Capacity of Multi-Antenna Channels in the Low-Power Regime

Antonia M. Tulino
Università degli Studi del Sannio,
82100 Benevento, Italy.
e-mail: `atulino@ee.princeton.edu`

Angel Lozano
Bell Laboratories,
Holmdel, NJ 07733, USA.
e-mail: `aloz@lucent.com`

Sergio Verdú
Princeton University,
Princeton, NJ 08540, USA.
e-mail: `verdu@ee.princeton.edu`

I. INTRODUCTION

Most studies of the capacity of multi-antenna architectures have thus far been restricted to a highly idealized canonical channel (uncorrelated zero-mean transfer coefficients.) Even with such simplified model, the capacity emerges as a formidable expression [1]. More insightful expressions have been obtained only asymptotically in the number of antennas [1, 2, 3]. Beyond this canonical model, the capacity of more realistic channels has been studied through simulation. Asymptotic analysis has also been applied [4], but the solutions obtained provide little insight. Since both the finite and the asymptotic capacities become particularly revealing in the high-power regime, insights have been drawn mostly in that realm.

In emerging mobile systems [5], however, users must operate very often in the low-power regime. Specifically, almost 40% of geographical locations experience signal-to-noise ratios (SNR) below 0 dB. Despite its relevance, the multi-antenna low-power regime had not been analyzed in depth until [6], where the figure of merit is not the SNR, but rather the normalized energy per information bit, $\frac{E_b}{N_0}$. Denoting by $\mathsf{C}(\frac{E_b}{N_0})$ the capacity, any system with power P, desired rate R (bits/s), and bandwidth B, must respect the fundamental limit

$$\frac{R}{B} \le \mathsf{C}\left(\frac{P}{RN_0}\right). \qquad (1)$$

As shown in [6], the key performance measures in the low-SNR regime are $\frac{E_b}{N_0}_{\min}$ and S_0 such that

$$\mathsf{C}(\tfrac{E_b}{N_0}) \approx \frac{S_0}{3\,\mathrm{dB}}\left(\frac{E_b}{N_0}|_{\mathrm{dB}} - \frac{E_b}{N_0}_{\min}|_{\mathrm{dB}}\right) \qquad (2)$$

Hence, $\frac{E_b}{N_0}_{\min}$ is the minimum energy per information bit required to convey information reliably while S_0 represents the capacity slope therein in bits/s/Hz/(3 dB).

Eq. (2) indicates that, in the low-SNR regime, the behavior of $\mathsf{C}\frac{E_b}{N_0}$ is highly linear. Fig. 1 reproduces the exact capacity along with its linear approximation for single-antenna and 4-antenna independently fading Rayleigh channels impaired by additive white Gaussian noise (AWGN). The linear approximation [6] is tight for rather ambitious levels of $\frac{E_b}{N_0}$.

This paper expands the findings of [6] using a channel model that realistically describes the conditions found in typical wireless systems. The focus is on channels that are known to the receiver, but unknown to the transmitter.

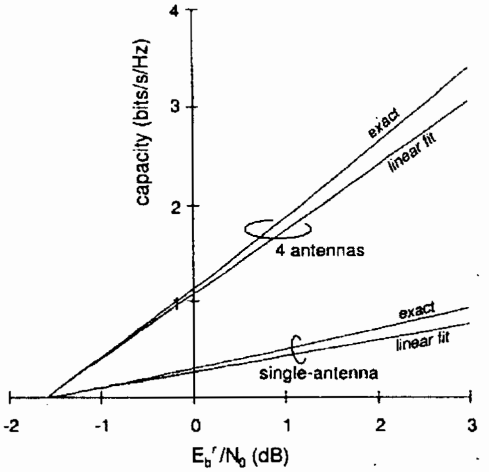

Figure 1: Comparison between the exact capacity as function of received $\frac{E_b}{N_0}$ and its low-SNR linear approximation for single-antenna and 4-antenna architectures.

II. DEFINITIONS

With n_{T} transmit and n_{R} receive antennas, a multi-antenna channel model with frequency-flat fading is

$$\mathbf{y} = \sqrt{\mathsf{g}}\,\mathbf{H}\,\mathbf{x} + \mathbf{n} \qquad (3)$$

where $\mathbf{x}$ is the n_{T}-dimensional transmit signal while $\mathbf{y}$ and $\mathbf{n}$ are the received signal and AWGN, both n_{R}-dimensional. The channel is represented by the $(n_{\mathrm{R}} \times n_{\mathrm{T}})$ random matrix $\sqrt{\mathsf{g}}\,\mathbf{H}$. The scalar $\sqrt{\mathsf{g}}$ has been factored out so as to yield a normalized matrix $\mathbf{H}$, the second-order moment of whose entries is unity. The single-sided spectral density of the noise is

$$N_0 = \frac{E[\|\mathbf{n}\|^2]}{n_{\mathrm{R}}}. \qquad (4)$$

When $n_{\mathrm{R}} = n_{\mathrm{T}}$ or when an equation applies to both transmitter and receiver, we use n to refer to the number of antennas generically.

The received SNR is given by

$$\mathsf{SNR} = \mathsf{g}\frac{E[\|\mathbf{H}\mathbf{x}\|^2]}{E[\|\mathbf{n}\|^2]} \qquad (5)$$

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whereas the normalized *transmitted* energy per bit is

$$\frac{E_b}{N_0} = \frac{E[\|\mathbf{x}\|^2]}{N_0(R/B)}. \tag{6}$$

Since the largest rate per unit bandwidth achievable with arbitrary reliability is precisely the capacity $C(\textsf{SNR})$, the minimum required transmitted energy per bit is

$$\frac{E_b}{N_0} = \frac{E[\|\mathbf{x}\|^2]}{N_0\, C(\textsf{SNR})}. \tag{7}$$

In addition to $\frac{E_b}{N_0}$, it is also useful to evaluate the corresponding *received* energy per bit:

$$\frac{E_b^r}{N_0} = n_\text{R} \frac{\textsf{SNR}}{C(\textsf{SNR})}. \tag{8}$$

III. CHANNEL MODEL

Since the fading tends to be either Rayleigh or Ricean, the entries of $\mathbf{H}$ can be modelled as complex Gaussian. With that, the characterization of $\mathbf{H}$ entails simply determining the mean and correlation between its entries.

A Rayleigh Channel

In the Rayleigh case, the mean of the elements of $\mathbf{H}$ is zero. Their correlation, in turn, can be obtained—in most cases—from the local correlation between transmit antennas and the local correlation between receive antennas, separately.[1] This separation turns the problem of determining the correlation between entries of $\mathbf{H}$ into the much more conventional problem of determining the correlation between antennas within each array.

The correlation coefficients between the n_T transmit antennas can be assembled into an $(n_\text{T} \times n_\text{T})$ matrix $\mathbf{\Theta}_\text{T}$ while the correlation coefficients between the n_R receive antennas can be assembled into a corresponding $(n_\text{R} \times n_\text{R})$ matrix $\mathbf{\Theta}_\text{R}$. The diagonal elements of $\mathbf{\Theta}_\text{T}$ and $\mathbf{\Theta}_\text{R}$ are, by definition, equal to 1. These Hermitian matrices can be used to generate properly correlated channels $\mathbf{H}$ via

$$\mathbf{H} = \mathbf{\Theta}_\text{R}^{1/2} \mathbf{H}_w \mathbf{\Theta}_\text{T}^{1/2} \tag{9}$$

where $\mathbf{H}_w$ is composed of independent unit-variance complex Gaussian random entries [7].

B Ricean Channel

The above model can be made Ricean by incorporating an additional deterministic term containing unit-magnitude entries. Such term is typically associated with a line-of-sight or diffracted component and thus

$$\mathbf{H} = \sqrt{\frac{1}{\mathsf{K}+1}} \mathbf{\Theta}_\mathsf{R}^{1/2} \mathbf{H}_w \mathbf{\Theta}_\mathsf{T}^{1/2} + \sqrt{\frac{\mathsf{K}}{\mathsf{K}+1}} \mathbf{a}_\text{R} \mathbf{a}_\text{T}^\dagger. \tag{10}$$

with the K-factor quantifying the ratio between the deterministic (cohesive) and the random (scattered) energies. The vectors $\mathbf{a}_\text{T}$ and $\mathbf{a}_\text{R}$, with norms n_T and n_R respectively, are the transmit and receive array responses to a plane wave.

C Properties

Let us define the *correlation number* associated with a n-dimensional correlation matrix $\mathbf{\Theta}$ as

$$\zeta(\mathbf{\Theta}) = \frac{\text{Tr}\{\mathbf{\Theta}^2\}}{n}. \tag{11}$$

Property 1. The correlation number is bounded by

$$1 \leq \zeta(\mathbf{\Theta}) \leq n \tag{12}$$

with the lower bound achieved if and only if the antennas are uncorrelated and the upper bound achieved if the antennas are fully correlated.

Property 2. In arrays whose geometry follows a regular uniform pattern (such as linear or circular), the correlation matrix is Toeplitz. If the number of antennas in such an array is increased from n_A to n_B while preserving the same structure, it can be verified that

$$\zeta(\mathbf{\Theta}_\text{B}) \geq \zeta(\mathbf{\Theta}_\text{A}) \tag{13}$$

with equality if and only if all antennas are perfectly uncorrelated.

IV. CAPACITY IN THE LOW-SNR REGIME

When $\mathbf{H}$ is unknown to the transmitter, the capacity is given by

$$C(\textsf{SNR}) = E[\log_2 \det(\mathbf{I} + \frac{\textsf{SNR}}{n_\text{T}} \mathbf{H}\mathbf{H}^\dagger)]. \tag{14}$$

Notice that our fading model is purposely frequency-flat. In frequency-selective environments, the channel can be decomposed into parallel non-interacting subchannels, each experiencing frequency-flat fading and having the same capacity as the overall channel.

From $C(\textsf{SNR})$, the capacity as a function of $\frac{E_b}{N_0}$ can be obtained through

$$\mathsf{C}(\tfrac{E_b}{N_0}) = C(\textsf{SNR}) \tag{15}$$

with $\textsf{SNR}$ the solution to

$$\frac{\textsf{SNR}}{C(\textsf{SNR})} = \mathsf{g}\frac{E_b}{N_0}. \tag{16}$$

Unfortunately, an explicit expression for $\mathsf{C}(\frac{E_b}{N_0})$ cannot be obtained. Recall, however, that in the low-SNR regime it is possible to approximate its logarithmic behavior very closely using (2). In linear scale, (2) becomes

$$\mathsf{C}(\tfrac{E_b}{N_0}) = S_0 \log_2 \left(\frac{\frac{E_b}{N_0}}{\frac{E_b}{N_0}_\text{min}} \right). \tag{17}$$

Consequently, characterizing $\mathsf{C}(\frac{E_b}{N_0})$ in the low-SNR regime requires simply that we obtain expressions for two parameters. These can be computed from the first and second derivatives of $C(\textsf{SNR})$ at $\textsf{SNR}=0$ to yield [6]

$$\frac{E_b}{N_0}_\text{min} = \frac{n_\text{T} \log_e 2}{\mathsf{g}\, E[\text{Tr}\{\mathbf{H}\mathbf{H}^\dagger\}]} \tag{18}$$

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[1] This separation applies if the immediate surroundings to each array are responsible for the correlation therein but have no impact on the correlation between the antennas at the other end of the link.

and

$$S_0 = 2\frac{E^2[\mathrm{Tr}\{\mathbf{H}\mathbf{H}^\dagger\}]}{E[\mathrm{Tr}\{(\mathbf{H}\mathbf{H}^\dagger)^2\}]}. \tag{19}$$

In the canonical case (Rayleigh uncorrelated channel), these expressions particularize to

$$\frac{E_b}{N_0}_{\min} = \frac{\log_e 2}{\mathrm{g}}\frac{1}{n_\mathrm{R}} \qquad S_0 = 2\frac{n_\mathrm{T} n_\mathrm{R}}{n_\mathrm{T}+n_\mathrm{R}} \tag{20}$$

from which we observe:

- The $\frac{E_b}{N_0}_{\min}$ depends on n_R, but not on n_T.
- S_0 is symmetric with respect to n_T and n_R.
- The slope in the low-SNR regime is at least as large as the slope at high SNR [1], given by $\min(n_\mathrm{T}, n_\mathrm{R})$, with equality only if $n_\mathrm{T}=n_\mathrm{R}$.
- The received minimum energy per bit is approximately -1.59 dB. More precisely, $\frac{E_b^r}{N_0}_{\min} = \log_e 2$, which is a fundamental property of Gaussian noise.

Using the more elaborate channel model presented in Section III, we obtain the following central result:

Proposition 1. *Consider a correlated Ricean channel known by the receiver and given by*

$$\mathbf{H} = \sqrt{\frac{1}{\mathsf{K}+1}}\boldsymbol{\Theta}_\mathrm{R}^{1/2}\mathbf{H}_w\boldsymbol{\Theta}_\mathrm{T}^{1/2} + \sqrt{\frac{\mathsf{K}}{\mathsf{K}+1}}\mathbf{a}_\mathrm{R}\mathbf{a}_\mathrm{T}^\dagger. \tag{21}$$

When $\mathbf{H}$ *is unknown to the transmitter,* $\frac{E_b}{N_0}_{\min}$ *and* S_0 *are given by*

$$\frac{E_b}{N_0}_{\min} = \frac{\log_e 2}{\mathrm{g}}\frac{1}{n_\mathrm{R}} \tag{22}$$

and by (23) at the bottom of the page.

Remarkably, the $\frac{E_b}{N_0}_{\min}$ is unaffected by the existence of antenna correlation and a Ricean term. Only S_0 reflects the structure of the channel. In the remainder, we study this slope in detail.

V. RAYLEIGH FADING

When $\mathsf{K}=0$, the slope particularizes to

$$S_0 = \frac{2\,n_\mathrm{T} n_\mathrm{R}}{n_\mathrm{T}\,\zeta(\boldsymbol{\Theta}_\mathrm{R}) + n_\mathrm{R}\,\zeta(\boldsymbol{\Theta}_\mathrm{T})} \tag{24}$$

where the effects of antenna correlation appear only through the correlation numbers of the transmit and receive arrays. Thus, a single scalar parameter uniquely quantifies the capacity impact of an entire correlation matrix.

Using property 1,

$$1 \le S_0 \le \frac{2\,n_\mathrm{T} n_\mathrm{R}}{n_\mathrm{T}+n_\mathrm{R}} \tag{25}$$

confirming that antenna correlation can only diminish the capacity. Although a first-order analysis of $C(\mathrm{SNR})$ would seem to indicate that the low-SNR capacity is unaffected by correlation [4], this is only true as far as the invariance of $\frac{E_b}{N_0}_{\min}$. At any SNR>0, antenna correlation *does* reduce capacity, but such reduction is not revealed by a first-order analysis.

Fixing rate and power, the bandwidth B required with correlation matrices $\boldsymbol{\Theta}_\mathrm{R}$ and $\boldsymbol{\Theta}_\mathrm{T}$ relative to the canonical bandwidth B_0 in the absence of correlation is

$$\frac{B}{B_0} = \frac{n_\mathrm{T}\,\zeta(\boldsymbol{\Theta}_\mathrm{R}) + n_\mathrm{R}\,\zeta(\boldsymbol{\Theta}_\mathrm{T})}{n_\mathrm{T}+n_\mathrm{R}}. \tag{26}$$

