

On the Capacity-Achieving Distribution of the Noncoherent Rician Fading Channel¹

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Abstract — **Transmission of information over a discrete-time memoryless Rician fading channel is considered. The noncoherent scenario, where neither the receiver nor the transmitter knows the fading coefficients, is assumed. If the input is subject to a peak-power constraint, it is shown that uniformly distributed phase is optimal and the capacity-achieving amplitude distribution is discrete with a finite number of mass points. It is also proven that if the input has only an average power constraint, the optimal input amplitude distribution has bounded support. Then, the Rician fading channel with uniform phase noise in the specular component is analyzed. It is shown that under an average power limitation, the optimal input amplitude distribution is discrete with a finite number of mass points.**

I. INTRODUCTION

With the recent development of powerful codes (e.g., Turbo codes, LDPC) that enabled us to operate very close to the Shannon capacity, information theoretic limits has gained practical relevance. Although these ultimate performance limits are well-studied for the additive white Gaussian noise (AWGN) channel, in depth analysis of the capacity and optimal input structure for noncoherent fading channels under practically related input and channel constraints has only been initiated recently. In [4], communication over a noncoherent discrete-time memoryless Rayleigh fading channel is considered and it is shown that under an average power limitation, the optimal amplitude distribution is discrete with a finite number of mass points. This shows that when the fading is not known, the optimal input has a markedly different structure than that of the unfaded AWGN channel where a Gaussian distribution is optimal. More recently, Katz and Shamai [5] considered the noncoherent AWGN channel and has shown that the optimal amplitude distribution is again discrete but with an infinite number of mass points. This result is extended to the block-independent case by Nuriyev and Anastasopoulos [6].

II. CHANNEL MODEL

We consider the following discrete-time memoryless Rician fading channel model

$$y_i = mx_i + a_i x_i + n_i \quad (1)$$

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where a_i and n_i are i.i.d. circular zero mean complex Gaussian random variables, independent of each other and of the input, with variances $E\{|a_i|^2\} = \gamma^2$ and $E\{|n_i|^2\} = N_0$, and m is a deterministic complex constant. x_i is the complex channel input and y_i is the complex channel output. $\{a_i\}$ and $\{n_i\}$ represent sequences of fading coefficients and background noise samples respectively. We consider the noncoherent scenario where $\{a_i\}$ are known by neither the receiver nor the transmitter.

The Rician fading channel model is particularly appropriate when there is a direct propagating line of sight (LOS) component in addition to the faded component arising from multipath propagation. Moreover, the Rician model includes both the unfaded Gaussian channel and the Rayleigh fading channel as two special cases. Hence, results obtained for this model provide a unifying perspective.

III. OPTIMAL INPUT DISTRIBUTION FOR THE PEAK-POWER LIMITED RICIAN FADING CHANNEL

In this section, we characterize the capacity-achieving input structure for the peak-power limited Rician fading channel. In communication systems, where RF amplifiers are employed at the transmitters, peak power limitations are imposed to avoid the nonlinear distortion caused by the saturation effect. Moreover, peak power consumption controls the battery life which is a major consideration in the design of portable radio units. Hence capacity results obtained under a peak-power constraint are more reflective of ultimate performance limits of practical communication systems.

For the Rician channel model (1), the capacity is the supremum of the input-output mutual information over the set of all input distributions satisfying the peak-power constraint, $|x|^2 \leq P$:

$$C = \sup_{\substack{F_x(\cdot) \\ |x|^2 \leq P \text{ a.s.}}} \int_{\mathbb{C}} \int_{\mathbb{C}} f_{y|x}(y|x) \log \frac{f_{y|x}(y|x)}{f_y(y)} dy dF_x(x)$$

where the conditional density of the output given the input,

$$f_{y|x}(y|x) = \frac{1}{\pi(\gamma^2|x|^2 + N_0)} \exp\left(-\frac{|y - mx|^2}{\gamma^2|x|^2 + N_0}\right),$$

is circular complex Gaussian. Moreover note that $f_y(y) = \int_{\mathbb{C}} f_{y|x}(y|x) dF_x(x)$ is the marginal output density, F_x is the distribution function of the input and \mathbb{C} denotes the complex domain. Next we state the main result on the optimal input structure.

Theorem 1 *For the noncoherent Rician fading channel (1) where the input is subject to a peak power constraint*

$|x|^2 \leq P$, uniformly distributed input phase that is independent of the amplitude is optimal and the capacity-achieving amplitude distribution is discrete with a finite number of mass points.

Proof: Optimality of uniform phase follows from the circular symmetry of the problem and the proof can be found in [1]. Using the approach employed in [4], it can be shown that F_0 is a capacity achieving amplitude distribution if and only if the following condition (Kuhn-Tucker condition) is satisfied

$$\int_0^\infty g(R, r) \log f_R(R, F_0) dR + \log(1 + r^2) + C + 1 \geq 0 \quad (2)$$

for all $0 \leq r \leq \sqrt{\alpha}$ with equality if $r \in E_0$ where E_0 is the set of points of increase of F_0 . In the above formulation, $R = \frac{1}{N_0}|y|^2$ and $r = \frac{\gamma}{\sqrt{N_0}}|x|$. $f_R(R; F_0) = \int_0^{\sqrt{\alpha}} g(R, r) dF_0(r)$ is the density function of R with the kernel given by $g(R, r) = \frac{1}{1+r^2} \exp\left(-\frac{R+Kr^2}{1+r^2}\right) I_0\left(\frac{2\sqrt{Kr}\sqrt{R}}{1+r^2}\right)$. Furthermore, F_0 is the distribution function of r , $K = \frac{|m|^2}{\gamma^2}$ is the Rician factor and $\alpha = \gamma^2 \frac{P}{N_0}$ is the normalized peak SNR.

To prove the theorem, we first find an upper bound on the left-hand-side (LHS) of the Kuhn-Tucker condition (2). To achieve this goal, we bound $f_R(\cdot, F_0)$ as follows.

$$f_R(R; F_0) = \int_0^{\sqrt{\alpha}} g(R, r) dF_0(r) \quad (3)$$

$$\leq e^{\left(-\frac{R}{1+\alpha} + \sqrt{R}\sqrt{\alpha}\right)} \int_0^{\sqrt{\alpha}} \frac{e^{\left(-\frac{Kr^2}{1+r^2}\right)}}{1+r^2} dF_0(r) \quad (4)$$

$$\leq e^{\left(-\frac{R}{1+\alpha} + \sqrt{R}\sqrt{\alpha}\right)} \quad \forall R \geq 0, \quad \forall F_0. \quad (5)$$

Upper bound (4) is easily verified by observing that for all $r \leq \sqrt{\alpha}$, $e^{\left(-\frac{R}{1+r^2}\right)} \leq e^{\left(-\frac{R}{1+\alpha}\right)}$ and $I_0\left(\frac{2\sqrt{Kr}\sqrt{R}}{1+r^2}\right) \leq e^{\left(\sqrt{K}\sqrt{R}\right)}$. Realizing that $g(R, r)$ is a non-central chi-square probability density function (with two degrees of freedom) in R , (5) leads to the following upper bound on the left-hand-side (LHS) of the Kuhn-Tucker condition (2):

$$\text{LHS} \leq -\frac{1 + (K+1)r^2}{1+\alpha} + \frac{\sqrt{K}\pi}{2} \sqrt{1+r^2} \left(1 + \frac{Kr^2}{1+r^2}\right) + \log(1+r^2) + C + 1 \quad \forall r \geq 0. \quad (6)$$

We proceed to the proof by assuming that the optimal input distribution F_0 has an infinite number of points of increase on a bounded interval. Next we extend the LHS of the Kuhn-Tucker condition (2) to the complex domain:

$$\Phi(z) = \int_0^\infty g(R, z) \log f_R(R, F_0) dR + \log(1+z^2) + C + 1$$

where $z \in \mathbb{C}$ and \log is the principle branch of the logarithm. Since the Kuhn-Tucker condition should be satisfied with equality at the points of increase of the optimal input distribution, from the above assumption, $\Phi(z) = 0$ for an infinite number of points having a limit point¹. Then by the Identity Theorem [8], $\Phi(z) = 0$ in

¹The Bolzano-Weierstrass Theorem [7] states that every bounded infinite set of real numbers has a limit point.

the whole region where it is analytic. By the Differentiation Lemma [9, Ch. 12], one can easily verify that $\Phi(z)$ is analytic in the region where $\text{Re}(1+z^2) > 0$ which includes the positive real line. Therefore we conclude that $\Phi(r) = 0 \quad \forall r \geq 0$. However, the right-hand-side of (6) diverges to $-\infty$ as $r \rightarrow \infty$ for any finite $\alpha, K \geq 0$. Therefore the condition $\Phi(r) = 0 \quad \forall r \geq 0$ can not be satisfied for any finite $\alpha, K \geq 0$. Hence we reach to a contradiction to the assumption that the the optimal input distribution has an infinite number of points of increase on a bounded interval and the theorem follows. \square

Recently, Verdú [2] has analyzed the spectral-efficiency/bit-energy tradeoff in the low-power regime for a general class of fading channels and has shown that if there are no constraints other than average power, the minimum bit energy, which is achieved at zero spectral efficiency, is $\log_e 2 = -1.59$ dB as long as the additive background noise is Gaussian. Verdú [2] has also defined the wideband slope as the slope of the spectral efficiency curve $C\left(\frac{E_b}{N_0}\right)$ in bits/s/Hz/3dB at zero spectral efficiency. The wideband slope closely approximates the growth of the spectral efficiency curve in the power-limited regime and hence is proposed as a new tool providing insightful results when bandwidth is a resource to be conserved.

Under quite general conditions on the input and the channel, Prelov and Verdú [3] obtained the exact asymptotic second-order behavior of the input-output mutual information for vanishing SNR. Using this asymptotic expression, we can prove the following proposition.

Proposition 1 *For the noncoherent Rician fading channel (1) where the input is subject to a peak power limitation $|x|^2 \leq P$, the normalized received bit energy required at zero spectral efficiency and wideband slope are*

$$\frac{E_b^r}{N_0} \Big|_{C=0} = \left(1 + \frac{1}{K}\right) \log_e 2 \quad \text{and} \quad S_0 = \frac{2K}{K+2} \quad (7)$$

respectively.

Note that both the bit energy at zero spectral efficiency and wideband slope depends only on the Rician factor $K = \frac{|m|^2}{\gamma^2}$ and for any finite $K > 0$, $\frac{E_b^r}{N_0} \Big|_{C=0} > -1.59$ dB and $0 < S_0 < 2$. As $K \rightarrow \infty$, the bit energy at zero spectral efficiency and wideband slope approach to -1.59 dB and 2, respectively, which are the values obtained in the unfaded Gaussian channel [2]. For the Rayleigh channel, where $K = 0$, $\frac{E_b^r}{N_0} \Big|_{C=0} = \infty$ indicating that the minimum bit energy is achieved at a nonzero spectral efficiency.

It is also of interest to consider the average power limited Rician fading channel (1). For this case, we have the following result.

Theorem 2 *For the Rician fading channel (1) where the input is subject to an average power constraint $E\{|x|^2\} \leq P$, uniformly distributed input phase that is independent of the amplitude is optimal and the capacity-achieving input amplitude distribution has bounded support.*

Proof: Optimality of uniform phase again follows from the circular symmetry of the problem. Similarly as in the proof of Theorem 1, F_0 is a capacity-achieving amplitude

distribution if and only if there exists $\lambda \geq 0$ such that the following condition (Kuhn-Tucker condition) is satisfied

$$\int_0^\infty g(R, r) \log f_R(R, F_0) dR + \log(1 + r^2) + \lambda(r^2 - \alpha) + C + 1 \geq 0 \quad \forall r \geq 0 \quad (8)$$

with equality if $r \in E_0$ where E_0 is the set of points of increase of F_0 . We will prove the proposition by contradiction. Hence we assume that the optimal amplitude distribution F_0 has unbounded support. With this assumption, for any finite $M \geq 0$, we obtain the following lower bound:

$$f_R(R; F_0) = \int_0^\infty g(R, r) dF_0(r) \geq \int_M^\infty \frac{1}{1+r^2} e^{\left(-\frac{R+Kr^2}{1+r^2}\right)} dF_0(r) \quad (9)$$

$$\geq e^{\left(-\frac{R}{1+M^2}\right)} \int_M^\infty \frac{1}{1+r^2} e^{\left(-\frac{Kr^2}{1+r^2}\right)} dF_0(r) \quad (10)$$

$$= D_{F_0, M} e^{\left(-\frac{R}{1+M^2}\right)} \quad \forall R \geq 0 \quad \forall M \geq 0, \quad (11)$$

where $0 < D_{F_0, M} \leq 1 \quad \forall M \geq 0$ and $\forall F_0$. Using (11), we obtain the following lower bound on the left hand side (LHS) of the Kuhn-Tucker condition (8):

$$\text{LHS} \geq \ln D_{F_0, M} - \frac{1}{1+M^2} - \frac{1+K}{1+M^2} r^2 + \log(1+r^2) + \lambda(r^2 - \alpha) + 1 + C, \quad \forall r \geq 0 \quad \forall M \geq 0.$$

For any $\lambda > 0^2$ and $D_{F_0, M} > 0$, we can choose M sufficiently large such that the above lower bound diverges to infinity as $r \rightarrow \infty$. However, if the optimal input has unbounded support, the LHS of the Kuhn-Tucker condition should be zero infinitely often as $r \rightarrow \infty$. This constitutes a contradiction and hence the theorem follows. \square

IV. OPTIMAL INPUT DISTRIBUTION FOR THE RICIAN CHANNEL WITH PHASE NOISE

In this section, we consider the following Rician fading channel

$$y_i = m e^{j\theta_i} x_i + a_i x_i + n_i. \quad (12)$$

where phase noise is introduced in the specular component. Here, $\{\theta_i\}$ is assumed to be a sequence of independent and identically distributed uniform random variables on $[-\pi, \pi)$. We again consider the noncoherent scenario where $\{a_i\}$ and $\{\theta_i\}$ are known by neither the receiver nor the transmitter. This model is relevant in mobile systems where rapid random changes in the phase are not compensated even in the specular component. Such a model is also suitable in cases where there is imperfect receiver side information about the fading magnitude. Moreover, it includes both the Rayleigh fading channel and the noncoherent AWGN channel as two special cases.

Since the phase information is completely destroyed in the channel (12), the input-output mutual information is $I(x; y) = I(|x|; |y|) = I(r; R)$ where we have again defined $R = \frac{1}{N_0} |y|^2$, $r = \frac{2}{\sqrt{N_0}} |x|$. The conditional distribution of R given r is

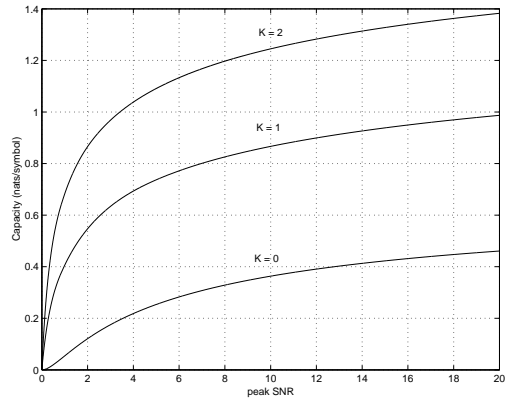


Figure 1: Capacity curves for the peak-power limited Rayleigh channel ($K = 0$) and Rician channels with $K = 1, 2$.

$$f_{R|r}(R|r) = \frac{1}{1+r^2} \exp\left(-\frac{R+Kr^2}{1+r^2}\right) I_0\left(\frac{2\sqrt{Kr}r\sqrt{R}}{1+r^2}\right)$$

where $K = \frac{|m|^2}{\gamma^2}$ is the Rician factor and I_0 is the zeroth order modified Bessel function of the first kind. We have the following result on the structure of the capacity-achieving input amplitude distribution.

