Capacity of MIMO Channels with One-sided Correlation

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***Abstract*— We present closed-form expressions for the marginal density distribution of the unordered eigenvalues of $\mathbf{H\Phi H}^\dagger$ where $\mathbf{\Phi}$ is an input covariance and $\mathbf{H}$ is a matrix representing a MIMO (multi-input multi-output) Rayleigh-faded channel with one-sided correlation at either end of the link, transmitter or receiver, with no constraints on the numbers of antennas therein. Using the foregoing distribution, we then derive analytical expressions for the capacity. The expressions found are evaluated through several examples conducted with correlation structures of practical interest.**

I. INTRODUCTION

The central idea in multi-antenna communication is to generate a MIMO (multi-input multi-output) channel by transmitting and receiving through antenna arrays [1]–[2]. The deployment of these arrays on small platforms (compact base stations, mobile terminals, portable computers, etc) almost inevitably results in some degree of correlation between the constituent antennas. This has motivated a multiplicity of efforts aimed at characterizing analytically the capacity and mutual information (with an isotropic input) of MIMO channels in the presence of correlation. These problems have been tacked primarily by means of asymptotic analysis: for large numbers of antennas in [3]–[6] and at low signal-to-noise ratio in [7]. Non-asymptotically, the mutual information has been characterized in integral form in [8] and explicitly in [9]–[11]. While the expression in [11] allows for simultaneous correlations at both transmitter and receiver, it is not immediately applicable if either end exhibits no correlation. This case, relevant because it models the frequent scenario where correlation is strongly dominant at either transmitter or receiver, is addressed in [9] and [10] albeit in the latter with the restriction that the correlation be at the end of the link with the most antennas. All these contributions obtain the mutual information through the moment-generating function.

In this paper, we focus on one-sided correlation but with emphasis on the distribution of the eigenvalues of $\mathbf{H\Phi H}^\dagger$ where $\mathbf{H}$ is the channel matrix and $\mathbf{\Phi}$ the input covariance. For channels with IID (independent identically distributed) Rayleigh-faded entries, this distribution is given in [1]. Here, we present its counterpart with one-sided correlation. Using this distribution, we then obtain new expressions for the capacity. Other communication-theoretical quantities that are functionals of this distribution (e.g., the signal-to-interference at the output of an MMSE receiver) can be also derived.

By explicitly incorporating the input covariance, we can express the capacity rather than simply the mutual information. The correlation is allowed to take place at either end of the link, a relevant point given that it is the end with the fewest antennas that constrains the number of spatial degrees of freedom.

Throughout the paper, we use $(\cdot)_{i,j}$ to denote the (i,j)-th entry of a matrix and $\lambda_j(\cdot)$ to indicate its j-th eigenvalue.

II. PROBLEM FORMULATION

Denoted by $n_{\rm T}$ and $n_{\rm R}$ the number of transmit and receive antennas, we consider the complex frequency-flat linear model

$$\mathbf{y} = \sqrt{g}\,\mathbf{Hx} + \mathbf{n}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the input and output vectors while $\mathbf{n}$ is white Gaussian noise. The channel is represented by the $(n_{\rm R}\times n_{\rm T})$ zero-mean random matrix $\sqrt{g}\,\mathbf{H}$ normalized such that

$$E[\mathrm{Tr}\{\mathbf{HH}^\dagger\}] = n_{\rm R} n_{\rm T}.$$

In terms of the correlation between the entries of $\mathbf{H}$, we adhere to the so-called *separable* model whereby [12]

$$\mathbf{H} = \mathbf{\Theta}_{\rm R}^{1/2}\mathbf{W}\mathbf{\Theta}_{\rm T}^{1/2} \quad (1)$$

where the $(n_{\rm R}\times n_{\rm T})$ matrix $\mathbf{W}$ has IID zero-mean unit-variance complex Gaussian random entries while the entries of $\mathbf{\Theta}_{\rm T}$ and $\mathbf{\Theta}_{\rm R}$ indicate, respectively, the correlation between transmit and between receive antennas.

The capacity-achieving input covariance, normalized by its energy per dimension, is denoted by

$$\mathbf{\Phi} = \frac{E[\mathbf{xx}^\dagger]}{\frac{1}{n_{\rm T}}E[\|\mathbf{x}\|^2]}$$

where the normalization ensures that $E[\mathrm{Tr}\{\mathbf{\Phi}\}]{=}n_{\rm T}$. The ergodic capacity is thus

$$C(\mathsf{SNR}) = E\left[\log_2\det\left(\mathbf{I} + \frac{\mathsf{SNR}}{n_{\rm T}}\,\mathbf{\Theta}_{\rm R}\mathbf{W}\mathbf{\Theta}_{\rm T}^{1/2}\mathbf{\Phi}\mathbf{\Theta}_{\rm T}^{1/2}\mathbf{W}^\dagger\right)\right]$$

where the expectation is over the distribution of $\mathbf{W}$ while

$$\mathsf{SNR} = g\frac{E[\|\mathbf{x}\|^2]}{\frac{1}{n_{\rm R}}E[\|\mathbf{n}\|^2]}$$

Denoting by $\mathbf{\Lambda}_{\mathrm{R}}$, $\mathbf{\Lambda}_{\mathrm{T}}$ and $\mathbf{P}$ the respective diagonal eigenvalue matrices of $\mathbf{\Theta}_{\mathrm{R}}$, $\mathbf{\Theta}_{\mathrm{T}}$ and $\mathbf{\Phi}$, and invoking the unitary invariance of $\mathbf{W}$ and the fact that—to achieve capacity—the eigenvectors of $\mathbf{\Phi}$ must coincide [13] with those of the transmit correlation, $\mathbf{\Theta}_{\mathrm{T}}$, the capacity can be rewritten as

$$C(\mathsf{SNR}) = E\left[\log_2 \det\left(\mathbf{I} + \tfrac{\mathsf{SNR}}{n_{\mathrm{T}}}\mathbf{\Lambda}_{\mathrm{R}}\mathbf{W}\mathbf{\Lambda}_{\mathrm{T}}\mathbf{P}\mathbf{W}^{\dagger}\right)\right]$$

where $\mathbf{P}$ defines the capacity-achieving power allocation [14]. If $\mathbf{\Lambda}_{\mathrm{T}}{=}\mathbf{I}$, then capacity is achieved when the input is isotropic ($\mathbf{P}{=}\mathbf{I}$). If $\mathbf{\Lambda}_{\mathrm{T}}{\neq}\mathbf{I}$, however, then in general $\mathbf{P}{\neq}\mathbf{I}$ [14].

The capacity, as well as other quantities of interest, is determined by the nonzero eigenvalues of $\mathbf{H\Phi H}^{\dagger}$, which are given by those of $\mathbf{\Lambda}_{\mathrm{R}}\mathbf{W}\mathbf{\Lambda}_{\mathrm{T}}\mathbf{P}\mathbf{W}^{\dagger}$ if $n_{\mathrm{T}}{\geq}n_{\mathrm{R}}$ and by those of $\mathbf{\Lambda}_{\mathrm{T}}\mathbf{P}\mathbf{W}^{\dagger}\mathbf{\Lambda}_{\mathrm{R}}\mathbf{W}$ if $n_{\mathrm{T}}{\leq}n_{\mathrm{R}}$. For notational convenience we therefore let $m{=}\min(n_{\mathrm{T}}, n_{\mathrm{R}})$ and $n{=}\max(n_{\mathrm{T}}, n_{\mathrm{R}})$ and introduce the matrix

$$\mathbf{S} = \mathbf{\Lambda}_m \mathbf{W} \mathbf{\Lambda}_n \mathbf{W}^{\dagger}$$

where $\mathbf{W}$ is $(m{\times}n)$ with zero-mean IID Gaussian entries while

$$\begin{array}{lll} \mathbf{\Lambda}_m = \mathbf{\Lambda}_{\mathrm{T}}\mathbf{P} \;\text{ and }\; \mathbf{\Lambda}_n = \mathbf{\Lambda}_{\mathrm{R}} & & n_{\mathrm{T}} \leq n_{\mathrm{R}} \\ \mathbf{\Lambda}_m = \mathbf{\Lambda}_{\mathrm{R}} \;\text{ and }\; \mathbf{\Lambda}_n = \mathbf{\Lambda}_{\mathrm{T}}\mathbf{P} & & n_{\mathrm{T}} \geq n_{\mathrm{R}} \end{array}$$

With that,

$$\begin{aligned} C(\mathsf{SNR}) &= E\left[\sum_{i=1}^{m} \log_2\left(1 + \tfrac{\mathsf{SNR}}{m}\lambda_i(\mathbf{S})\right)\right] \\ &= m\, E\left[\log_2\left(1 + \tfrac{\mathsf{SNR}}{m}\lambda(\mathbf{S})\right)\right] \qquad (2) \end{aligned}$$

where λ denotes an arbitrary eigenvalue of $\mathbf{S}$ and the expectation is over its distribution.

Since the entries of $\mathbf{W}$ are jointly Gaussian, the entries of $\mathbf{Z}{=}\mathbf{\Lambda}_{\mathrm{R}}^{1/2}\mathbf{W}\mathbf{\Lambda}_{\mathrm{T}}^{1/2}\mathbf{P}^{1/2}$ are also jointly Gaussian with distribution [15]

$$f_{\mathbf{Z}}(\mathbf{\Xi}) = \frac{e^{-\mathrm{Tr}\{\mathbf{\Lambda}_{\mathrm{R}}^{-1}\mathbf{\Xi}\mathbf{\Lambda}_{\mathrm{T}}^{-1}\mathbf{P}^{-1}\mathbf{\Xi}^{\dagger}\}}}{\pi^{n_{\mathrm{T}}n_{\mathrm{R}}}\det(\mathbf{\Lambda}_{\mathrm{R}})^{n_{\mathrm{T}}}\det(\mathbf{\Lambda}_{\mathrm{T}}\mathbf{P})^{n_{\mathrm{R}}}}$$

and, as a consequence, the distribution of $\mathbf{S}$ is [15]

$$f_{\mathbf{S}}(\mathbf{\Xi}) = \frac{\det(\mathbf{\Xi})^{n-m} e^{-\mathrm{Tr}\{\mathbf{\Lambda}_m^{-1}\mathbf{\Xi}/q\}}\, {}_0\boldsymbol{F}_0(\mathbf{I}{-}q\mathbf{\Lambda}_n^{-1}, \mathbf{\Lambda}_m^{-1}\mathbf{\Xi}/q)}{\Gamma_m(n)\det(\mathbf{\Lambda}_m)^n \det(\mathbf{\Lambda}_n)^m} \tag{3}$$

where $q{>}0$ is a positive constant, ${}_0\boldsymbol{F}_0(\cdot,\cdot)$ is the hypergeometric function of exponential type of two square matrix arguments of different dimensions [15], and $\Gamma_p(q)$ with $p{\leq}q$ is the complex multivariate Gamma function [16]

$$\Gamma_p(q) = \pi^{p(p-1)/2}\prod_{\ell=1}^{p}(q-\ell)!$$

Starting from (3), we set out to obtain the density distribution of the unordered eigenvalues of $\mathbf{S}$, from which the capacity in (2) can be then computed. As detailed in the Appendix, we leverage the technique pioneered in [17] to evaluate (2) by means of the Mellin transform [18].

We shall consider channels where either $\mathbf{\Lambda}_{\mathrm{R}}{=}\mathbf{I}$ or $\mathbf{\Lambda}_{\mathrm{T}}\mathbf{P}{=}\mathbf{I}$. Both cases basically mirror each other with the caveat that, while $\mathbf{\Lambda}_{\mathrm{R}}$ is determined exclusively by the receive correlations, $\mathbf{\Lambda}_{\mathrm{T}}\mathbf{P}$ involves both the transmit correlations and the resulting power allocation [14]. In either case, the eigenvalues (diagonal entries) are required to be nonzero and distinct. Nonetheless, as will be illustrated in the examples, zero eigenvalues or eigenvalues with plural multiplicity can be accommodated through simple perturbation techniques.

The proofs of the various results are relegated to the Appendix.

III. Characterization of the Unordered Eigenvalues

In this section, we provide expressions for the marginal distribution of the unordered eigenvalues of $\mathbf{S}$. Let us start with correlations at the end of the link with the fewest antennas.

Theorem 1: Let $n{\geq}m$ with $\mathbf{\Lambda}_n{=}\mathbf{I}$ and with $\mathbf{\Lambda}_m{\neq}\mathbf{I}$ where $\theta_1, \ldots, \theta_m$ are its m distinct eigenvalues. The marginal density distribution of an unordered eigenvalue of $\mathbf{S}$ is

$$f_{\lambda}(\xi) = L\sum_{i=1}^{m}\sum_{j=1}^{m}\xi^{n-m+j-1}e^{-\xi/\theta_i}\mathcal{D}(i,j) \tag{4}$$

with

$$L = \frac{1}{m\prod_{\ell=1}^{m}(n-\ell)!\det(\mathbf{\Lambda}_m)^n\prod_{k<\ell}^{m}(\frac{1}{\theta_\ell}-\frac{1}{\theta_k})}$$

and with $\mathcal{D}(i,j)$ the (i,j)-cofactor of an $(m{\times}m)$ matrix whose (ℓ,k)-th entry is

$$\frac{(n-m+k-1)!}{\theta_\ell^{-n+m-k}}$$

Proof: See Appendix.

The above distribution can be illustrated using a typical correlation structure encountered in wide-area mobile communication systems deployed in rural or suburban areas.

Example 1: Consider a mobile terminal radiating from $n_{\mathrm{T}}{=}4$ transmit antennas with $\mathbf{\Theta}_{\mathrm{T}}{=}\mathbf{I}$. Then, let the receiving base station have $n_{\mathrm{R}}{=}2$ antennas with

$$(\mathbf{\Theta}_{\mathrm{R}})_{i,j} = e^{-0.05\,d^2(i-j)^2} \tag{5}$$

which corresponds to a d-wavelength antenna separation and a broadside (truncated) Gaussian power azimuth spectrum with 2° root-mean-square spread. For $d{=}2$, the marginal density distribution of an arbitrary eigenvalue of $\mathbf{S}{=}\mathbf{H\Phi H}^{\dagger}$, given by (4), is depicted in Fig 1. Also shown is the histogram of 10000 random realizations obtained via Montecarlo, which closely matches the analytical function.

Next, we turn our attention to the scenario where the correlation is at the end of the link with the most antennas.

Theorem 2: Let $n{>}m$ with $\mathbf{\Lambda}_m{=}\mathbf{I}$ and with $\mathbf{\Lambda}_n{\neq}\mathbf{I}$ where $\theta_1, \ldots, \theta_n$ are its n distinct eigenvalues. The marginal density

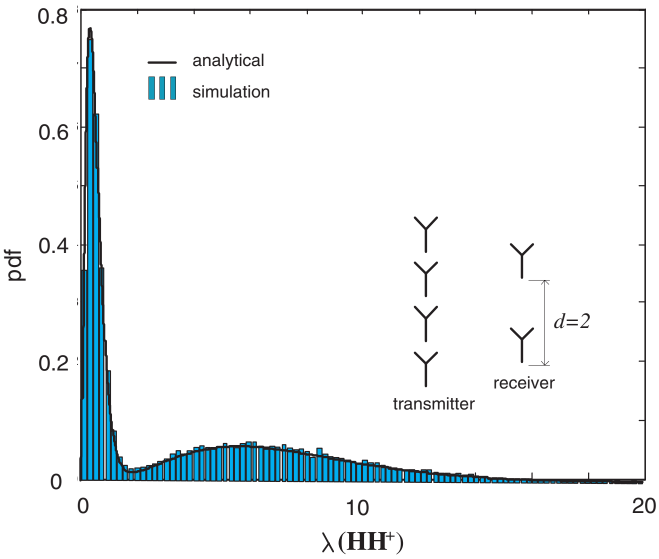

Fig. 1. Marginal density distribution of an unordered eigenvalue of $\mathbf{S}$ with n_{T}=4, n_{R}=2, $\mathbf{\Lambda}_{\mathrm{T}}$=$\mathbf{I}$ and with $\mathbf{\Lambda}_{\mathrm{R}}$ the diagonal eigenvalue matrix of $\mathbf{\Theta}_{\mathrm{R}}$ in (5).

distribution of an unordered eigenvalue of $\mathbf{S}$ is

$$f_\lambda(\xi) = L \sum_{i=1}^{m} \sum_{j=1}^{m} \xi^{j-1} \mathcal{D}(i,j) \Big(\theta_{n-m+i}^{n-m-1} e^{-\xi/\theta_{n-m+i}} - \sum_{\ell=1}^{n-m} \sum_{k=1}^{n-m} (\mathbf{\Psi}^{-1})_{k,\ell}\, \theta_{n-m+i}^{k-1} \theta_{\ell}^{n-m-1} e^{-\xi/\theta_\ell} \Big)$$

with

$$L = \frac{\det(\mathbf{\Psi})}{m \prod_{k<\ell}^{n} (\theta_\ell - \theta_k) \prod_{\ell=1}^{m-1} \ell!} \tag{6}$$

where $\mathbf{\Psi}$ is the $(n-m)\times(n-m)$ Vandermonde matrix [19]

$$\mathbf{\Psi} = \begin{bmatrix} 1 & \theta_1 & \dots & \theta_1^{n-m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \theta_{n-m} & \dots & \theta_{n-m}^{n-m-1} \end{bmatrix}$$

while $\mathcal{D}(i,j)$ is the (i,j)-cofactor of the $(m\times m)$ matrix whose (ℓ,k)-th entry equals

$$(k-1)! \left(\theta_{n-m+\ell}^{n-m+k-1} - \sum_{p=1}^{n-m} \sum_{q=1}^{n-m} (\mathbf{\Psi}^{-1})_{p,q} \theta_{n-m+\ell}^{p-1} \theta_q^{n-m+k-1} \right)$$

Proof: See Appendix.

IV. Characterization of the Capacity

First, we evaluate the capacity with correlation at the most constrained end of the link in terms of spatial degrees of freedom.

Theorem 3: Consider a Rayleigh-faded channel with correlation at the end of link with the fewest antennas. Denoting by $\theta_1, \dots, \theta_m$ the m distinct eigenvalues of the corresponding correlation matrix, the capacity is given by

$$C(\mathsf{SNR}) = \frac{m\,L}{\log_e 2} \sum_{j=1}^{m} \sum_{i=1}^{m} \frac{\mathcal{D}(i,j)}{\theta_i^{-n+m-j}} \left(\mathcal{R}_{i,j} + \mathcal{T}_{i,j}\right) \tag{7}$$

with

$$\mathcal{R}_{i,j} = \delta! \left(\log_e \frac{\mathsf{SNR}\,\theta_i}{m} + \sum_{u=1}^{\delta} \frac{1}{u} - \gamma \right) + \sum_{k=1}^{\delta} \frac{(-1)^{k+1} (\delta-k)!}{k \left(\mathsf{SNR}\,\theta_i/m\right)^k}$$

$$\mathcal{T}_{i,j} = (-1)^{\delta} \sum_{k=1}^{\infty} \frac{\log_e \frac{\mathsf{SNR}\,\theta_i}{m} + \sum_{u=1}^{k-1} \frac{1}{u} - \gamma + \frac{1}{k+\delta}}{(\delta+k) \left(\mathsf{SNR}\,\theta_i/m\right)^{\delta+k} (k-1)!}$$

where L and $\mathcal{D}(i,j)$ are as in Theorem 1 while δ=$n-m+j-1$ and $\gamma\approx 0.5772$ is the Euler-Mascheroni constant.

Proof: See Appendix.

Before proceeding with the converse scenario, we proceed to exemplify Theorem 3.

Example 2: Consider the same scenario of Example 1. The capacity for various antenna separations, d, is shown in Fig. 2 alongside corresponding Montecarlo simulations. (The case d=10, in particular, results in the eigenvalues of $\mathbf{\Theta}_{\mathrm{R}}$ being 1.01 and 0.99, which can be seen as a perturbation of $\mathbf{\Theta}_{\mathrm{R}}$=$\mathbf{I}$.) In every case, the agreement is excellent.

Theorem 4: Consider a Rayleigh-faded channel with correlation at the end of link with the most antennas. Denoting by $\theta_1, \dots, \theta_n$ the n distinct eigenvalues of the corresponding correlation matrix, the capacity is given by

$$C = \frac{m\,L}{\log_e 2} \sum_{i=1}^{m} \sum_{j=1}^{m} \mathcal{D}(i,j) \Big(\theta_{\tau+i}^{\tau+j-1} \left(\mathcal{R}_{\tau+i,j-\tau} + \mathcal{T}_{\tau+i,j-\tau}\right) - \sum_{\ell=1}^{\tau} \sum_{k=1}^{\tau} (\mathbf{\Psi}^{-1})_{k,\ell}\, \theta_{\tau+i}^{k-1} \theta_\ell^{\tau+j-1} \left(\mathcal{R}_{\ell,j-\tau} + \mathcal{T}_{\ell,j-\tau}\right) \Big)$$

where τ=$n-m$, L is as in Theorem 2, and $\mathcal{R}_{p,q}$ and $\mathcal{T}_{p,q}$ are as in Theorem 3.

Proof: See Appendix.

Example 3: Consider a mobile terminal radiating from n_{T} transmit antennas whose correlation is

$$(\mathbf{\Theta}_{\mathrm{T}})_{i,j} = J_0(\pi |i-j|^2) \tag{8}$$

where $J_0(\cdot)$ is the zero-order Bessel function of the first kind. This corresponds to a half-wavelength antenna spacing and a uniform power azimuth spectrum, representative of a dense scattering environment. Let the receiving base station have n_{R}=2 antennas, sufficiently separated such that $\mathbf{\Theta}_{\mathrm{R}}$=$\mathbf{I}$. The capacity for n_{T}=3 and n_{T}=4 is shown in Fig. 3 along with corresponding Montecarlo simulations.

V. Conclusions

Closed-form expressions for the marginal distribution of the unordered eigenvalues of $\mathbf{H\Phi H}^\dagger$ with one-sided correlation have been presented and, from this distribution, the corresponding capacity has been analytically characterized.

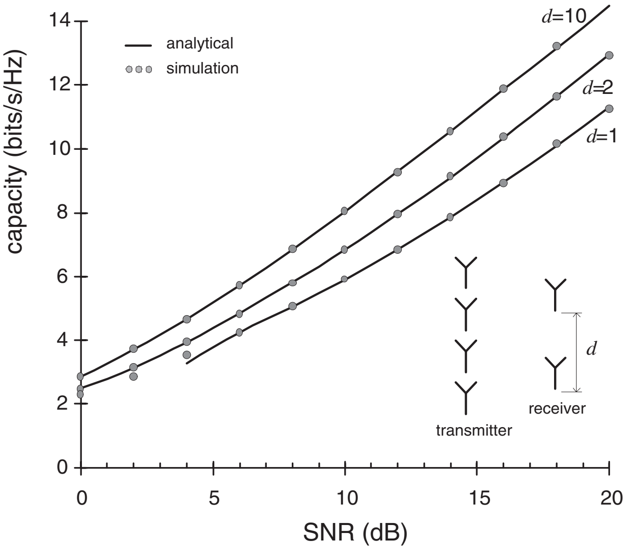

Fig. 2. $C(\mathsf{SNR})$ for n_{T}=4 and n_{R}=2 with $\mathbf{\Theta}_{\mathrm{T}}$=$\mathbf{I}$ and $\mathbf{\Theta}_{\mathrm{R}}$ as in (5).

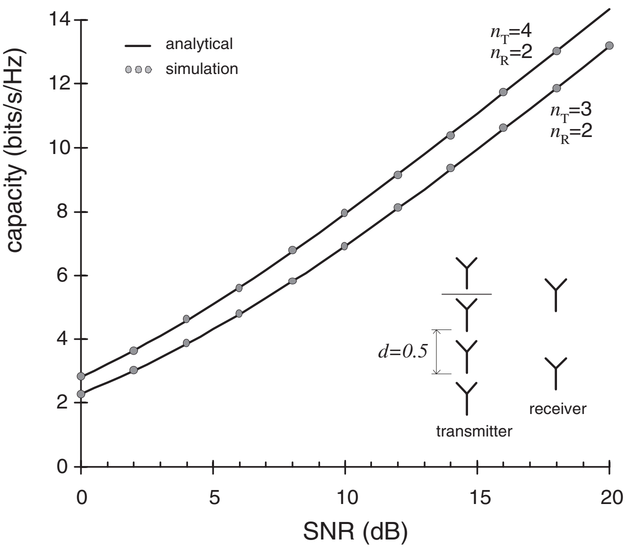

Fig. 3. $C(\mathsf{SNR})$ for n_{T}=3,4 and n_{R}=2 with $\mathbf{\Theta}_{\mathrm{T}}$ as in (8) and $\mathbf{\Theta}_{\mathrm{R}}$=$\mathbf{I}$.

It must be noted that, while the evaluation of the capacity expressions is in general straightforward, it becomes numerically demanding at low signal-to-noise ratios, especially if some of the eigenvalues of the correlation matrix are close to zero. This difficulty can be appreciated in Fig. 2, where for d=1 the eigenvalue spread of $\mathbf{\Theta}_{\mathrm{R}}$ becomes $\lambda_{\mathsf{max}}(\mathbf{\Theta}_{\mathrm{R}})/\lambda_{\mathsf{min}}(\mathbf{\Theta}_{\mathrm{R}})\approx 40$ and the lack of sufficient numerical precision precluded obtaining the capacity below about 4 dB.

Appendix

We evaluate (2) via

$$C(\mathsf{SNR}) = m\,\mathcal{M}^{-1}\left(\frac{\Gamma(s)\Gamma(1-s)\mathcal{M}_{f_\lambda}(1-s)}{s\left(\frac{\mathsf{SNR}}{m}\right)^s}\right) \qquad (9)$$

where $f_\lambda(\cdot)$ is the density distribution of a generic eigenvalue of $\mathbf{S}$ and $\mathcal{M}_{f_\lambda}(\cdot)$ is its Mellin transform

$$\mathcal{M}_{f_\lambda}(s) = \int_0^\infty \xi^{s-1} f(\xi)\, d\xi$$

with $\mathcal{M}^{-1}(\cdot)$ denoting the inverse Mellin Transform and $\Gamma(\cdot)$ the standard Gamma function [20].

A. Proof of Theorems 1 and 3

For $\mathbf{\Lambda}_n$=$\mathbf{I}_n$, the distribution of $\mathbf{S}$ in (3) leads to the following expression for the joint distribution of the ordered eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m$ [16]:

$$f_{\mathbf{\Lambda}}(\mathbf{\Xi}) = \frac{\pi^{m(m-1)}\prod_{k=1}^{m}\xi_k^{n-m}\prod_{k<\ell}^{m}(\xi_k-\xi_\ell)^2\,{}_0\boldsymbol{F}_0(-\mathbf{\Upsilon}^{-1},\mathbf{\Xi})}{\Gamma_m(n)\Gamma_m(m)\det(\mathbf{\Lambda}_m)^n} \qquad (10)$$

with $\mathbf{\Lambda}$=diag$\{\lambda_1,\ldots,\lambda_m\}$, $\mathbf{\Xi}$=diag$\{\xi_1,\ldots,\xi_m\}$, and with $\mathbf{\Upsilon}^{-1}$=diag$\{\frac{1}{\theta_1},\ldots,\frac{1}{\theta_m}\}$ containing the reciprocals of the eigenvalues in $\mathbf{\Lambda}_m$. The joint distribution of the unordered eigenvalues is obtained dividing (10) by $m!$

When $\theta_1,\ldots,\theta_m$ are distinct, we can express the hypergeometric function in (10) as [21]

$${}_0\boldsymbol{F}_0(-\mathbf{\Upsilon}^{-1},\mathbf{\Xi}) = \frac{\Gamma_m(m)\det(\mathbf{B})}{\pi^{m(m-1)/2}\prod_{k<\ell}^{m}(\xi_k-\xi_\ell)\prod_{k<\ell}^{m}(\frac{1}{\theta_\ell}-\frac{1}{\theta_k})}$$

with $(\mathbf{B})_{i,j}$=$e^{-\xi_i/\theta_j}$. Substituting in (10),

$$f_{\mathbf{\Lambda}}(\mathbf{\Xi}) = \frac{\prod_{k=1}^{m}\xi_k^{n-m}\prod_{k<\ell}^{m}(\xi_k-\xi_\ell)\det(\mathbf{B})}{\prod_{\ell=1}^{m}(n-\ell)!\det(\mathbf{\Lambda}_m)^n\prod_{k<\ell}^{m}(\frac{1}{\theta_\ell}-\frac{1}{\theta_k})}.$$

Recall here that $\prod_{k<\ell}^{m}(\xi_k-\xi_\ell)$ is the determinant of a matrix whose (i,j)-th entry is ξ_i^{n-j}. To obtain a handy expression for the joint p.d.f of the unordered eigenvalues, we expand this determinant as well as $\det(\mathbf{B})$ via Laplace expansion and then integrate the result with respect to $m-1$ eigenvalues by virtue of [22],[8, Corollary 1] obtaining the distribution of a single arbitrary eigenvalue

$$f_\lambda(\xi) = L\sum_{i=1}^{m}\sum_{j=1}^{m}\xi^{n-m+j-1}e^{-\xi/\theta_i}\mathcal{D}(i,j)$$

as claimed in Theorem 1. Its Mellin transform is

$$\mathcal{M}_{f_\lambda}(s) = L\sum_{i=1}^{m}\sum_{j=1}^{m}\mathcal{D}(i,j)\frac{\Gamma(s+n-m+j-1)}{\theta_i^{m-s-n-j+1}}$$

The capacity is obtained from (9), which in this case yields[1]

$$C = m\,L\sum_{k=1}^{m}\sum_{\ell=1}^{m}\frac{\mathcal{D}(k,\ell)}{2\mathrm{j}\pi\omega_k^\delta}\int_{a-\mathrm{j}\infty}^{a+\mathrm{j}\infty}\frac{\Gamma(s)\Gamma(1-s)\Gamma(\delta+1-s)}{s\,(\theta_k\frac{\mathsf{SNR}}{m})^s}ds$$

[1]Caution: here j denotes the imaginary unit, j=$\sqrt{-1}$.

where $-1<a<0$ is a real constant while $\delta=n-m-\ell-1$. Since the integral is as in [17, Eq. 35], it can be calculated using the fact that the function $(\rho s)^{-1}\Gamma(s)\Gamma(1-s)\Gamma(\delta-s+1)$ admits [17, Eq. 36–38]:

- Double poles at $s=\delta+k$ with $k=0,1,\dots$ with residues

$$(-1)^{\delta}\frac{\log_e\rho+\sum_{u=1}^{k}\frac{1}{u}-\gamma+\frac{1}{\delta+k+1}}{(\delta+k+1)\rho^{\delta+k+1}k!} \tag{11}$$

- Simple poles at $s=k$ with $k=1,\dots\delta-1$ with residues

$$(-1)^{k+1}\frac{(\delta-k)!}{k\,\rho^{k}} \tag{12}$$

- Double pole at $s=0$ with residue

$$\delta!\left(\log_e\rho+\sum_{u=1}^{\delta}\frac{1}{u}-\gamma\right) \tag{13}$$

from which (7) follows.

