction with the initial step provided in Section 2 where we showed that if the power is allocated optimally between the relay and the transmitter, then we have \(\beta_1 = \alpha_1 = \alpha_0 = \alpha_K\). The details are provided in [4].

To derive an explicit expression for the capacity of a \(K\)-relay channel we give by (9)-(11) we need to find a set \(\{r_k\}_{0,k,K+1}\) such that \(R_0 = R_1 = \cdots = R_K\). While the problem is in general very difficult, under the assumption of optimal power allocation we can use Theorem 3 to simplify it significantly. We begin by re-writing (9) under the assumption that \(\alpha_k = \alpha_1\). This results in

\[
R_k - C \left( R_0 \sum_{j=0}^{K-1} \frac{\alpha_j + 1}{\alpha_j} \right) \tag{19}
\]

Setting \(R_k = R_{k+1}\), we get

\[
\frac{\sum_{j=0}^{K-1} (j+1) \alpha_j}{\sum_{j=0}^{K-1} N_j} \leq \frac{\sum_{j=0}^{K} (j+1) \alpha_j}{\sum_{j=0}^{K} N_j} \tag{20}
\]

which yields

\[
\frac{(k+1) \alpha_k}{\sum_{j=0}^{K} N_j} \leq \frac{1}{\sum_{j=0}^{K} N_j} \left( \frac{1}{\sum_{j=0}^{K} N_j} - \frac{1}{\sum_{j=0}^{K} N_j} \right) \tag{21}
\]

\[
\frac{(k+1) \alpha_k}{\sum_{j=0}^{K} N_j} \leq \frac{1}{\sum_{j=0}^{K} N_j} \left( \frac{\sum_{j=0}^{K} N_j - 1}{\sum_{j=0}^{K} N_j} \right) \tag{22}
\]
Thus,  

$$a_k = \frac{1}{(k+1)^2} \sum_{j=0}^{k} (j+1)^2 a_j$$  

(23)

where  

$$a_k \overset{\text{def}}{=} \frac{N_k}{\sum_{j=0}^{k} N_j}$$  

(24)

and one can easily check that if we set $K = 1$ we obtain the solution obtained in Section 2 by direct solution of the single relay optimization problem. Additionally, we note that if $N_k > 0$, $a_k > 0$ and $a_k > 0$.

Equation (23) provides us with an iterative approach to generate the sequence $a_0, a_1, \ldots$ for any $K$. For each $K$, the starting point, $a_0$, needs to be chosen so that $\sum_{k=0}^{K} a_k = 1$. However, in practice, we can always start with $a_0 = 1$, thus producing the same (infinite) sequence for any $K$. Then, for each $K$, we can re-normalize $a_0, \ldots, a_K$ to satisfy the constraint.

By choosing $\alpha_k$'s in this way, we make sure that $R_0 = R_1 = \ldots = R_K$. Then the capacity of the $K$-relay channel can then be written simply as

$$C_K = R_0 = C \left( \sum_{k=0}^{K} \frac{P_k N_k}{\sum_{j=0}^{k} N_j} \right) = C \left( \sum_{k=0}^{K} (1 + a_k) \frac{P_k N_k}{\sum_{j=0}^{k} N_j} \right)$$  

(25)

where we can now define

$$J(K, N_0, \ldots, N_K) \overset{\text{def}}{=} \alpha_k \prod_{k=0}^{K} (1 + a_k)$$  

(26)

to be the maximum SNR improvement provided by having some fixed amount of power $P_k$ at the transmitter and $K$ relays. This quantity depends on the total number of relays available and the noise power at each of the $K$ noise stages. For a given sequence $\{N_k\}$ and a given $K$, it can be computed recursively using (23). Figure 2 shows the improvement achievable with $N_c = N_k$ for the SNR improvement in dB is plotted in Figure 2 for up to 1000 relays.

