Decoupling of CDMA Multiuser Detection via the Replica Method

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Abstract

In CDMA, the optimal multiuser detector in mean square sense is the conditional mean estimator (CME) with respect to the input distribution. It outputs the expected value of the transmitted symbol conditioned on the received signal and the channel state (spreading sequences, fading, etc.). In this paper, we first note that every multiuser detector can be reformulated as the CME for a certain postulated CDMA channel with a postulated input distribution which may be different from the actual channel and inputs. Using the replica method developed in statistical physics, a class of generalized CME, referred to as the replica-averaged CME (RACME), can be obtained. The RACME is a natural extension of the replica-averaged maximum likelihood estimator (RAME) and has the same form for both single-user and multiuser channels. The RACME is a universal multiuser detector. It can be shown that the RACME detector is equivalent to the optimum detector in mean square sense for the channel with the postulated input distribution.

1 Introduction

In a full-duplex communication (FDMA), multiple-access interference (MAI) arises due to the non-orthogonal spreading sequences from all users. Numerous multiuser detection techniques have been proposed to mitigate the MAI to various degrees. Regardless of the input distribution, the optimal detector in mean square sense is the conditional mean estimator (CME), which outputs the expected value of the input symbol conditioned on the received signal and the channel state (i.e., the spreading sequences, the received signal amplitudes, etc.). In case of linear inputs, individual optimum detection which achieves the

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CDMA Multiuser Detection
The Replica Method

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Abstract

Multiuser detector in many square sense is the conditioning of the input distribution, which outputs unital symbols conditioned on the received signaling sequence, fading, etc. In this paper, we first vector can be reformulated as the CME for a certain with a postulated input distribution which may be different and inputs. Using the replica method developed of generalized CME first and applied to randomly a unified framework as the large-system limit. In this work, the single-user channel setup at the generator level is equivalent to a Gaussian channel followed by a The degradation factor in the effective SNR of the channel access interference is the multiuser efficiency, equal point equation. The spectral efficiency of such a separate channel is derived. Based on a general upper bound and lower bound, the multiuser efficiency is also applicable to MIMO channels such as systems.

CDMA, multi-access interference (MAI) arises due signals from all users. Neuronal multiuser detection schemes to the MAI to various degrees. Regardless of the MAI in CME is the conditional mean of the expected value of the input symbols conditioned a localized state (i.e., the spreading sequences, the received

3 Collaborative participation in the Communications and Information Technology Laboratories at Princeton University (DAAD19-03-1-0011). The U.S. Government is authorized to reproduce and distribute privately without limitation, any copyright notice attached signal amplitude, etc.). In case of binary inputs, the CME is a soft version of the individually optimum detector which achieves the minimum bit-error-rate (BER) [1,2].

A new step-by-step we may establish in this paper is that every multiuser detector can be regarded as a CME for a certain postulated CDMA channel and input distribution, which may be different than the actual channel and inputs. In other words, a multiuser detector is nothing but a generalized CME, which is the optimum detector for a postulated CDMA system, and henceforth may be suboptimal for the actual system to mismatch between the postulated and the actual probability laws.

The reformulation of multiuser detection as a generalized CME provides us with a unified framework in the treatment of multiuser communication problems. Indeed, by introducing an infinite-postulated channel and input distribution, it is possible to analyze a wide class of multiuser detectors and arrive at general conclusions. By tuning the postulated channel and inputs, the result can be easily particularized to user detectors of interest. Taking advantage of the above new formulation, we present in this paper a large-system characterization of the input-output relationship of a class of generalized CME with each applied to randomly spread CDMA. By a large system we refer to the limit that both the number of users and the spreading factor tend to infinity with a fixed ratio.

It has been shown that the output of a wide class of linear multiuser detectors conditioned on the input conveys a distribution in Gaussian random variables in the large-system limit [3]. The reason is that the MAI in the detection output is a superposition of interference and thus central limit theorems apply. Since the MAI seen by all users are statistically the same in the large-system limit regardless of the input distribution, and the individual signal-to-noise ratio (SNR), the large-system performance of linear detectors can be fully characterized by a single parameter, called multiuser efficiency, which is the degradation in the effective SNR due to the MAI. Fortunately, the multiuser efficiency of a finite-size linear system can be written as an explicit function of the singular values of the spreading matrix, the empirical distribution of which converges to a known Gaussian as the matrix size goes to infinity. Therefore, the large-system multiuser efficiency can be obtained as an integral with respect to the limiting singular-value distribution, which, by using the Stodolsky transformation, is found as the solution to a fixed-point equation (c.f. [1,3]). As far as linear multiuser detectors are concerned, the multiuser channel can be effectively decomposed into single-user Gaussian channels. Much less success has been reported in the application of random matrix theory and central limit theorems in analyzing multiuser detectors that fall out of the above group of linear detectors, mainly due to lack of explicit expressions of the performance measures in terms of the singular values. For instance, the optimum detection output is notoriously non-Gaussian in the large-system limit. In this paper, we study the large-system input-output relationship of a class of generalized CME that corresponds to an arbitrary postulated input distribution and a postulated CDMA channel that differs from the actual channel only in the noise variance. Surprisingly, it is found that under arbitrary inputs and fading, the single-user channel seen at the generalized CME output is equivalent to a scalar Gaussian channel followed by a strictly nonnegative decision function. Indeed, the fact that the output of multiuser detectors conditioned on the input relates to a Gaussian distribution through a deterministic (linear or nonlinear) function has been pointed out in a different context.

The concept of multiuser efficiency was first introduced in binary encoded transmission to refer to the degradation of the minimum bit-error-rate (BER) relative to a single-user channel calculated in the equivalent SNR limit.
long evaded discovery. It is also found in this paper that the effective SNR under this equivalent Gaussian channel is equal to the input SNR times the same multipath efficiency for all users, which achieves a first-order equivalent equation. By appropriate choice of the postulated input and noise variance, the results can be particularized to obtain the multipath efficiency of the matched filter, decorrelator, MMSE detector, as well as the penalty and individual optimum detectors. In all, the multipath system under sequence decoding can be effectively decoupled into single-user Gaussian channels that interact only through the multipath efficiency.

The foundation of the above simple large-system characterization is the so-called "self-averaging" property, namely, the dependence of the performance measures on the spreading sequences vanishes as the system size becomes without bound. This is a direct outcome of the asymptotic equivalence property (AEP). The "asympotic" property is also known as the free energy in statistical physics, is introduced in this paper using the replica method. The replica method has its origin in spin glass theory in statistical physics, and was first used by Ruga in a multithouse detector to obtain the large-order uncorrected maximum bit-error-rate and spectral efficiency with optimal inputs [5]. The replica method has been used successfully in many problems in statistical physics, as well as neural networks and coding theory, while rigorous proof of this replica method is an ongoing effort in mathematics and physics communities.

Because of the decoupling shown in this paper, the capacity of the single-user channel seen at the generalised CME output is equal to the mutual information of the equivalent scalar Gaussian channel under the same inputs. The spectral efficiency under optimum joint decoding is also derived and it is found that regardless of the input distribution, maximum-likelihood decoding with a CME front-end against the joint uncorrelated noise achieves the optimum spectral efficiency.

From a practical viewpoint, this paper presents new results on the performance of CDMA under arbitrary signaling such as m-PSK. More importantly, the MAB, which often exhibits very complicated structure, is characterized by a single parameter, the multipath efficiency. The spectral efficiency achievable by coded systems is also easily quantified by means of this parameter. Thus, our results offer valuable insights into the design and analysis of coded and uncoded CDMA systems.

2 CDMA and Multiuser Detection

Consider the K-user CDMA system with spreading factor N depicted in Fig. 1. At each interval the vector of input symbols from all users $X = \{X_1, \ldots, X_K\}$ contains independent identically distributed (i.i.d.) entries with distribution $p_X$, which has zero mean and unit variance. The individual instantaneous SIRs $\{T_{i,k}\}_{i,k}$ are i.i.d. with distribution $\mu_i$ of finite moments, hereafter referred to as the SIR distribution. Let the spreading sequence of user $k$ be denoted by $s_k = [s_{k1}, s_{k2}, \ldots, s_{Kn}]$, where $s_k$ is a real random variable with zero mean, unit variance, and finite moments. The $\sum_i$ spreading vector is denoted by $S = [\sqrt{N} s_1, \ldots, \sqrt{N} s_N]$. Assuming symbol synchronous transmission, we have the following memoryless CDMA channel:

$$Y = SX + W$$

where $W \sim \mathcal{N}(0, I)$. The characteristics of the channel

$$P_Y(X,Y|S) = (2\pi)^{-\frac{N}{2}} \exp \left(-\frac{1}{2} |Y|_2^2 \right)$$

The most efficient use of channel (1) in terms of mutual information, which is the capacity subject to a rate constraint, is achieved by optimum joint decoding, one often breaks the received vector into a set of separate decodings as shown in Fig. 1. In this case, the multiuser detector front-end is viewed as a single-user detector output sequence for an individual user in decoding, given the user's own information; hence the name.

One particular choice of the multiuser detector is

$$\hat{X}_k = E \{ X | Y, S \}$$

which achieves the minimum mean square error (MSE) with respect to the posterior probability distribution $p_X|Y$. Given a distribution $\psi$ of channel (4) through the Bayes formula,

$$p_X|Y \sim \psi$$

The CME can be generalized by taking the sum with respect to the posterior probability distribution, $p_X|Y$ may be different from the true one. Let the input $X$ of the postulated channel be $\psi$ and $p_X \psi$ a respective posterior probability distribution commonly denoted as

$$p_X \psi (x, Y, S) = (2\pi)^{-\frac{N}{2}} \exp \left(-\frac{1}{2} |Y|_2^2 \right)$$

By choosing an appropriate probability measure generated CME in many different multiuser detected channel differs from the true channel (1).
found in this paper that the effective SNR under this input to the MMSE detector is the same as before, the cost of the detector, and the performance of the receivers, the results can be related to the performance of the ML detector, decorrelator, and MRCSE channel. As the receiver order increases, the effective SNR increases without bound. This is a direct application of the effective SNR (EES) property. The “optimal rate” determined by the input to the MMSE detector is equal to the output of the equalizer in statistical theory in a statistically model for the multivariate Gaussian channel with correlated inputs [3].

The result of this problem in statistical physics and closely resemble the problem of three steps, as we know, while a rigorous proof of the result is beyond the scope of this paper. However, given in this paper, the capacity of the uncorrelated channel is the channel capacity, and it is equal to the mutual information of the equivalent system. The spectral efficiency under optimum and optimum input is not known. However, it is known that regardless of the input distribution, the channel capacity is achieved by a coded system, and the spectral efficiency of the system is also equal to the channel capacity.

Our results offer valuable insights into the uncorrelated channel. Furthermore, we also model the multiple-input multiple-output (MIMO) channel with higher-dimensional MIMO channels such as multiple-signalizing and various detection techniques.

**Linear Detection**

**Figure 1:** CDSMA channel with separate decoding.

we have the following inequalities (CDSMA channel):

\[ Y = SX + W \]

where \( W \sim \mathcal{N}(0, I) \). The characteristic of the channel is described as:

\[ p_{X,Y|S}(x,y|s) = \mathcal{N}(x|sY - s) \]

The most efficient use of channel (1) in terms of capacity is achieved by optimal joint decoding, where the total capacity subject to a certain input distribution is determined by the mutual information \( I(X; Y|S) \). Practically, due to the prohibitive complexity of joint decoding, one often uses the process to a multivariate detector front end followed by separate decoding as shown in Fig. 1. In this case, the CDSMA channel together with the multivariate detector front end is viewed as a single-source channel for each user. The detection output sequence for an individual user is used for decoding even this user's own information; hence the loss in capacity.

Our particular choice of the multivariate detector is the conditional mean estimator:

\[ \hat{X} = \mathbb{E}(X|Y, S) \]

which achieves the minimum mean square error. Hereafter, angle brackets denote expectation with respect to the posterior probability distribution \( p_{X|Y, S} \), which is determined by the input distribution \( p_S \) and the conditional Gaussian density function \( p_{X|Y, S} \) of channel (1) through the Bayes formula. The CME can be generalized to taking the conditional expectation as in (3) with respect to the posterior probability distribution of a “postulated” CDSMA system which may be different from the true one. Let the input distribution and channel characteristic of the postulated channel be \( p_S \) and \( p_{X|Y, S} \) respectively, which, in turn, determines the postulated posterior probability distribution \( p_{X|Y, S} \). The generalization CME is conveniently denoted as:

\[ \hat{X} = \mathbb{E}(X|Y, S) \]

By choosing an appropriate probability measure \( q \), it is possible to particularize the generalized CME to many different multivariate detectors of interest. In this paper, the postulated channel differs from the true channel (1) by only the noise variance:

\[ p_{X|Y, S}(x|y, s) = \mathcal{N}(x|sY - s) \]

\[ \hat{X} = \mathbb{E}(X|Y, S) \]

\[ \text{Var}(X) \]

\[ \text{Var}(Y) \]
Here, \( \mu \) serves as a control parameter. Also, the postulated inputs are linked with an arbitrary input distribution \( q_Y \) of zero-mean and unit variance.

Suppose that the postulated input distribution \( q_Y \) is standard Gaussian. It is not difficult to see that the generalized CME (4) outputs a linear filtering of the received signal:

\[
\mathcal{L}(X) = S^T S + \mu^T \mu - S^T Y.
\]

If \( \sigma \to \infty \), the generalized CME estimate is consistent with the matched filter output. If \( \sigma = 1 \), (6) is exactly the soft output of the linear MMSE-detector. If \( \sigma \to \infty \), (6) converges to the output of the decorrelator.

Alternatively, suppose that the postulated input distribution \( q_Y \) is identical to \( p_Y \). If \( \sigma = 1 \), then \( \lim_{\sigma \to \infty} \mathcal{L}(X) \) is the jointly optimal (or maximum likelihood) detection (2). If \( \sigma = 1 \), the smoothed measure \( q \) coincides with the true measure \( p \), and the CME outputs \( \mathcal{L}(X) \) is a soft version of the individually optimum multiuser detector (2). Also worth mentioning is that, if \( \sigma \to \infty \), the generalized CME reduces to the matched filter.

3 Decoupling of CDMA Multiuser Detection

3.1 Main Results

Consider a canonical scalar Gaussian channel:

\[
Z = \sqrt{r} X + \frac{1}{\sqrt{r}} W
\]

where \( \Gamma > 0 \) is the input SIR, \( \sigma > 0 \) the inverse noise variance and \( W \sim \mathcal{N}(0,1) \). Given \( \zeta > 0 \), we consider also a postulated Gaussian channel with input SIR \( \Gamma \) and inverse noise variance \( \zeta \). Let the input distribution to this postulated channel be \( q_Y \).

Then the underlying measure of the postulated channel is \( q_Y \). The straightforward estimate of the postulated channel is characterized by the posterior probability distribution \( q_Y \), namely, it takes in an input \( X \) and outputs a random variable \( X \) according to \( q_Y \).

The generalized CME estimate of \( X \) given \( Z \) is therefore:

\[
\mathcal{L}(X) = \mathcal{L}(X|Z; \Gamma) = \frac{1}{\sqrt{r}} \log \mathcal{N}(x|\Gamma \zeta - 1, \Gamma \zeta - 1).
\]

Consider now a concatenation of the scalar Gaussian channel (7) and the retrochannel of the postulated channel as depicted in Fig. 2. The generalized CME is also included.

Let the input to the Gaussian channel (7) be denoted by \( X \). To deconvolve it from the retrochannel output \( X' \). We define the mean square error of the variables respectively:

\[
\varepsilon(U, \Gamma) = E \left[ (X - \lambda)^2 \right] \quad \varepsilon(W, \zeta) = E \left[ (X' - \lambda)^2 \right]
\]

Claim 1: Let the generalized CME of the CDMA postulated input distribution \( q_Y \) and noise variance, the distribution of the multiuser detector output \( \lambda \) (obtained with SIR \( \Gamma \)) is identical to the distribution of the equivalent scalar Gaussian channel (7) and, with input SIR \( \Gamma \) = \( \Gamma_1 \) (where the multiuser efficiency \( \varepsilon \) of the postulated scalar channel satisfy the coupled)

\[
\varepsilon^2 = 1 + \beta \varepsilon \mathcal{N}(0, \Gamma)
\]

where the expectations are taken over the SIR distribution \( (\mu, \sigma) \) are chosen to maximize the free energy

\[
\mathcal{F} = -E \left[ \sum_{i=1}^{M} \log (\lambda_i) - \log \mathcal{N}(\lambda_i|\Gamma \zeta - 1, \Gamma \zeta - 1) \right] + \frac{1}{2} \log \mathcal{N}(\lambda|\Gamma \zeta - 1, \Gamma \zeta - 1) - \log
\]

Claim 1: asserts that the multiple-access channel can be decoupled into scalar Gaussian channels in the sense that the multiuser CME is asymptotically a function of a Gaussian random variable. It is in general numerically. Multiple-access may exist, which phase transitions in statistical physics. Among these is the case that gives the smallest value of the free operational meaning in the communication problem.

The design function (i.e., the CME) in Fig. 2 is sequential in both detection and information theory.

Corollary 1: In the large-system limit, the sampled CME output is equal to the input-output mutual information of the Gaussian channel (7) with the same input distribution variance \( \sigma \) as the multiuser efficiency determined by

\[
\mathcal{F}(\gamma) = \log \mathcal{N}(\gamma|\Gamma, \Gamma)
\]

Clearly, the overall spectral efficiency under this joint decoding gain is given by the following:

\[
\mathcal{E}(\gamma) = \mathcal{F}(\gamma)
\]

\[ \text{Divergence} = \sum p(x) \log \left( \frac{p(x)}{q(x)} \right) \]

\[ p(x) = \text{Post-Process Distribution} \]

\[ q(x) = \text{Relative Entrapment of the Metric} \]

\[ \text{Mean Entrapment of the Metric} \]

\[ \text{Optimal Separating Decision} \]

\[ C_\text{opt}(\cdot) = \arg \min_{C} \mathbb{E} \left[ \text{Cost}(\cdot) \right] \]

\[ \text{Cost}(\cdot) = \text{Divergence} \]

\[ \text{Optimal Spectral Efficiency} \]

\[ \text{Joint Decoding Gain} \]

\[ \text{Joint Decoding Gain} \geq \text{Optimal Spectral Efficiency} \]

\[ C_\text{opt}(\cdot) = C_\text{opt}(\cdot) \]
Claim 2 The gain of optimum joint decoding over the multuser CME followed by separate decoding in the large-system spectral efficiency of the CDMA channel (1) is
\[
\frac{C_{\text{CME}}(\theta) - C_{\text{Mona}}(\theta) - \log(\theta) - \log(\theta) = D(N(0, \theta)) + N(0, 1))}{10}.
\]
where $\theta$ is the CME multuser efficiency.

The expression (15) coincides with the expression found originally in [6] in the case of Gaussian inputs and later in [7] in the case of binary inputs. Interestingly, the spectral efficiencies under joint and separate decoding are also related by the following generalization of a result in [6].

Proposition 1 Under every equal distribution $p_x$,\n\[
C_{\text{Mona}}(\theta) = \int_0^\theta \frac{1}{f} C_{\text{Mona}}(\theta') d\theta'.
\]

3.2 Discussions
We can receive an interference decoder that decodes the users successively in which the symbol decoded symbols are fed to a new channel with the same channel information. Suppose the users are decoded in reverse order, then the generalized CME for user $k$ is only $k - 1$ interfering users. Hence the performance for user $k$ under successive decoding is identical to that of the CME applied to a CDMA system with $k$ instead of $N$ users. The multuser efficiency experienced by user $k$ is $\bar{\gamma}(k)$ where we use the fact that it is a function of the load $\bar{\gamma}$ for the generalized CME for user $k$. It is easy to see that the overall spectral efficiency converges almost surely:
\[
\frac{1}{N} \sum_{k=1}^N \left( \frac{1}{d} \log(\bar{\gamma}(k)) \right) = \mathbb{E} \left( \int_0^\theta C(\Gamma, \theta) d\theta \right).
\]

Note that the above result on successive decoding is true for arbitrary input distribution and generalized CME detection. In the special case of the CME, for which the postulated inputs and channel are identical to the actual input and channel, the right-hand side of (17) is equal to $C_{\text{CME}}(\theta)$ by Proposition 1. We can summarize this principle as

Proposition 2 In the large-system limit, successive decoding with a CME front end against joint multihop users achieves the optimal CDMA channel capacity under arbitrary input distributions.

Proposition 2 is a generalization of the previous result that a successive decoder with a linear MMSE front end against multihop users achieves the capacity of the CDMA channel under Gaussian inputs [6, 7]. Indeed, this result is an outcome of the chain rule of mutual information, which holds for all inputs and arbitrary number of users.

3.3 Analysis via the Replica Method
The statistical-inference problem faced by the decoder is depicted in Fig. 3. The input and output of the channel $y_{\text{sys}}$, under state $S$, is denoted by $x_{\text{sys}}$ and $y_{\text{sys}}$ respectively. A

\[\begin{align*}
\text{Source} & : \quad p_x \quad & \text{Decoder} & : \quad p_{\text{out}} \quad & \text{Input} & : \quad y_{\text{sys}}
\end{align*}\]

Figure 3: Canonical channel, notations

generic vector $y_{\text{sys}}$ as input of a parallel input $X$ with distribution $p_x$. The meaningful way to find the expected value of $X$ conditioned on $(Y_{\text{sys}})$ postulated measure differs than $\theta$, may be other statistics or a particular choice that corresponds to the postulated channel, $\rho_{\text{sys}}$. Outputs a random variable $X$ according to $p_x$ output $(X_{\text{sys}})$ is the expected value of the realization $X$. The policy studies the distribution of the input $X_{\text{sys}}$ with a large-system limit, where both for a factor $K$ need to infinity but with $K/N$ converging. $X_{\text{sys}}$ to denote the input to distinguish it from the input-output relationship is independent on the system, it becomes increasingly predictable from the realization of the spreading matrix of the system, is known as the self-averaging property in statistics of the asymptotic eigenvalue property. In the case of non-self-averaging properties, the spreading sequences, microscopic quantifications on the spectral efficiency, are averaged over data, over the large-system limit.

In order to recover the input-output relationship, denoted on the channel state:\n\[
\begin{align*}
E \left( x_{\text{sys}} \cdot X_{\text{sys}} \right) & \cdot S, \\
\text{theorem} \text{of the reversed order} X_{\text{sys}}, \text{denoted as} X_{\text{sys}} \cdot \left( X_1, \ldots, X_N \right) \text{is a Markov chain, it can be shown moments of the head and tail of the Markov chain:}
\end{align*}
\]

\[
E \left( x_{\text{sys}} \cdot X_{\text{sys}} \right) = \frac{1}{N} \mathbb{E} \left( X_{\text{sys}} \right).
\]

which can be evaluated by integrating over the distributions $p_{\text{out}}(X_{\text{sys}} | S)$. According to the AEP, the randomness in the $X_{\text{sys}}$, independent of $S$, the large-system limit, the $X_{\text{sys}}$.

\[
\frac{1}{N} \mathbb{E} \left( \log p_{\text{out}}(X_{\text{sys}} | S) \right).
\]

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\[ P_X(\theta) = \int \frac{1}{2} P_{\text{lim}}(f) \, df. \] (16)

\[ P_Y(\theta) = \int \frac{1}{2} P_{\text{lim}}(f) \, df. \] (17)

**Figure 3:** Canonical channel, retrochannel and generalized CME.

For the specific case of the canonical channel, the right-hand-side of the previous result is a convex sum of unrecorded users where the capacity of the CDMA channel is [2]. Indeed, this result is meaningful of the channel holds for all inputs and arbitrary number of users.

**Propose Method**

**Fixed**: In the diagram, the state \( S \) is denoted by \( X_0 \) and \( Y \) respectively. A 1 agrees with the set of information sources.

\[ E \{ X_0, Y \} = \begin{cases} 0 & x_0 = 0, \ldots, 1 \end{cases} \] (18)

\[ E \{ X_0, Y \} = \begin{cases} 0 & x_0 = 0, \ldots, 1 \end{cases} \] (19)

\[ \frac{1}{R} \log q_Y(Y|S) \] (20)
scales as \( K \to \infty \). The limit is known as the free energy in statistical physics, which is defined by:

\[
\mathcal{F} = \lim_{K \to \infty} \left( \frac{1}{K} \log \mathbb{E} \left[ \exp \left( \frac{1}{K} \log \mathbb{P}_c (Y | S) \right) \right] \right),
\]

(21)

Essentially, in the large-system limit, almost all realizations of the received signal are "typical", and it suffices to calculate the moments in the typical set. The expected value of the logarithm in (21) is an open problem, which can be formulated equivalently as:

\[
\mathcal{F} = \lim_{K \to \infty} \frac{1}{K} \log \mathbb{E} \left[ \exp \left( \frac{1}{K} \log \mathbb{P}_c (Y | S) \right) \right].
\]

(22)

Using the replicas of the retroactive channel output, we can evaluate

\[
- \lim_{K \to \infty} \frac{1}{K} \log \mathbb{E} \left[ \exp \left( \frac{1}{K} \log \mathbb{P}_c (Y | S) \right) \right] = \lim_{K \to \infty} \frac{1}{K} \log \mathbb{E} \left[ \prod_{i=1}^{K} \mathbb{P}_c (y_i | x_i, S) \right]
\]

(23)

as a function of the integer replica number \( n \). Equations in statistical physics, the replica trick asserts that the order of the limit and the derivative in (22) can be exchanged, and that the resulting expression from (23) is also valid as long as the vicinity of \( n = 0 \). Therefore, this is asymptotic at \( n = 0 \) in the free energy. Note that \((Y, S)\) is indexed by the transmitted symbols \( X_i \), by taking expectation over \( Y \) first and then averaging over the \( S \), one finds that

\[
- \frac{1}{K} \log \mathbb{E} \left[ \exp \left( \frac{1}{K} \log \mathbb{P}_c (Y | S) \right) \right] = - \frac{1}{K} \log \mathbb{E} \left[ \exp \left[ \frac{1}{K} \mathbb{P}_c (Q) - G^c (Q) \right] \right]
\]

(24)

where \( G^c (Q) \) is some function of the SIRs, the transmitted symbols and their replicas. By first conditioning on the \((x + 1) \times (x + 1)\) correlation matrix \( Q \) of the replicas, the central limit theorem helps to reduce (23) to

\[
- \frac{1}{K} \log \mathbb{E} \left[ \exp \left[ \frac{1}{K} \mathbb{P}_c (Q) - G^c (Q) \right] \right] = - \frac{1}{2} \frac{\partial^2}{\partial Q^2} \mathbb{E} \left[ \log \mathbb{P}_c (Q) \right].
\]

(25)

where \( \mu_{Q}^2 \) is some function of the correlation matrix \( Q \), and \( \mu_{Q}^2 \) is its probability measure. Large deviations can be invoked to show that (25) converges as \( K \to \infty \) to

\[
- \frac{1}{2} \frac{\partial^2}{\partial Q^2} \mathbb{E} \left[ \log \mathbb{P}_c (Q) \right] = \mathcal{F} (Q),
\]

(26)

where \( \mathcal{F} (Q) \) is the rate function of the measure \( \mu_{Q}^2 \) [16].

To overcome the exponential (26) over a \((x + 1)^2\)-dimensional space is a hard problem. The technique to circumvent this is to assume replica symmetry, namely, that the supremum in \( Q \) is symmetric over all a-priori distributed. The resulting expression is then over a few parameters, and the free energy can be calculated.

The joint moments (19) of the input and the retroactive channel output can be evaluated for all "typical" realizations of the received signal. The result is that \( (X_i) \) can be regarded as the generalized conditional mean estimate of a scalar Gaussian channel with input \( X_i \), hence the proof of Claim I. Given the input distribution \( \mu_x \), the total capacity under optimum joint decoding can be

\[
\mathcal{C}_{\text{total}} = \mathcal{F} (Q) - \frac{1}{2} \log (2\pi e).
\]

(27)

Claim 2 is proved by finding the free energy under the assumption that the postulated measure \( \mathbb{P}_c \) is identical to the actual measure \( p \).

4 Conclusion

Using the replica method, a family of specialized cut-off the large-system limit, which includes well-known asymptotics. MMSE detector, the jointly and fixed result is the decoding of Gaussian CDMA clean detect, limit and into scalar Gaussian channels. A the spectral efficiency of CDMA channels expressed. It is straightforward to particularize the results or as an in QAM, which can be useful not only for the important special case of the canonical single-user.

References


is known as the free energy in statistical physics, which
\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \left[ \exp \left( J \sum_{i=1}^{N} Y[S_i] \right) \right] = \frac{1}{k} \log \mathbb{E} \left[ \exp \left( J \sum_{i=1}^{N} Y[S_i] \right) \right]
\]
(21)
limit, almost all realizations of the received signal are
the moments in the typical set. The expected value
problem, which can be formulated equivalently as
\[
\lim_{N \to \infty} \frac{1}{N} \log \mathbb{E} \left[ \exp \left( J \sum_{i=1}^{N} Y[S_i] \right) \right]
\]
(22)
and output, \( \mathbb{E} \) can evaluate
\[
\mathbb{E} \left[ \exp \left( J \sum_{i=1}^{N} Y[S_i] \right) \right] = \exp \left( \sum_{i=1}^{N} \mathbb{E} \left[ Y[S_i] \right] \right)
\]
(23)
under \( n \). Directions in statistical physics, the exp-
unstable, and the derivative in (22) cannot be exchanged, and
in (25) is also valid at least in the vicinity of \( n = 0 \), and
\( n = 0 \) at the free energy. Note that \( (Y[S]) \) is induced
By taking expectation over \( Y \) first and then averaging
\[
\frac{1}{k} \log \mathbb{E} \left[ \exp \left( J \sum_{i=1}^{N} Y[S_i] \right) \right] = \frac{1}{k} \log \mathbb{E} \left[ \exp \left( J \sum_{i=1}^{N} Y[S_i] \right) \right]
\]
(24)
\( \mathbb{E} \), the transmitted symbols and their replicas. By
\( (n+1) \)-correlation matrix \( Q \) of the replicas, the central
\[
\mathbb{E} \left[ \exp \left( J \sum_{i=1}^{N} Y[S_i] \right) \right] = \exp \left( \sum_{i=1}^{N} \mathbb{E} \left[ Y[S_i] \right] \right)
\]
(25)
the correlation matrix \( Q \) and \( \rho_{\beta}^{(n)} \) is its probability
\( (J \sum_{i=1}^{N} Y[S_i]) \) converges as \( N \to \infty \) to
\[
(1 - J \sum_{i=1}^{N} Y[S_i] - \exp(J \sum_{i=1}^{N} Y[S_i]))
\]
(26)
of the measure \( \rho_{\beta}^{(n)} \) [18]. Seeking the extremum (26) is a hard problem. The technique to circumvent this
\( (J \sum_{i=1}^{N} Y[S_i]) \) converges as \( N \to \infty \) to
\[
\frac{1}{k} \log \mathbb{E} \left[ \exp \left( J \sum_{i=1}^{N} Y[S_i] \right) \right] = \frac{1}{k} \log \mathbb{E} \left[ \exp \left( J \sum_{i=1}^{N} Y[S_i] \right) \right]
\]
(27)
energy under the assumption that the postulated
\[
\delta \mathcal{F}_{\text{approx}} = -\frac{1}{2} \log(2\pi e)
\]
4 Conclusion
Using the replica method, a family of generalized conditional mean estimators is studied in the large-system limit, which includes well-known detectors such as the matched filter, decorrelator, MMSE detector, the jointly and individually optimal detector. One major result is the decoupling of Gaussian CDMA channel correlated with a multiuser detector front end into scalar Gaussian channels. Another result is several formulas for the spectral efficiency of CDMA channels expressed in terms of the multiuser efficiency. It is straightforward to particularize this result to any practical input constraints, such as QAM, which can be useful not only for the design and analysis of CDMA, but for the important special case of the canonical single-user multiantenna array channel.

References

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