Example 1. *Consider* $n_\mathrm{T}=2$ *and* $n_\mathrm{R}=1$. *The low-SNR capacity in a Rayleigh channel is given by*

$$\mathsf{C}\left(\tfrac{E_b}{N_0}\right) \approx \frac{4}{3+|\rho|^2}\log_2\left(\frac{\mathrm{g}}{\log_e 2}\frac{E_b}{N_0}\right) \tag{27}$$

with ρ *the transmit antenna correlation coefficient. Remarkably, correlation has a rather limited impact in this case: with full correlation, 75% of the canonical capacity can still be attained. The bandwidth expansion factor incurred because of such correlation is*

$$\frac{B}{B_0} = 1 + \frac{|\rho|^2}{3}. \tag{28}$$

Further insight on the capacity of correlated Rayleigh channels impaired by AWGN can be gathered from the slope in (24) and the fact that the $\frac{E_b}{N_0}_{\min}$ is unaffected:

Corollary 1. *If* $n_\mathrm{T}=n_\mathrm{R}$, *the slope particularizes to*

$$S_0 = \frac{2}{\zeta(\boldsymbol{\Theta}_\mathrm{R}) + \zeta(\boldsymbol{\Theta}_\mathrm{T})}\,n. \tag{29}$$

This would seem to indicate that the capacity still scales linearly with the number of antennas, as in the canonical channel, but with a reduced slope [4]. This is, however, not the case. Since $\zeta(\boldsymbol{\Theta}_\mathrm{T})$ and $\zeta(\boldsymbol{\Theta}_\mathrm{R})$ are functions of n, in the presence of correlation the capacity is no longer linear on n.

Corollary 2. *The low-SNR slope is equivalent to that of* $n_\mathrm{T}^{\mathrm{eq}}$ *and* $n_\mathrm{R}^{\mathrm{eq}}$ *uncorrelated antennas given by*

$$n_\mathrm{T}^{\mathrm{eq}} = \frac{n_\mathrm{T}}{\zeta(\boldsymbol{\Theta}_\mathrm{T})} \qquad n_\mathrm{R}^{\mathrm{eq}} = \frac{n_\mathrm{R}}{\zeta(\boldsymbol{\Theta}_\mathrm{R})}. \tag{30}$$

$$S_0 = \frac{2\,n_\mathrm{T} n_\mathrm{R}\,(\mathsf{K}+1)^2}{n_\mathrm{T}\,\zeta(\boldsymbol{\Theta}_\mathrm{R}) + n_\mathrm{R}\,\zeta(\boldsymbol{\Theta}_\mathrm{T}) + \mathsf{K}^2 n_\mathrm{T} n_\mathrm{R} + 2\mathsf{K}\left(n_\mathrm{T}\frac{\mathrm{Tr}\{\mathbf{a}_\mathrm{R}\mathbf{a}_\mathrm{R}^\dagger\boldsymbol{\Theta}_\mathrm{R}\}}{n_\mathrm{R}} + n_\mathrm{R}\frac{\mathrm{Tr}\{\mathbf{a}_\mathrm{T}\mathbf{a}_\mathrm{T}^\dagger\boldsymbol{\Theta}_\mathrm{T}\}}{n_\mathrm{T}}\right)} \tag{23}$$

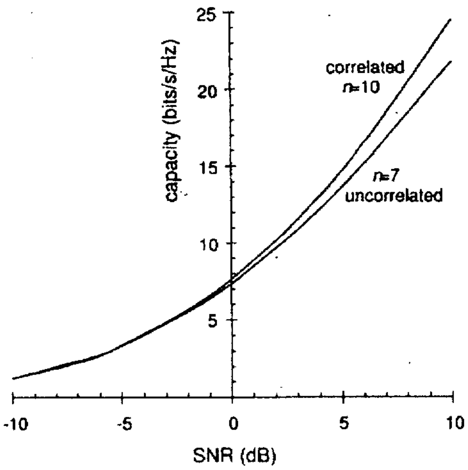

Figure 2: Capacity with $n_T = n_R = 10$ correlated antennas vs. capacity with $n_T^{eq} = n_R^{eq} = 7$ uncorrelated antennas.

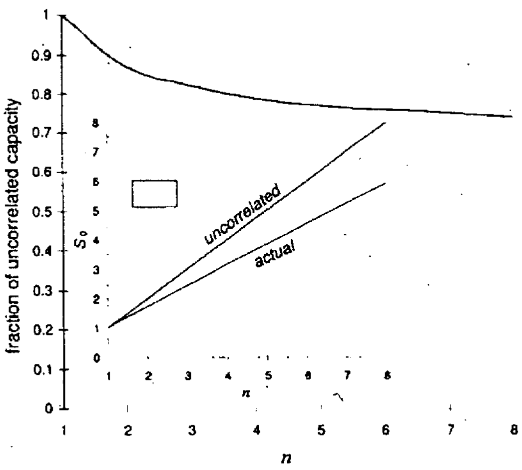

Figure 3: Fraction of uncorrelated capacity achievable as function of the number of antennas ($n_T = n_R$).

Example 2. *Consider a transmitting base station and a receiving terminal, both with uniform linear arrays with antenna spacing 4 and 0.5 wavelenths, respectively. Consider a broadside Gaussian power spectrum at the base with a 2° rms angular spread and a uniform spectrum over 360° at the terminal. The corresponding correlation numbers with $n=10$ are $\zeta(\Theta_T)=1.41$ and $\zeta(\Theta_R)=1.43$. Hence, the capacity ought to equal that of a canonical channel with $n_T^{eq}=7.1$ and $n_R^{eq}=6.98$. The capacities of the actual 10-antenna configuration and its 7-antenna canonical equivalent are displayed in Fig. 2. The equivalence remains very tight for a wide range of* SNR *levels.*

An additional result follows from (28) and property 3:

Corollary 3. *With Toeplitz correlation matrices, the bandwidth expansion factor increases monotonically with the number of antennas given fixed antenna spacing.*

Hence, even though the canonical capacity grows linearly with the number of antennas, only a diminishing fraction of such capacity can actually be attained.

Example 3. *Consider the same scenario of Example 2. As shown in Fig. 3, the attainable fraction of canonical capacity decreases from 100% with $n=1$ down to 74% with $n=8$. The comparison between the canonical and actual capacity slopes is displayed in the inset. While still healthy, the actual slope is mildly curved as opposed to linear in the number of antennas.*

VI. RICEAN FADING

In Ricean conditions, the deterministic component compounds to the effects of antenna correlation. Since the latter was studied in the previous section, we now evaluate the impact of the former by setting $\Theta_T = \mathbf{I}$ and $\Theta_R = \mathbf{I}$. From Proposition 1, the low-SNR slope becomes

$$\mathcal{S}_0 = \frac{2(\mathsf{K}+1)^2}{\mathsf{K}^2 + (2\mathsf{K}+1)\frac{n_T + n_R}{n_T n_R}}. \tag{31}$$

For large K, we observe that as $\mathsf{K}\to\infty$, $\mathcal{S}_0 \to 2$, which is the low-SNR slope of a scalar deterministic channel [6]. Together with (22), we can conclude that, in the presence of a strong Ricean component unknown to the transmitter, multiple transmit antennas are irrelevant and multiple receive antennas are only relevant in terms of $\frac{E_b}{N_0}_{\min}$. Interestingly, if (and only if) one or both of the arrays has a single antenna, the capacity is higher in strong Ricean channels than in uncorrelated Rayleigh conditions, even though the transmitter is unaware of the Ricean statistics. It can also be checked from (31) that:

Corollary 4. *With two uncorrelated antennas at both transmitter and receiver*

$$\mathsf{C}(\tfrac{E_b}{N_0}) \approx 2\log_2\left(\frac{2\,\mathsf{g}}{\log_e 2}\frac{E_b}{N_0}\right) \tag{32}$$

regardless of a possible unknown Ricean component.

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