Correlation Number: A New Design Criterion in Multi-Antenna Communication

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***Abstract*—This paper establishes a connection between antenna correlation and the Shannon capacity of multi-antenna channels. At low and moderate signal-to-noise ratios, which constitute the most relevant regime in contemporary mobile systems, a fundamental parameter emerges out of such connection. We illustrate how this parameter, for which we coin the term *correlation number*, can be used as a design criterion for diversity arrays. In addition, we present correlation numbers from experimental measurements conducted in New York City.**

I. Motivation

Antenna diversity is one of the most effective mechanisms to mitigate the deleterious effects of fading in multipath radio channels. Although, historically, such form of diversity had usually been placed at the receiver [1], transmit diversity has also been popularized as of late [2]. Moreover, combined transmit and receive diversity has been shown to offer great potential for spectrally efficient communication [3], [4].

Given an antenna array, though, effective diversity is obtained only if the antennas within the array are sufficiently uncorrelated. Although, in order to reduce their correlation, antennas are typically separated by some distance and often cross-polarized, a certain degree of correlation is nonetheless unavoidable. The impact of such correlation has been studied over the years [5] in terms of its impact on error probability and signal-to-noise ratio (SNR).[1] In this paper, we undertake the analysis of the relationship between antenna correlation and the most fundamental of performance measures: the information-theoretic (Shannon) capacity. Focusing on the range of low and moderate SNR, which constitutes the most relevant operational regime in contemporary mobile systems, our goal is to provide analytical insight on this relationship and, through it, on how antenna correlation affects the interplay of power, bandwidth and rate.

The growing relevance of the low-SNR regime stems from the facts that:

1) Geometrically, a majority of users lie in the periphery of their cells.
2) Efficient bandwidth utilization requires aggressive frequency reuse across neighboring cells and sectors, which causes very high levels of out-of-cell interference.

Consequently, users operate very often at low SNR while only rarely at high SNR. In emerging third-generation mobile data systems [6], [7], for instance, almost 40% of users experience SNR levels below 0 dB while less than 10% display levels above 10 dB.

II. Multi-Antenna Capacity at Low SNR

With n_T transmit and n_R receive antennas, a baseband discrete-time model for the multi-antenna channel with frequency-flat fading is[2]

$$\mathbf{y} = \sqrt{\mathrm{g}}\,\mathbf{H}\,\mathbf{x} + \mathbf{n}$$

where $\mathbf{x}$ is the n_T-dimensional transmit signal while $\mathbf{y}$ and $\mathbf{n}$ are the received signal and the white[3] Gaussian noise, both n_R-dimensional. The channel is represented by the $(n_R \times n_T)$ random matrix $\sqrt{\mathrm{g}}\,\mathbf{H}$. The scalar $\sqrt{\mathrm{g}}$ has been factored out so as to yield a normalized identically distributed[4] matrix $\mathbf{H}$, the power of whose entries is unity, and thus $1/\mathrm{g}$ indicates the average path-loss. Throughout the paper, we focus on those scenarios where the channel is known to the receiver but unknown to the transmitter. The single-sided spectral density of the noise is

$$N_0 = \frac{E[\|\mathbf{n}\|^2]}{n_R}.$$

The average received SNR per antenna is given by

$$\mathsf{SNR} = \mathrm{g}\frac{E[\|\mathbf{Hx}\|^2]}{E[\|\mathbf{n}\|^2]}$$

and the capacity as function of SNR is denoted by $C(\mathsf{SNR})$. The normalized received energy per bit, in turn, is

$$\frac{E_b^r}{N_0} = n_R \frac{\mathsf{SNR}}{(R/B)}.$$

where R and B are, respectively, the rate (in bits/s) and the bandwidth. Since the largest rate per unit bandwidth achievable

[1] Throughout the paper, the term "noise" encompasses both thermal noise and co-channel interference.

[2] In frequency-selective environments, the channel can always be decomposed into a number of parallel non-interacting subchannels, each experiencing frequency-flat fading and having the same capacity as the aggregate channel.

[3] The analysis can be generalized to colored noise [8].

[4] The analysis can also be extended to non-identically distributed channel matrices [9].

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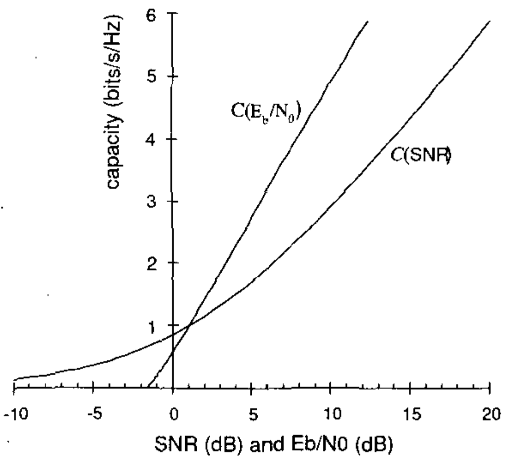

Fig. 1. Capacity as function of SNR and $\frac{E_b^r}{N_0}$ with Rayleigh fading and $n_T=n_R=1$.

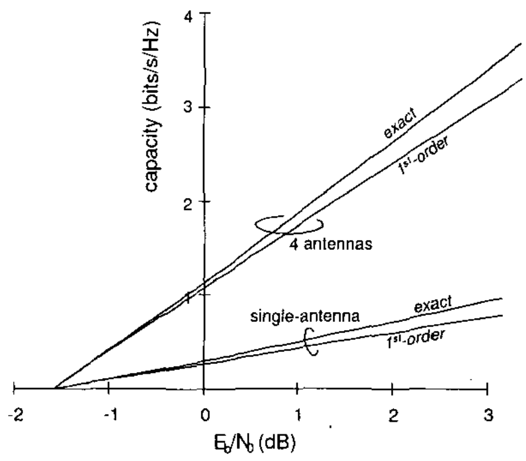

Fig. 2. Comparison between the exact capacity as function of $\frac{E_b^r}{N_0}$ and its low-SNR first-order expression for single-antenna and uncorrelated 4-antenna architectures.

with arbitrary reliability is precisely the capacity, the minimum required transmitted energy per bit is

$$\frac{E_b^r}{N_0} = \frac{E[\|\mathbf{x}\|^2]}{N_0\, C(\text{SNR})}. \quad (1)$$

As function of $\frac{E_b^r}{N_0}$, the capacity is denoted by $\mathsf{C}(\frac{E_b^r}{N_0})$.

Both $C(\text{SNR})$ and $\mathsf{C}(\frac{E_b^r}{N_0})$ are useful representations of the capacity and they are uniquely related through (1). As the SNR grows, both $C(\text{SNR})$ and $\mathsf{C}(\frac{E_b^r}{N_0})$ *exhibit a similar behavior: they* quickly approach a certain—in fact the same [10]—slope. As an example, Fig. 1 depicts the capacity of a Rayleigh-faded scalar channel (i.e. $n_T=n_R=1$), where the slope becomes 1 bit/s/Hz/(3 dB) as the SNR gets large. At low SNR, though, only $\mathsf{C}(\frac{E_b^r}{N_0})$ admits a useful first-order expression and thus it becomes a much more convenient representation therein [8]. This first-order expression is given by [11]

$$\mathsf{C}(\tfrac{E_b^r}{N_0}) \approx \frac{S_0}{3\,\text{dB}} \left(\frac{E_b^r}{N_0}|_{\text{dB}} + 1.59 \right) \quad (2)$$

evidencing that reliable communication requires that the received energy per bit be at least -1.59 dB above the noise floor.[5] The key performance measure in (2) is S_0, the capacity slope in bits/s/Hz/(3 dB) for $\frac{E_b^r}{N_0} \downarrow -1.59$ dB.

In contrast with $C(\text{SNR})$ and $\mathsf{C}(\frac{E_b^r}{N_0})$, whose analysis is very involved even neglecting antenna correlation [4] and especially when accounting for it [12], insightful closed-form expressions for S_0 can be obtained rather generally [11]. From these expressions, as we will see next, the impact of antenna correlation on the low-SNR capacity can be determined.

[5]Exactly, the minimum normalized energy per bit required is $\log_e 2$ expressed in dB.

TABLE I

$\frac{E_b^r}{N_0}$ BELOW WHICH (4) IS OVER 90% ACCURATE AND FRACTION OF CELL LOCATIONS ON WHICH IT CAN BE ACHIEVED.

antennas	E_b^r/N_0	locations
1	0.1 dB	24%
2	1.9 dB	60%
4	3 dB	72%
8	3.4 dB	76%

III. The Correlation Number

A. Definition

Consider the standard Rayleigh-faded multi-antenna channel model [13], [14]

$$\mathbf{H} = \mathbf{\Theta}_R^{1/2} \mathbf{W} \mathbf{\Theta}_T^{1/2} \quad (3)$$

where the entries of $\mathbf{W}$ are independent zero-mean unit-variance complex Gaussian random variables whereas $\mathbf{\Theta}_T$ and $\mathbf{\Theta}_R$ are $(n_T \times n_T)$ and $(n_R \times n_R)$ antenna correlation matrices. Specifically, the entries of $\mathbf{\Theta}_T$ indicate the correlation between each pair of transmit antennas while those of $\mathbf{\Theta}_R$ indicate the correlation between each pair of receive antennas. When $\mathbf{H}$ is identically distributed, the diagonal entries of both $\mathbf{\Theta}_T$ and $\mathbf{\Theta}_R$ equal 1.

For the channel in (3), a very compact formula for S_0 can be found [8] yielding, for low and moderate levels of SNR,

$$\mathsf{C}(\tfrac{E_b^r}{N_0}) \approx \frac{2\, n_T\, n_R}{n_T\, \zeta(\mathbf{\Theta}_R) + n_R\, \zeta(\mathbf{\Theta}_T)} \left(\frac{\frac{E_b^r}{N_0}|_{\text{dB}} + 1.59}{3\,\text{dB}} \right) \quad (4)$$

where, given an $(n \times n)$ antenna correlation matrix $\mathbf{\Theta}$, either transmit or receive, $\zeta(\mathbf{\Theta})$ is a real scalar defined as

$$\zeta(\mathbf{\Theta}) \stackrel{\text{def}}{=} \frac{\text{Tr}\{\mathbf{\Theta}^2\}}{n}$$

TABLE II

SPACING BETWEEN ADJACENT BASE ANTENNAS IN EXAMPLE 1.

antennas	2	4	6	8
d (wavelengths)	4.8	4.4	4.2	4.1

TABLE III

SPACING BETWEEN ADJACENT BASE ANTENNAS IN EXAMPLE 2.

antennas	2	4	6	8
d (wavelengths)	4	4.4	4.5	4.6

for which we coin the term *correlation number*, bound to lie within $[1,n]$. The correlation number equals 1 when the antennas are uncorrelated and it equals the number of antennas when they are fully correlated.

Note that the behavior of (4) is dominated by the largest of the quantities $\frac{\zeta(\Theta_{\rm T})}{n_{\rm T}}$ and $\frac{\zeta(\Theta_{\rm R})}{n_{\rm R}}$. If these quantities are unbalanced, there is little sensitivity to the smallest of them and thus, from a design viewpoint, it is important that they are properly equalized.

If $n_{\rm T}=n_{\rm R}=n$, (4) particularizes to

$$\mathsf{C}(\tfrac{E_b^r}{N_0}) \approx \frac{2n}{\zeta(\Theta_{\rm R})+\zeta(\Theta_{\rm T})} \left(\frac{\frac{E_b^r}{N_0}|_{\rm dB} + 1.59}{3\,{\rm dB}} \right) \quad (5)$$

which, in the absence of correlation, scales linearly with n. This slope equals its high-SNR counterpart [3], [4] thus dispelling the misconception [15], [16] that multiple transmit and receive antennas are not as valuable at low SNR as they are at high SNR. The opposite is, in fact, the case.

Fig. 2 reproduces the exact capacity along with its first-order expression in (5) for $n=1$ and $n=4$ (no correlation). The levels below which the difference is less than 10% of the exact capacity are listed in Table I, parameterized by the number of antennas. For the emerging wireless data systems described in [6], [7], Table I also shows that the linear approximation is over 90% accurate for a very large fraction of the cell locations, increasingly so as the number of antennas grows. More importantly, these are precisely the locations wherein multi-antenna techniques are more likely to be relevant.

It is also noteworthy to point out that, although the unique capacity-achieving distribution is Gaussian, the right-hand side of (4) can be obtained with QPSK on each antenna [11, Theorem 14] evidencing that such simple signaling becomes progressively optimum as the SNR diminishes.

B. The Correlation Number as Design Criterion

Notice that, in (4), each correlation number uniquely quantifies the impact of an entire correlation matrix. Different correlation matrices mapping onto the same correlation number are thus equivalent in terms of low-SNR capacity and, as a result, the correlation number can be used as a design criterion in terms of configuring the geometry of diversity arrays. In order to illustrate this role, we next furnish some examples.

Example 1. Consider a base station and a terminal, both equipped with uniform linear arrays. Consider a broadside Gaussian power spectrum at the base with a 2° root-mean-square angular spread, for which the correlation between two antennas whose indices differ by i can be approximated [17] by

$$\rho_{\rm base}(i) \approx e^{-0.05\,d^2 i^2} \quad (6)$$

with d the spacing, in wavelengths, between adjacent antennas. At the terminal, consider a uniform spectrum over 360° and a typical compact antenna spacing of 0.5 wavelengths, which results [1] in

$$\rho_{\rm term}(i) = J_0(\pi i) \quad (7)$$

with $J_0(\cdot)$ the zero-order Bessel function of the first kind. From (7), the terminal correlation number can be computed as function of the number of antennas therein. In order to equalize the correlation numbers at base and terminal, d in (6) must be as listed in Table II.

Example 2. Consider again a base station whose antenna correlation is dictated by (6). With two antennas spaced by $d=4$, the correlation number equals 1.2. If additional antennas are incorporated while constraining the correlation number to remain constant, d must be as listed in Table III.

Notice that, in order to keep the correlation numbers constant—and thus (5) linear on n—the spacing between contiguous antennas has to be increased as n grows. In the next section, we uncover this and other implications.

C. Additional Implications

Using (4), the impact of antenna correlation on the interplay of power, bandwidth and rate, can be established. Fixing the rate R and power P, the bandwidth B required with correlation matrices $\Theta_{\rm T}$ and $\Theta_{\rm R}$ relative to the bandwidth B_0 required in the absence of correlation is

$$\frac{B}{B_0} = \frac{n_{\rm T}\,\zeta(\Theta_{\rm R}) + n_{\rm R}\,\zeta(\Theta_{\rm T})}{n_{\rm T}+n_{\rm R}}. \quad (8)$$

Conversely, if we fix R and B, the power penalty that results from correlation is

$$\frac{P|_{\rm dB} - P_0|_{\rm dB}}{3\,{\rm dB}} = \frac{R}{2B}\left(\frac{\zeta(\Theta_{\rm T})-1}{n_{\rm T}} + \frac{\zeta(\Theta_{\rm R})-1}{n_{\rm R}} \right)$$

which, in contrast with (8), depends on the operating point (R/B).

In arrays whose geometry follows a regular pattern (such as linear or circular), the antenna correlation between any pair of antennas is determined only by their relative positions and the corresponding correlation matrix is Toeplitz. Buttressing the results found in Example 2, the correlation number of such arrays can be shown to grow monotonically with the number of antennas if the spacing between adjacent ones is preserved [8]. Conversely, only a diminishing fraction of the uncorrelated capacity can be attained, if their spacing is help constant, as the number of antennas grows large. For $n_{\rm T}=n_{\rm R}=n$, consequently, the capacity in (5) is sublinear on n except in the absence of correlation.[6]

[6]In the presence of correlation, only asymptotically in the number of antennas does the capacity with correlation become linear on n [8], [15].

TABLE IV

MEASUREMENT SET-UP

carrier frequency	2.11 GHz
base station height	100 m
terminal height	1.5 m
antenna gain	4 dBi
mutual coupling	< -30 dB
Ricean K-factor	< 0.1

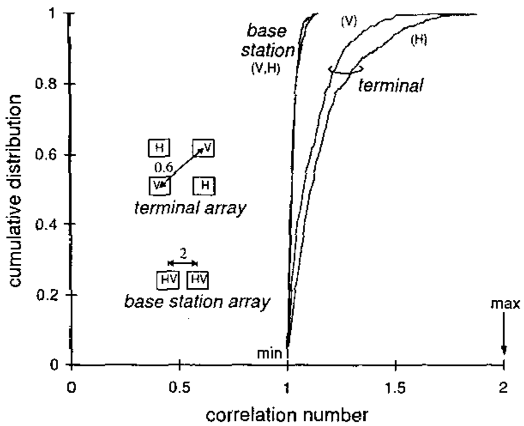

Fig. 3. Measured correlation numbers at base station and terminal for $n_T = n_R = 2$ as function of the polarization (V and H). Antenna spacing (wavelenths) is 2 and 0.6 at the base and terminal, respectively.

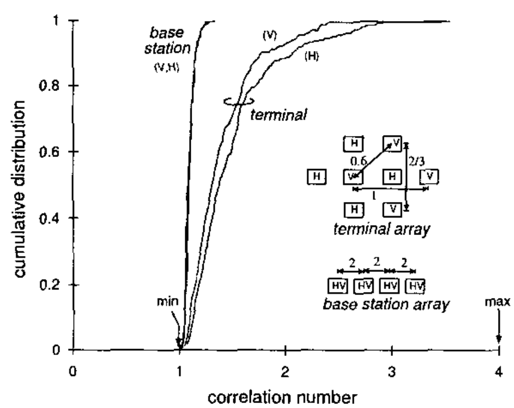

Fig. 4. Measured correlation numbers at base station and terminal for $n_T = n_R = 4$ as function of the polarization (V and H). Antenna spacing (wavelenths) is 2 at the base and 0.6−1 at the terminal, respectively.

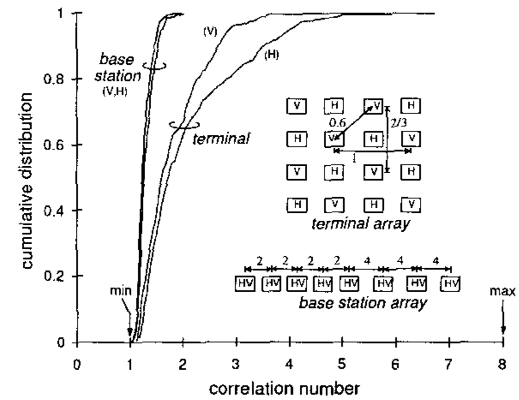

Fig. 5. Measured correlation numbers at base station and terminal for $n_T = n_R = 8$ as function of the polarization (V and H). Antenna spacing (wavelenths) is 2−4 at the base and 0.6−1.12 at the terminal, respectively.

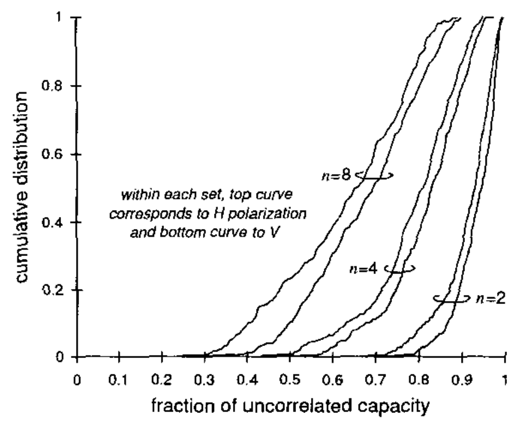

Fig. 6. Fraction of uncorrelated capacity attainable on either polarization with measured correlation matrices for n=2, 4, and 8. Antenna spacing (wavelenths) is 2−4 at the base and 0.6−1.12 at the terminal, respectively.

D. Experimental Evidence

In order to complement the analysis, we present experimental data gathered in New York City and reported in [18]. A narrowband transmitter, acting as a base station, was installed on a high-rise building while a receiver, mounted on the side of a van, was driven around Manhattan. Each array contained 16 antennas, 8 on each of two polarizations: vertical (V) and horizontal (H). A set of 367 pairs of base station and terminal correlation matrices were processed, with each polarization treated separately. For more details, see Table IV.

Fig. 3–5 display the distribution of transmit and receive correlation numbers with n=2, n=4, and n=8 active antennas on each polarization, respectively. The geometry of the arrays is also shown, with the minimum antenna spacing being 2 wavelengths at the base and 0.6 wavelengths at the terminal.

We observe that the base antennas exhibit extremely low levels of correlation while the terminal antennas display stronger correlation, clearly a consequence of the smaller spacing and the compact rectangular geometry. Also noteworthy is the fact that the horizontal polarization consistently exhibits a higher level of correlation.

Finally, Fig. 6 quantifies the fraction of uncorrelated capacity attainable, on each polarization, with n=2, n=4, and n=8. As anticipated (even though the correlation matrix of the terminal array is non-Toeplitz and that of the base array is only Toeplitz for n=2 and n=4), such fraction decreases monotonically with n. On average, over 90% of the uncorrelated capacity is attained with n=2 co-polarized antennas, but only about 65% with n=8. According to [18] a similar decrease is found at SNR=10 dB indicating that the validity of this result seemingly extends beyond the low-SNR regime.

IV. Summary

Using the capacity as function of the normalized energy per bit as a conduit, the degree of diversity of an antenna array—used at either transmitter or receiver—has been found to depend on a parameter, the *correlation number*, which is immediately obtained from the correlation between the array antennas. Besides enabling very valuable analytical insight, the correlation number emerges as a design criterion for diversity arrays. The process has been exemplified and further complemented with correlation numbers obtained experimentally during an extensive multi-antenna channel measurement campaign in New York City.

Acknowledgment

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