B. Proof of Theorems 2 and 4

For $\mathbf{\Lambda}_n\neq\mathbf{I}_n$ and arbitrary $\mathbf{\Lambda}_m$, the joint distribution of the entries of $\mathbf{S}$ is given in [23, Eq. (3)] leading, for $\mathbf{\Lambda}_m=\mathbf{I}$, to the joint distribution of the unordered eigenvalues of $\mathbf{S}$

$$f_{\mathbf{\Lambda}}(\mathbf{\Xi})=\frac{\prod_{k<\ell}^{m}(\xi_\ell-\xi_k)\det(\mathbf{\Delta}_n)}{\prod_{\ell=1}^{m}\ell!\prod_{k<\ell}^{n}(\theta_\ell-\theta_k)} \tag{14}$$

with

$$\mathbf{\Delta}_n=\begin{pmatrix} 1\ \dots\theta_1^{\tau-1} & \theta_1^{\tau-1}e^{-\frac{\xi_1}{\theta_1}} & \dots\theta_1^{\tau-1}e^{-\frac{\xi_m}{\theta_1}} \\ 1\dots\theta_2^{\tau-1} & \theta_2^{\tau-1}e^{-\frac{\xi_1}{\theta_2}} & \dots\theta_2^{\tau-1}e^{-\frac{\xi_m}{\theta_2}} \\ \vdots\ \vdots\ \ \vdots & \vdots & \vdots\qquad\vdots \\ 1\dots\theta_n^{\tau-1} & \theta_n^{\tau-1}e^{-\frac{\xi_1}{\theta_n}} & \dots\ \theta_n^{\tau-1}e^{-\frac{\xi_m}{\theta_n}} \end{pmatrix}$$

where $\tau=n-m$. In order to apply [8, Corollary 1] we have to express the determinants in the numerator of (14) as the product of two determinants of matrices of the same dimension m. To this end, we first partition Δ_n as

$$\mathbf{\Delta}_n=\begin{bmatrix}\mathbf{\Psi} & \mathbf{A}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} \tag{15}$$

where $\mathbf{\Psi}$, defined in Theorem 2, is the principal $(n-m)$ minor of $\mathbf{\Delta}_n$. Applying the Inversion Lemma for block matrices [19],

$$\det(\mathbf{\Delta_n})=\det(\mathbf{\Psi})\det(\mathbf{D}-\mathbf{C}\mathbf{\Psi}^{-1}\mathbf{A})$$

where the (i,j)-th entry of $(\mathbf{D}-\mathbf{C}\mathbf{\Psi}^{-1}\mathbf{A})$ is given by

$$\theta_{\tau+i}^{\tau-1}e^{-\xi_j/\theta_{\tau+i}}-\sum_{\ell=1}^{\tau}\sum_{k=1}^{\tau}(\mathbf{\Psi}^{-1})_{k,\ell}\theta_{\tau+i}^{k-1}\theta_\ell^{\tau-1}e^{-\xi_j/\theta_\ell}$$

Integrating over m-1 eigenvalues and applying [8, Corollary 1] to the product of the two determinants $\det(\mathbf{D}-\mathbf{C}\mathbf{\Psi}^{-1}\mathbf{A})$ and $\prod_{k<\ell}^{m}(\xi_\ell-\xi_k)$, we obtain the claim of Theorem 2. The Mellin transform of (6), $\mathcal{M}_{\mathsf{f}_\lambda}(s)$, is given by

$$L\sum_{j=1}^{m}\sum_{i=1}^{m}\mathcal{D}(i,j)\left(\theta_{n-m+i}^{n-m+s+j-2}\Gamma(s+j-1)\right.$$
$$\left.-\sum_{\ell=1}^{n-m}\sum_{k=1}^{n-m}(\mathbf{\Psi}^{-1})_{k,l}\theta_\ell^{n-m+s+j-2}\theta_{n-m+i}^{k-1}\Gamma(s+j-1)\right)$$

with L and $\mathcal{D}(i,j)$ as in Theorem 2. This Mellin transform leads again to an integral of the form in [17, Eq. 35], which can be solved using (11–13) to obtain the claimed capacity.

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