Fading Channels in the Power Limited Regime

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Abstract — **The tradeoff of spectral efficiency vs energy-per information bit (normalized to the noise level E_b/N_0) is the key measure of the capacity of channels in the power-limited regime. This paper finds the fundamental bandwidth-power tradeoff of a general class of channels in the wideband regime in which the spectral efficiency is small but nonzero.**

I. