Capacity of Antenna Arrays with Space, Polarization and Pattern Diversity

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Abstract — **We present an analytical characterization of the multi-antenna capacity in the limit of a large number of antennas. In contrast with previous studies, the entries of the channel matrix are not restricted to be identically distributed thus incorporating diversity mechanisms that are otherwise excluded, such as those based on the use of antennas with distinct polarizations and radiation patterns. In addition to the capacity, first-order expressions in the low- and high-power regimes are also evaluated both asymptotically and non-asymptotically.**

I. MOTIVATION

The analysis of the Shannon capacity of multi-dimensional communication channels such as those created by antenna arrays is greatly facilitated in the asymptotic regime, as the number of dimensions—in this case antennas—grows large. The limiting value of the capacity has, besides theoretical interest, practical appeal: it has been shown in various contexts that convergence to such limit occurs rapidly and thus the asymptotic capacity usually serves as excellent approximation to the ergodic capacity of arrays of virtually any size [1]. As a result, the asymptotic multi-antenna capacity has been the object of much attention but, nearly always, with the entries of the associated channel matrix identically distributed. Although this assumption is very convenient from an analytical standpoint, it poses a severe constraint on the structure of the arrays: it implies that the power transfer must be the same from each transmit to each receive antenna and, therefore, it restricts the validity of the solution to homogeneous arrays whose antennas are identical. Only the space diversity provided by antenna spacing can be accounted for. In practice, however, space diversity is often supplemented or even replaced by other diversity mechanisms (e.g. [2]), mainly:

Polarization diversity. Antennas with orthogonal polarizations provide low levels of correlation with minimum or no spacing and render the wireless link robust to polarization rotations [3]. Although very effective, polarization diversity alone does not suffice once the number of antennas exceeds the number of orthogonal polarizations, which is very small [4]. Hence, in large arrays polarization diversity must be combined with space diversity.

Pattern diversity. Antennas with different radiation patterns (or with rotated versions of the same pattern) may also be used to discriminate different multipath components and reduce correlation. As with polarization diversity, though, this mechanism must be combined with space diversity if the number of antennas is large.

Different polarizations and/or patterns create asymmetries in the power transfer between antennas [3] and, as a result, the channel matrix is no longer identically distributed. In this paper, we incorporate these asymmetries by removing the identical-distribution constraint. Using recent results on random matrix theory, reported in [5], we evaluate the asymptotic capacity of several classes of non-identically distributed multi-antenna channels. In addition to the capacity, first-order expressions in the low- and high-power regimes are also characterized.

Throughout the paper, $(\mathbf{A})_i$ indicates the i-th column of a matrix $\mathbf{A}$ and $(\mathbf{A})_{i,j}$ indicates its (i,j)-th entry.

II. CAPACITY & KEY PERFORMANCE MEASURES

Given n_{T} transmit and n_{R} receive antennas, the baseband model with frequency-flat fading[1] is

$$\mathbf{y} = \sqrt{\mathsf{g}}\,\mathbf{H}\,\mathbf{x} + \mathbf{n}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the transmit and received signal vectors while $\mathbf{n}$ is additive noise with one-sided spectral density $N_0{=}E[\|\mathbf{n}\|^2]/n_{\mathrm{R}}$. The channel is embodied by the scalar factor g and the normalized zero-mean Gaussian matrix $\mathbf{H}$, the power of whose entries can be assembled into a matrix $\mathbf{P}$ such that $E[|(\mathbf{H})_{i,j}|^2]{=}(\mathbf{P})_{i,j}$ which satisfies[2]

$$\sum_{j=1}^{n_{\mathrm{T}}}\sum_{i=1}^{n_{\mathrm{R}}}(\mathbf{P})_{i,j} = n_{\mathrm{T}}n_{\mathrm{R}}. \tag{1}$$

In white Gaussian noise and with the channel unknown to the transmitter, the ergodic capacity is [6, 7]

$$C = E\left[\log_2 \det\left(\mathbf{I} + \tfrac{\mathsf{SNR}}{n_{\mathrm{T}}}\,\mathbf{H}\mathbf{H}^{\dagger}\right)\right] \tag{2}$$

where SNR represents the average signal-to-noise ratio observed over the n_{R} receive antennas

$$\mathsf{SNR} = \mathsf{g}\frac{E[\|\mathbf{x}\|^2]}{N_0}.$$

[1]When the fading is frequency selective, the channel can be decomposed into parallel non-interacting subchannels, each experiencing flat fading and having the same ergodic capacity as the aggregate channel.

[2]If $\mathbf{H}$ is identically distributed, each of its entries has unit power.

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Also of interest are the total (normalized) power injected into the channel by the j-th transmit antenna

$$P_{\mathrm{T}}(j) \triangleq \sum_{i=1}^{n_{\mathrm{R}}} (\mathbf{P})_{i,j}$$

and the total (normalized) power collected by the i-th receive antenna

$$P_{\mathrm{R}}(i) \triangleq \sum_{j=1}^{n_{\mathrm{T}}} (\mathbf{P})_{i,j}.$$

Notice that the normalization imposed on $\mathbf{H}$ by (1) puts different channels on an equal footage in terms of average power, i.e. any difference is factored out of $\mathbf{H}$ and absorbed into the SNR through g.

For the purpose of asymptotic analysis, we let $\frac{n_{\mathrm{T}}}{n_{\mathrm{R}}}=\beta$ and define the capacity per receive antenna

$$\mathcal{C} \triangleq \frac{C}{n_{\mathrm{R}}}$$

to which, in the remainder, we apply a body of new results on random matrix theory introduced in [5]. We shall make extensive use of the *asymptotic power profile*

$$\mathcal{P}(r,t) \triangleq \lim_{n_{\mathrm{R}}\to\infty} \mathcal{P}^{(n_{\mathrm{R}})}(r,t)$$

with $r \in (0,1]$ and $t \in (0,\beta]$ and with $\mathcal{P}^{(n_{\mathrm{R}})}$ given by

$$\mathcal{P}^{(n_{\mathrm{R}})}(r,t) \triangleq (\mathbf{P})_{i,j} \qquad \tfrac{i}{n_{\mathrm{R}}} \le r < \tfrac{i+1}{n_{\mathrm{R}}}, \quad \tfrac{t}{n_{\mathrm{R}}} \le t < \tfrac{j+1}{n_{\mathrm{R}}}.$$

In order to evaluate the tradeoffs between power and bandwidth [8], it is often convenient to express the capacity, not only as function of the SNR, but also as function of the normalized energy per bit

$$\frac{E_b}{N_0} = \frac{\mathsf{SNR}}{\mathcal{C}(\mathsf{SNR})}.$$

Unfortunately, the function $\mathsf{C}(\frac{E_b}{N_0})$ is known in closed-form only for scalar unfaded channels. For most purposes, nonetheless, its first-order expressions in the low- and high-SNR regimes suffice [9]. Moreover, as we shall see, these expressions often amend themselves to analysis even non-asymptotically.

At low SNR, the capacity can be posed as [10, 8]

$$\mathsf{C}(\frac{E_b}{N_0}) = \frac{S_0}{3\,\mathrm{dB}} \left(\frac{E_b}{N_0}|_{\mathrm{dB}} - \frac{E_b}{N_0}_{\min}|_{\mathrm{dB}} \right) + \epsilon$$

with ϵ vanishing faster than the main term as $\frac{E_b}{N_0} \downarrow \frac{E_b}{N_0}_{\min}$.

$\frac{E_b}{N_0}_{\min}$ indicates the minimum energy per bit required for reliable communication, which at the receiver is [10]

$$\frac{E_b^r}{N_0}_{\min} = \log_e 2$$

whereas, at the transmitter [8]

$$\frac{E_b}{N_0}_{\min} = \frac{\log_e 2}{\mathsf{g}\, n_{\mathrm{R}}}.$$

In either case, the key performance measure is S_0, the low-SNR slope in bits/s/Hz/(3 dB).

At high SNR, in turn, the capacity approaches [11]

$$\mathsf{C}(\frac{E_b}{N_0}) = \frac{S_\infty}{3\,\mathrm{dB}} \frac{E_b}{N_0}|_{\mathrm{dB}} + o(\frac{E_b}{N_0}|_{\mathrm{dB}})$$

with the key performance measure being S_∞, the high-SNR slope in bits/s/Hz/(3 dB).

III. Multi-Antenna Gaussian Channels

A. Uncorrelated Non-Identically Distributed Entries

For channels where antenna correlation is negligible but heterogeneous arrays are involved, we present the following result.

Proposition 1 *Consider a normalized channel $\mathbf{H}$ with uncorrelated entries and asymptotic power profile $\mathcal{P}$. The capacity per antenna converges, as $n_{\mathrm{R}}\to\infty$, to*

$$\begin{aligned} \mathcal{C}(\beta,\mathsf{SNR}) \;=\;& \beta E\left[\log_2\left(1+\mathsf{SNR}\,\Gamma(\mathsf{T},\mathsf{SNR})\right)\right] \\ &+E\left[\log_2\left(1+E\left[\mathcal{P}(\mathsf{R},\mathsf{T})\,\Upsilon(\mathsf{T},\mathsf{SNR})|\mathsf{R}\right]\right)\right] \\ &-\beta E\left[\Gamma(\mathsf{T},\mathsf{SNR})\,\Upsilon(\mathsf{T},\mathsf{SNR})\right]\log_2 e \end{aligned} \tag{3}$$

where the expectations are with respect to the random variables R *and* T*, independent and uniformly distributed in* $[0,1]$ *and* $[0,\beta]$*, respectively, and with*

$$\begin{aligned} \Gamma(t,\mathsf{SNR}) &= \frac{1}{\beta} E\left[\frac{\mathcal{P}(\mathsf{R},t)}{1+E\left[\mathcal{P}(\mathsf{R},\mathsf{T})\Upsilon(\mathsf{T},\mathsf{SNR})|\mathsf{R}\right]}\right] \\ \Upsilon(t,\mathsf{SNR}) &= \frac{\mathsf{SNR}}{1+\mathsf{SNR}\,\Gamma(t,\mathsf{SNR})}. \end{aligned}$$

Proof *See [5].*

The quantities Γ and Υ have actual engineering meaning. Specifically, $\mathsf{SNR}\,\Gamma(t,\mathsf{SNR})$ represents the asymptotic signal-to-noise at the output of a linear MMSE receiver as function of the normalized transmit antenna index, t. The mean-square error at the output of such receiver, in turn, equals $\Upsilon(t,\mathsf{SNR})/\mathsf{SNR}$.

Exact expressions for S_0 and S_∞ can be found for arbitrary n_{T} and n_{R}. In particular,

$$S_0 = \frac{2\, n_{\mathrm{T}}^2 n_{\mathrm{R}}^2}{\sum_{j=1}^{n_{\mathrm{T}}} P_{\mathrm{T}}^2(j) + \sum_{i=1}^{n_{\mathrm{R}}} P_{\mathrm{R}}^2(i)} \tag{4}$$

which satisfies

$$1 \le S_0 \le \frac{2\, n_{\mathrm{T}} n_{\mathrm{R}}}{n_{\mathrm{T}} + n_{\mathrm{R}}}$$

where the lower bound is attained when the entire power is transferred from a single transmit to a single receive antenna while the upper bound corresponds to equal power transfer from each transmit to each receive antenna. The more balanced the powers, the higher S_0.

Equally revealing is S_∞, which can be rewritten as

$$S_\infty = \min(n'_{\mathrm{T}}, n'_{\mathrm{R}}) \tag{5}$$

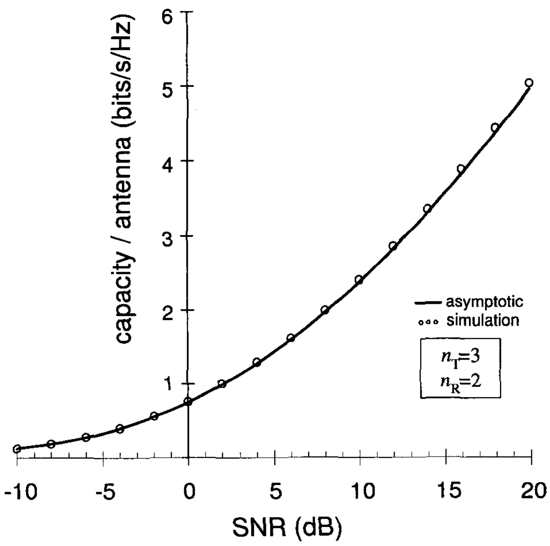

Figure 1: $\mathcal{C}(\text{SNR})$ for the power matrix $\mathbf{P}$ in (6).

with n'_T the number of transmit antennas for which $P_T(j)>0$ and n'_R the number of receive antennas for which $P_R(i)>0$.

The asymptotic result in Proposition 1 suggests that the capacity with fixed n_T and n_R can be approximated by (3) with $\beta=\frac{n_T}{n_R}$ and $\mathcal{P}=\mathcal{P}^{(n_R)}$. The next example illustrates how this process yields extremely accurate values even when n_T and n_R are very small.

Example 1 *Let $n_T=3$ and $n_R=2$ with no correlation and*

$$\mathbf{P} = \begin{bmatrix} 3.6 & 0.36 & 0.04 \\ 1.8 & 0.18 & 0.02 \end{bmatrix}. \qquad (6)$$

$\mathcal{C}(\text{SNR})$ is plotted in Fig. 1. Also shown, as a reference, is the result of a Monte-Carlo evaluation of (2).

Proposition 2 *The asymptotic capacity per antenna can also be expressed as*

$$\mathcal{C} = \int_0^\beta \log_2(1+\text{SNR}\,F(t,\text{SNR}))\,dt$$

with F satisfying

$$F(t,\text{SNR}) = \frac{1}{\beta}E\left[\frac{\mathcal{P}(\mathsf{R},t)}{1+\text{SNR}\,\frac{\beta-t}{\beta}E\left[\frac{\mathcal{P}(\mathsf{R},\mathsf{T})}{1+\text{SNR}\,F(t,\text{SNR})}|\mathsf{R}\right]}\right]$$

where R and T are two independent random variables uniformly distributed in $[0,1]$ and $[y,\beta]$, respectively.

Proof: *See [5].*

The quantity $\text{SNR}\,F(t,\text{SNR})$, with $\frac{t}{n_R}\le t<\frac{j+1}{n_R}$, indicates the signal-to-noise exhibited by the j-th transmit antenna at the output of a linear MMSE receiver prior to which interference from antennas $(1,\ldots,j-1)$ has been perfectly cancelled. This interprets the well-known result that, if each transmit antenna radiates a separately encoded signal, then a receiver that successively detects and cancels these signals can achieve capacity [12, 9]. The rate (normalized by n_R) conveyed by each transmit antenna as function of t is $\log_2(1+\text{SNR}\,F(t,\text{SNR}))$.

The general solution presented in Proposition 1 simplifies greatly if $\mathbf{P}$ exhibits certain structures.

Definition 1 *An $(n\times m)$ matrix $\mathbf{A}$ taking values in $\mathcal{A}\subset\mathbb{R}$ is asymptotically row-regular if, $\forall\xi\in\mathcal{A}$,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}1\{(\mathbf{A})_{i,j}<\xi\}$$

does not depend on i.

Definition 2 *A matrix $\mathbf{A}$ is asymptotically column-regular if $\mathbf{A}^T$ is asymptotically row-regular.*

Asymptotic row (resp. column) regularity thus entails every row (resp. column) having the same asymptotic empirical distribution.

Definition 3 *A matrix $\mathbf{A}$ is asymptotically doubly-regular if it is asymptotically both row-regular and column-regular.*

In such matrices, the average of every row is equal to the average of every column, i.e.

$$\lim_{m\to\infty}\frac{1}{m}\sum_{j=1}^{m}(\mathbf{A})_{i,j} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(\mathbf{A})_{i,j}.$$

Examples of double-regularity are (*i*) a square Toeplitz matrix, and (*ii*) $(\mathbf{A})_{i,j}=\tau_{i-j}$ with τ_q a periodic sequence.

Most arrays of practical interest result in channels where $\mathbf{P}$ is doubly-regular.

Proposition 3 *Consider a normalized channel $\mathbf{H}$ with uncorrelated entries and power matrix $\mathcal{P}$. If $\mathbf{P}$ is doubly-regular, the asymptotic capacity per antenna is given by*

$$\begin{aligned}\mathcal{C}(\beta,\text{SNR}) &= \log_2\left(1+\text{SNR}-\tfrac{1}{4}\mathcal{F}(\beta,\tfrac{\text{SNR}}{\beta})\right)\\ &+ \beta\log_2\left(1+\tfrac{\text{SNR}}{\beta}-\tfrac{1}{4}\mathcal{F}(\beta,\tfrac{\text{SNR}}{\beta})\right)\\ &- \beta\frac{\log_2 e}{4\,\text{SNR}}\mathcal{F}(\beta,\tfrac{\text{SNR}}{\beta})\end{aligned}$$

with the auxiliary function

$$\mathcal{F}(x,y)\triangleq\left(\sqrt{1+y(1+\sqrt{x})^2}-\sqrt{1+y(1-\sqrt{x})^2}\right)^2.$$

Proof: *See [5].*

This result states that the function $\mathcal{C}(\text{SNR})$ derived in [9] for independent identically distributed channels is, in fact, also valid if the channel is non-identically distributed so long as $\mathbf{P}$ is doubly-regular. (Different antennas, however, may result in a different value for g translating, in turn, into a different SNR.)

Particularizing (4) and (5) to channels where $\mathbf{P}$ is doubly-regular,

$$S_0 = \frac{2\, n_T n_R}{n_T + n_R} \qquad\qquad S_\infty = \min(n_T, n_R).$$

A second structure of interest for $\mathbf{P}$ is the outer product of two vectors, in which case $\mathcal{P}(r,t)=\mathcal{P}_R(r)\mathcal{P}_T(t)$ and the capacity equals that of a channel whose correlation is "separable".

B. Separable Correlation

Definition 4 *The correlation of a matrix* $\mathbf{H}$ *is "separable" if* $r_{\mathbf{H}}(i,j;i',j')\triangleq E\left[(\mathbf{H})_{i,j}(\mathbf{H})^*_{i',j'}\right]$ *can be expressed as the product of two marginal correlations that are functions, respectively, of (i,j) and (i',j').*

Within the context of multi-antenna channels [13], $r_{\mathbf{H}}(i,j;i',j')=(\mathbf{\Theta}_R)_{i,i'}(\mathbf{\Theta}_T)_{j',j}$ where $\mathbf{\Theta}_T$ and $\mathbf{\Theta}_R$ are $(n_T \times n_T)$ and $(n_R \times n_R)$ matrices indicating the correlation between each pair of transmit antennas and each pair of receive antennas, respectively. With that,

$$\mathbf{H} = \mathbf{\Theta}_R^{1/2}\mathbf{W}\mathbf{\Theta}_T^{1/2} \tag{7}$$

with $\mathbf{W}$ containing zero-mean unit-variance independent complex Gaussian random variables. When $\mathbf{H}$ is identically distributed, the diagonal entries of $\mathbf{\Theta}_T$ and $\mathbf{\Theta}_R$ are equal to 1. Here, this assumption is not needed.

Proposition 4 *If the channel is given by (7), its capacity per antenna converges to*

$$\begin{aligned}\mathcal{C}(\beta, \mathsf{SNR}) &= \beta E\left[\log_2(1+\Lambda_T \mathsf{SNR}\,\Gamma)\right] \\ &\quad + E\left[\log_2(1+\Lambda_R\, \mathsf{SNR}\,\Upsilon)\right] - \beta\, \mathsf{SNR}\,\Gamma\Upsilon \log_2 e\end{aligned}$$

with Γ *and* Υ *the solutions to*

$$\Gamma = \frac{1}{\beta}E\left[\frac{\Lambda_R}{1+\Lambda_R\,\mathsf{SNR}\,\Upsilon}\right] \qquad\qquad \Upsilon = E\left[\frac{\Lambda_T}{1+\Lambda_T \mathsf{SNR}\,\Gamma}\right]$$

where Λ_R *and* Λ_T *are independent random variables distributed according to the asymptotic density eigenvalue distributions of* $\mathbf{\Theta}_R$ *and* $\mathbf{\Theta}_T$*, respectively.*

Proof: *See [5].*

In contrast with the solution in [14], where β was constrained to be unity, Proposition 4 is general in β.

As before, exact non-asymptotic expressions for $\mathcal{S}_0$ and $\mathcal{S}_\infty$ can be found:

$$S_0 = \frac{2\, n_T n_R}{n_T\zeta(\mathbf{\Theta}_R) + n_R\zeta(\mathbf{\Theta}_T)} \qquad\qquad S_\infty = \min(n''_T, n''_R)$$

with $\zeta(\mathbf{\Theta}_T)=\frac{\mathrm{Tr}\{\mathbf{\Theta}_T^2\}}{n_T}$ and $\zeta(\mathbf{\Theta}_R)=\frac{\mathrm{Tr}\{\mathbf{\Theta}_R^2\}}{n_R}$ denoting the transmit and receive correlation numbers introduced in [8] and with n''_T and n''_R the number of non-zero eigenvalues of $\mathbf{\Theta}_T$ and $\mathbf{\Theta}_R$, respectively.

The applicability of Proposition 4 extends, beyond the model in (7), to channels with uncorrelated entries whose asymptotic power profile can be factored as $\mathcal{P}(r,t)=\mathcal{P}_R(r)\mathcal{P}_T(t)$. In that case, Λ_R and Λ_T are distributed as $\mathcal{P}_R(\mathsf{R})$ and $\mathcal{P}_T(\mathsf{T})$ with R and T uniform in $[0,1]$ and $[0,\beta]$, respectively.

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