Theorem 3 For the noncoherent Rician fading channel with phase noise (12) where the input is subject to an average power constraint $E\{|x|^2\} \leq P$, the capacity-achieving input amplitude distribution is discrete with a finite number of mass points.

Proof: See [1].

As an extension to the recent results [4] and [5], we have proven the discreteness of the capacity-achieving distribution in a unifying setting where there is both multipath fading and a specular component with random phase. In [5], it is shown that the optimal amplitude distribution for the noncoherent AWGN channel is discrete with an infinite number of mass points. An interesting conclusion of the above result is that the presence of an unknown multipath component induces an optimal distribution that has a finite number of mass points.

V. NUMERICAL RESULTS

Numerical results for the capacity-achieving input distribution are obtained using the approach employed in [4]. First the input-output mutual information is maximized assuming a fixed number of mass points and then the maximizing input distribution is tested with the sufficient and necessary Kuhn-Tucker condition. For the peak-power limited Rician fading channel ($K > 0$), numerical results indicate that for sufficiently low SNR values, the optimal amplitude distribution has a single mass at \sqrt{P} and hence all the information is carried on the uniform phase. For the Rayleigh channel ($K = 0$), an equiprobable two mass point distribution where one mass is at the origin and the other mass at the peak level is capacity-achieving in the low-power regime.

Figure 1 plots the capacity curves for the peak-power limited Rayleigh channel and Rician channels with $K =$

²The impossibility of $\lambda = 0$ is shown in [4].

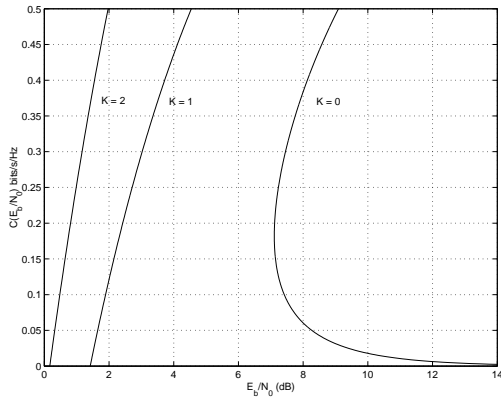


Figure 2: Spectral Efficiency $C(\frac{E_b}{N_0})$ bits/s/Hz vs. $\frac{E_b}{N_0}$ (dB) for the peak-power limited Rayleigh channel ($K = 0$) and Rician channels with $K = 1, 2$.

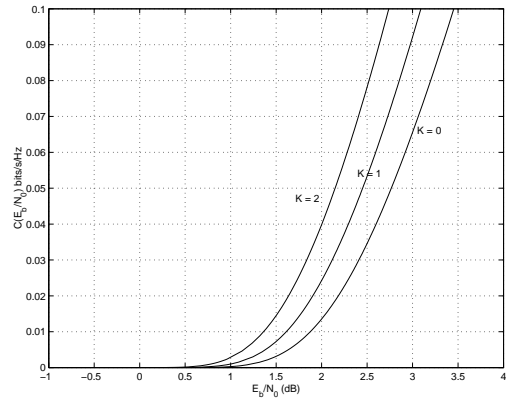


Figure 4: Spectral Efficiency $C(\frac{E_b}{N_0})$ bits/s/Hz vs. $\frac{E_b}{N_0}$ (dB) for the Rayleigh channel ($K = 0$) and Rician channels ($K = 1, 2$) with phase noise.

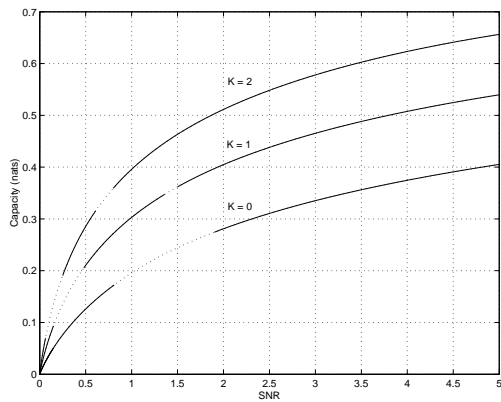


Figure 3: Capacity curves for the Rayleigh channel ($K = 0$) and Rician channels ($K = 1, 2$) with phase noise.

1, 2 as a function of the peak SNR. Note that the Rayleigh channel capacity curve has a zero slope at zero SNR. Fig. 2 plots the spectral-efficiency/bit-energy curves in the low-power regime. For the Rayleigh channel, the minimum bit energy is ~ 7.1078 dB and it is achieved at a nonzero spectral efficiency, $C^* \simeq 0.182$ bits/s/Hz. As observed in [2], one should avoid operating in the region $C < C^*$ where decreasing the spectral efficiency only increases the required bit energy. For Rician channels with $K = 1$ and 2, minimum bit energies are achieved at zero spectral efficiency. Note that for $K = 1$, $\frac{E_b^r}{N_0} \Big|_{C=0} = (1 + \frac{1}{K}) \log_e 2 = 1.418$ dB and $S_0 = \frac{2K}{K+2} = \frac{2}{3}$ and for $K = 2$, $\frac{E_b^r}{N_0} \Big|_{C=0} = 0.169$ dB and $S_0 = 1$.

Numerical results for the average power limited Rician fading channel with phase noise indicates that a two mass point input distribution with one mass at the origin is capacity-achieving in the low-power regime and as $\text{SNR} \rightarrow 0$, the nonzero mass point migrates away from the origin while its probability decreases sufficiently fast to satisfy the average power constraint. This type of input

is called *flash signaling* in [2]. Fig. 3 plots the capacity curves for the average power limited Rayleigh channel ($K = 0$) and Rician channels with $K = 1, 2$. The dashed segments are conjectured capacity curves. In these regions, a new mass point emerges with a very small probability and hence numerical optimization methods do not provide stable results.

Verdú [2] has shown that for a general class of average power limited fading channels, the bit energy at zero spectral efficiency is $\log_e 2 = -1.59$ dB. Moreover, if there is imperfect receiver side information, the wideband slope is zero. Fig. 4 plots the spectral-efficiency/bit-energy curves in the low-power regime. Indeed, we see that for all K , the wideband slope is zero, and below 1 dB we do not observe a significant gain in spectral efficiency when the Rician factor is increased.

REFERENCES

- [1] M. C. Gursoy, H. V. Poor, S. Verdú, "The Noncoherent Rician Fading Channel – Part I : Structure of the Capacity-Achieving Input," submitted to *IEEE Trans. Wireless Commun.*
- [2] S. Verdú, "Spectral efficiency in the wideband regime," *IEEE Trans. Inform. Theory*, vol. 48, pp. 1319-1343, June 2002.
- [3] V. Prelov and S. Verdú, "Second order asymptotics of mutual information," 2002, preprint.
- [4] I. Abou-Faycal, M. D. Trott, and S. Shamai (Shitz), "The capacity of discrete-time memoryless Rayleigh fading channels," *IEEE Trans. Inform. Theory*, vol. 47, pp. 1290-1301, May 2001.
- [5] M. Katz and S. Shamai (Shitz), "On the capacity-achieving distribution of the discrete-time non-coherent additive white Gaussian noise channel," *Proc. 2002 IEEE Int'l. Symp. Inform. Theory*, Lausanne, Switzerland, June 30-July 5, 2002, p. 165.
- [6] R. Nuriyev and A. Anastasopoulos, "Capacity and coding for the noncoherent block-independent AWGN channel," to appear in ISIT 2003.
- [7] W. Rudin, *Principles of Mathematical Analysis*. McGraw Hill: New York, 1964.
- [8] K. Knopp, *Theory of Functions*. Dover: New York, 1945.
- [9] S. Lang, *Complex Analysis 2nd. Ed.* Springer-Verlag: New York, 1985.