COHERENT-DETECTION OPTICAL COMMUNICATION SYSTEMS: A PROOF OF THE GAUSSIAN BIT ERROR RATE

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Abstract. It is shown that the asymptotic probability of error of a binary equiprobable hypothesis test for coherent Poisson point-processes with rate \( \lambda_0(t) = \lambda_1(t) = \lambda(t) \cdot e^{-\lambda(t) \cdot t} \) where \( \lambda(t) \) is equal to the error probability of optimum discriminant signal detection in additive white Gaussian noise when the signal covariates with the square root of the point-process rates. This result proves the folk theorem that coherent-detection optical communication systems have Gaussian bit error rate.

The minimum probability of error of equiprobable binary signal-detection in additive white Gaussian noise is equal to \( Q[\sqrt{2s}] \), where \( s \) denotes the energy of the difference between both signals and \( s^2 \) is the noise power spectral density. In contrast, there exists no scalar parameter that characterizes the performances of hypothesis testing for observed Poisson point-processes, and the rates must be known for all times in order to determine error probability. Moreover, unless the log likelihood ratio is conditionally Poisson (e.g. if one rate dominates the other and their ratio is piecewise constant), no closed or semi-closed expressions are known for the probability error.

Fortunately, the cutoff rate, \( \beta_0 \), of a Poisson point-process channel exhibits a much more manageable behavior. Snyder and Rhodes \cite{Snyder} have shown that \( \beta_0 \) coincides with the cutoff rate of a \( \frac{1}{2} \) white Gaussian noise channel whose signals are equal to the square root of the point-process rates. The purpose of this paper is to show that the error probabilities of both channels also coincide in problems where the point-process rates are uniformly large. Consequently, it is shown that in the limit the error probability depends on the point-process rates only through the energy of the difference between their square roots.

Specifically, it is shown below that the minimum error probability of a binary equiprobable hypothesis test between the point-process rates \( \lambda_0(t) = \lambda_1(t) = \lambda(t) \cdot e^{-\lambda(t) \cdot t} \) i.e. \( i = 0, 1 \) approaches \( Q[\sqrt{2s}] \) as \( t \to \infty \). This work was partially supported by the U.S. Office of Naval Research under contract N00014-83-K-0145.

\( i \to \infty \). The asymptotic rate is:

\[
\lambda(t) = \langle \lambda(t) \rangle \cdot e^{-\langle \lambda(t) \rangle \cdot t}
\]

where \( \langle \rangle \) is the instantaneous rate.

\( i \) is 

\[
Q[\sqrt{2s}]
\]

\( \lambda_0(t) = \lambda_1(t) = \lambda(t) \cdot e^{-\lambda(t) \cdot t} \)

i)

where \( \langle \rangle \) is the instantaneous rate and \( \alpha \) and \( \beta \) are non-integer.

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COMMUNICATION SYSTEMS: BIT ERROR RATE

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Probability of error of a binary equiprobable
point-process with rate n is the error probability of optimum
Gaussian noise when the signals coincide.
This result proves the full theorem of
Gaussian bit error rates.

The binary signal-detectors in addition
where $d^2$ denotes the energy of the
power spectral density. In contrast,
- The probability of hypothesis testing
must be known for all times in order
- The likelihood ratio is conditionally
- Poincaré’s theorem is also equal to

The aim of this paper is to show that
the problems of the point-process
that in the limit the error probability
the energy of the difference between
minimum error probability of a binary
the point-process rates

where $Q$ is the Q-function.

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$z \to \infty$. This asymptotic rate model is motivated by two important situations arising in
optical digital communications systems.

\[ P_e = Q \left[ \mu \left( \frac{1}{2} \sqrt{E_{b}} - \xi \right) / \sqrt{4} \right] \]  

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i) In intensity modulated direct-detection systems the rate of the stream of electrons
put out by the photodetector is modeled by [2]:

\[ k_n(t) = a \left| E_n(t) \right|^2 \beta \]

where $E_n(t)$ is the instantaneous complex amplitude of the incident electric field,
and $a$ and $\beta$ are non-negative constants of the photodetector. Identifying
\[ z = -a \beta \left| E_n(t) \right|^2 \]

results in the model described by the Gaussian bit error rate.