We observe from Figure 2 that as we increase the number of relays we get less and less improvement out of the additional relays. For example, we gain approximately 18 dB from the first 100 relays and only about another 10 dB from the next 900. A natural question to ask is whether the improvement which we can attain increases without bound.

In the case of equal signal strengths the answer turns out to be no; adding the $K^\text{th}$ relay stage

$$\lim_{K \to \infty} J(K, N, \ldots, N) = J(K-1, N)$$

The proof of this statement is provided in [4].

Thus, as the number of relays grows, the SNR ideally as log(80679K) = log K + log 80679 = log K in the SNR growth as the distributed computations 3 for 1 through 100 relays.

5 Wideband performance in relay channel

In a recent paper [5], Vardi demonstrated that we account when wideband performance of communication link is optimal. These results were shown [6] that in the wideband channel for both the multi-access Gaussian channel and the broadcast channel, the capacity is not achievable by any finite number of relays. However, in the case of the degraded Gaussian channel, the capacity is not achievable by any finite number of relays.

We illustrate this by considering the signal-to-noise ratio (SNR) $Q$ and $\alpha_k$ to be the signal-to-noise ratio between the input and output of the $K$-th stage. While $Q$ may not always be the largest number of relays, it is not a measure of the capacity. Then for the single-relay channel, the capacity

$$C = C(Q, Q)$$

Thus, $\lim_{Q \to \infty} C = C(Q, Q)$, resulting in

$$\frac{E_s}{N_{\text{max}}} = \log_2 C(Q, Q)$$

Turning now to TDM, we note that we can allocate the capacity of the transmitter to relay the signal to the relay or to relay the message to the relay. The capacity is given by

$$C_{\text{TDM}}(Q) = \max_{\theta \in \Theta} \log_2 \left[ \frac{1}{\theta} C(Q) + (1-\theta) C(Q) \right]$$
In the case of equal strength noises the answer turns out to be yes. In this case the SNR improvement offered by adding the $K^{th}$ relay stage satisfies
\[
\lim_{K \to \infty} J(K, N, \ldots, N) - J(K-1, N, \ldots, N) = \frac{\alpha}{\alpha^2} = 0.6079
\]
\[\text{(27)}\]

The proof of this statement is provided in [4].

Thus, as the number of relays grows, the SNR improvement, in dB, grows asymptotically as $\log_2 0.6079N = \log K - \log 0.6079 = \log K - 2.164B$. We define this loss of slope in the SNR growth as the distributed computation penalty and this is plotted in Figure 3 for $K = 1$ through 100 relays.

5 Wideband performance in the Gaussian degraded relay channel

In a recent paper [5], Verdú demonstrated that second-order effects must be taken into account when wideband performance of communication systems is considered. Using these results it was shown [6] that in the wideband regime TDMA is strictly sub-optimal for both the multi-access Gaussian channel and the broadcast Gaussian channel. This is so despite the fact that it does achieve the optimal $\frac{E_b}{N_0}$ defined as
\[
\frac{E_b}{N_0_{\text{max}}} = \lim_{\alpha \to \infty} \frac{\log_2 \frac{E_b}{N_0}}{C(\alpha)}
\]
\[\text{(28)}\]

where $C$ is the capacity of the channel as a function of SNR.

However, in the case of the degraded Gaussian relay channel, time division multiplexing (TDMA) does not even achieve the same $\frac{E_b}{N_0_{\text{max}}}$ as the information theoretically optimal communication scheme. We illustrate this using the single-relay channel. Define $Q_{\text{opt}}$ to be the signal-to-noise ratio between the transmitter and the first relay stage. While $Q$ may not always be the best measure of the signal-to-noise ratio, it is sufficient for our purposes since any reasonable measure of SNR should go to 0 as $Q$ goes to 0.

Then for the single-relay channel, the capacity as a function of $Q$ is given by (2) as
\[
C_1 = C(\alpha Q)
\]
\[\text{(29)}\]

Thus, $\lim_{Q \to \infty} C(Q) = \alpha$, resulting in
\[
\frac{E_b}{N_0_{\text{max}}} = \frac{\log_2 2}{\alpha}
\]
\[\text{(30)}\]

Turning now to TDMA, we note that we can alternately between two modes of communication: the transmitter can transmit directly to the receiver or the transmitter can transmit to the relay which then relays the message to the receiver. In the second instance, the transmitter and relay transmissions have to be time-multiplexed as well. Let $C_2(Q)$ be the capacity function for direct transmitter-to-receiver communication and $C_2(Q)$ the capacity function for communication using the relay. Then the TDMA capacity function is given by
\[
C_{\text{TDMA}}(Q) = \max_{\alpha \in [0,1]}[\alpha C_2(Q) + (1-\alpha)C_2(Q)] = \max_{\alpha \in [0,1]}[C_2(Q)C_2(Q)]
\]
\[\text{(31)}\]
Thus, the optimal strategy is to either always use a relay or to never use it. Which of the two options is to be chosen in a specific scenario depends on the values of $\alpha$ and $\beta$. In order to specify this more precisely we need to examine the functions $\mathcal{C}_Q$ and $\mathcal{C}_I$, respectively.

$$\mathcal{C}_Q(Q) = C \left( \frac{1}{1 + Q} \right)$$  
(32)

If we are using the relay to communicate, then we must time-share the channel between the relay and the transmitter. Thus, the capacity function in this case is given by

$$\mathcal{C}_I(Q) = \max_{\alpha \in \mathbb{C}} \min \{ \alpha C(Q), (1 - \alpha) C(I(Q)) \}$$  
(33)

The minimum point of (33) is achieved by setting the two arguments of the minimum equal which results in

$$\mathcal{C}_I(Q) = \frac{C(I(Q)) + C(I(0))}{2} = \frac{1}{\sqrt{\frac{C(Q)}{C(0)}}}$$  
(34)

Since both $\mathcal{C}_Q$ and $\mathcal{C}_I$ are monotonically increasing and concave, (28) and (31) give us

$$\frac{E_{R}(Q)}{E_{R}(0)} = \frac{\log_2 \left( \frac{1 + \frac{1}{\alpha} \min(1, \frac{1}{\eta})}{\alpha} \right)}{\log_2 \frac{C(Q)}{C(I(0))}}$$

(35)

Equation (35) demonstrates that in the wideband regime the optimal CDM strategy is to use the relay if and only if it is capable of transmitting at a higher power than the transmitter. This strategy is clearly worse than the optimal communication strategy and [4] shows that $\alpha^* < \frac{1}{\pi \sqrt{\frac{Q}{Q + 1}}} \frac{1}{\eta}$ for all $0 < \eta < \infty$.

### References


### Abstract

It is known that some of the spectacular capacity point-to-point rich scattering channel, namely TDMA, can also be obtained in a multi-user environment. With linear growth in the sum-capacity when the number of users grows linearly, but with fewer users than antennas. We ask the following question: When is the linear growth of the sum-capacity of a channel with $d=2$ antennas, which are linearly arrayed, with bandwidth $b$ and number of users $n$.

1 Introduction

The linear growth of capacity with the number of users in the receiver over a point-to-point wireless rich-scatter environment is an area of interest in designing simple techniques to achieve this goal. One approach is to use TDMA or CDMA, to service a multi-user environment. However, this is not the only way to achieve this goal. For example, using multiple antennas in isolation for designing multiple access protocols with high data rates in [8] suggest that high sum-rates are possible and combines multiple antennas with known-separated transmit to multiple users.

A linear growth in sum-capacity (maximum users) in a rich scattering environment can be achieved if linear and for the downlink (one-to-many) [9–11] when channel. We obtain the constant of proportionality of the number of antennas and users grows simultaneously. Although cellular systems generally have more antennas and users are also being considered for the IEEE 802.11 standard.