ii) In so-called coherent-detection optical communication systems [3], a strong
coherent-generated field \( \left| E(t) \right| \), \( \beta \), is added to the received electromagnetic
field \( E_n(t) \), \( \beta \), and the rate of the photodetector output process is

\[ k_n(t) = a \left| E(t) \right| \beta \left| E_n(t) \right| \beta \]

which corresponds to

\[ z = -a \beta \left| E(t) \right| \beta \left| E_n(t) \right| \beta \cos(\varphi(t) - \delta) \]

and

\[ k_n(t) = a \left| E(t) \right| \beta \left| E_n(t) \right| \beta \cos(\varphi(t) - \delta) \]

Letting \( E \to \infty \) and applying the result
we obtain the error probability conditioned on \( E(t) \), given by

\[ P_e = Q \left[ \mu \left( \frac{1}{2} \sqrt{E_{b}} - \xi \right) / \sqrt{4} \right] \]

Note that (1) is the minimum probability of error for any receiver that knows
\( k_n(t) \) and \( \mu(t) \) (and consequently the phase difference \( \varphi(t) - \delta \)).

In the cases where the modulation is either linear \( \varphi(t) = \xi(t) - \delta \) or
exponential \( E_n(t) = E(t) \), the square of the argument of the Q-function in

(1) reduces to

\[ Q \left[ \mu \left( \frac{1}{2} \sqrt{E_{b}} - \xi \right) / \sqrt{4} \right] \]
Proposition: Consider the following pair of equivalent hypotheses:

\( H_0, H_1 \) is a Poisson counting process with rate \( \lambda_0(t, z) \), \( z \in \Omega \), and \( \lambda_0(t, z) \geq \lambda_1(t, z) \).

Suppose that \( \lambda_1(t, z) = \lambda_0(t, z) + \varepsilon(t, z) \), where \( \varepsilon(t, z) \) is a real number.

Let \( \Phi = \int_{\Omega} \lambda_0(t, z) \, d\mu(t, z) \). Then, the minimum probability of error of the above test satisfies

\[
\lim_{\varepsilon \to 0} P_e(\varepsilon) = \Phi(\varepsilon) \tag{2}
\]

Proof: Let \( \Omega \) denote the observation space. For each \( \varepsilon > 0 \), let \( \mu_d(t, z) \), \( z \in \Omega \), denote the sample function density of a Poisson point-process with rate \( \lambda_d(t, z) \), with respect to \( d\mu(t, z) \), the probability measure generated by a univariate Poisson point-process, and define the hitting probability density

\[
q(x, z) = \exp(-B(x)) \cdot \frac{1}{2} \sum \{ z \mid \min(\lambda_0(t, z), x) \}
\]

where \( B(x) \) is the Bhattacharyya distance between \( \mu_d(t, z) \) and \( \mu_d(t, z) \):

\[
B(x) = \int_{\Omega} \mu_d(t, z) \, d\mu(t, z) \tag{3}
\]

Since \( \varepsilon > 0 \) the likelihood ratio \( \rho(x, z) = \frac{1}{2} \sum \{ z \mid \min(\lambda_0(t, z), x) \} \) is defined everywhere in \( \Omega \), and the minimum probability of error is given by

\[
P_e(\varepsilon) = \int_{\Omega} \left[ \exp(-B(x)) \cdot \frac{1}{2} \sum \{ z \mid \min(\lambda_0(t, z), x) \} \right] \, d\mu(t, z) \tag{4}
\]

From this, we obtain

\[
P_e(\varepsilon) = \int_{\Omega} \exp(-B(x)) \cdot \frac{1}{2} \sum \{ z \mid \min(\lambda_0(t, z), x) \} \, d\mu(t, z) \tag{5}
\]

Since \( \lambda_0(t, z) \) is a Poisson process, we have

\[
\Phi(\varepsilon) = \int_{\Omega} \exp(-B(x)) \cdot \frac{1}{2} \sum \{ z \mid \min(\lambda_0(t, z), x) \} \, d\mu(t, z) \tag{6}
\]

We show now that the change of measure

\[
\Phi(\varepsilon) = \exp(-B(x))
\]

converges everywhere in \( \Omega \) to \( \exp(-B(x)) \).

From the density evaluated at the event

\[
\Phi(t, x) \tag{6}
\]

we can write

\[
\Phi(t, x) = \exp(-B(x))
\]

where we have employed the Chernoff theorem:

\[
\rho(x, z) = \exp(-B(x)) \sum_{z \in \Omega} \frac{1}{2} \sum \{ z \mid \min(\lambda_0(t, z), x) \}
\]

At each event \( x \in \Omega \), the test in equation (3) is equal to

\[
\rho(x, z) = \int_{\Omega} \frac{1}{2} \sum \{ z \mid \min(\lambda_0(t, z), x) \} \, d\mu(t, z)
\]

This completes the proof.
We show now that the characteristic function

$$\Psi_{x}(z) = \int e^{ixz} \varphi(x; z) \, dx$$  

(8)

converges everywhere in \( \mathbb{R} \) to \( \exp(-|z|^{2}/2) \) as \( z \to \infty \). Using the fact that the sample function density evaluated at the unordered realization \( (x_1, \ldots, x_n) \) is equal to \( \mathbb{P} \):

$$\rho_n(x_1, \ldots, x_n; z) = \int f_{\mathcal{U}|\mathcal{E}}(x_i; x_0) \prod_{i=1}^{n} \lambda_i(x_i; z)$$  

(7)

we can write

$$\Psi_{x}(z) = \exp \left( \int B(x) \right) \prod_{i=1}^{n} \lambda_i(x_i; z)$$

$$= \exp \left( \int \sum_{i=1}^{n} \left( f(x, z) \left( x_i - \frac{1}{2} \right)^{2} \lambda_i(x_i; z) \right) \right) \prod_{i=1}^{n} \lambda_i(x_i; z)$$

$$= \exp \left( \sum_{i=1}^{n} \left( f(x, z) \left( x_i - \frac{1}{2} \right)^{2} \lambda_i(x_i; z) \right) \right) \prod_{i=1}^{n} \lambda_i(x_i; z)$$

$$= \exp \left( j \omega (x, z) - \frac{1}{2} \left( x - \frac{1}{2} \right)^{2} \right) \prod_{i=1}^{n} \lambda_i(x_i; z)$$

(8)

where we have employed the Correlation parameter notation

$$\rho(x, z) = \int \lambda(x, z) \varphi(x; z) \, dx$$

(9)

At each time \( t \in [0, T] \), the integral in the exponent of the right-hand side of (8) is equal to

$$\int \lambda(x, z) \varphi(x; z) \, dx$$

(10)
Therefore, on taking the limit of the right-hand side of (8) it follows

\[ \lim_{\mathcal{F}, \mathcal{G} \downarrow 0} \Phi \left( \frac{z}{\sqrt{2}} \right) = \exp \left( -\frac{1}{2} \mathbf{v}^2 \right). \]  

(11)

So the log likelihood-ratio under \( \gamma(x, z) \) converges in distribution as a zero mean Gaussian random variable with variance equal to \( 4 \delta^2 \). Therefore, the continuity and boundedness of \( f \) imply (4) that the limit of the right-hand side of (9) is equal to

\[ \lim_{\mathcal{F} \downarrow 0} P_e(x) = \lim_{\mathcal{G} \downarrow 0} \exp \left( \frac{1}{2} \mathbf{v}^2 \right) \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} \mathbf{v}^2 \right) \phi(x - \mathbf{v}) \, dx \]

\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} \mathbf{v}^2 \right) \phi(x - \mathbf{v}) \, dx = \Phi \left( \frac{x}{\sqrt{2}} \right) \]

\( \square \)

It has been shown that the asymptotic minimum probability of error of binary hypothesis testing for Poisson point process observations coincides with the error probability of optimum discrimination of the square root of the point-process rates included in additive white Gaussian noise. The main significance of this result is that the error rate analysis of optical fibre communication systems based on heterodyne or homodyne coherent-detection (which appear to be increasingly important in applications) is equivalent to the analysis of digital signaling over additive white Gaussian noise channels. Interestingly, this result appears to be widely accepted by practitioners in the field on grounds of analytical convenience and agreement with experimental data (e.g., [8], [9]).

Attempts to dispatch this result on grounds that "Poisson in Gaussian is 'the limit'" are doomed to failure; the observed point-process in Poisson is not a Gaussian random process in the limit, and the log-likelihood ratio, although as shown above asymptotically Gaussian under the adequate probability measure, is not Poisson. Also, the fact that high-density shot noise (superposition of triplets of an arbitrary waveform delayed by the point-process arrival times) is a Gaussian process \( \psi \) appears to be of little use since no closed-form expression is known for the error probability of a test for mean and autocorrelation.
\[ p(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]  

\[ \mathbb{E}(x) = \mu \]  

\[ \mathbb{V}(x) = \sigma^2 \]  

\[ f(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]  

\[ \int f(x) dx = 1 \]  

\[ \mathbb{E}(x^2) = \mu^2 + \sigma^2 \]  

\[ \mathbb{V}(x^2) = \sigma^2 \]  

References:


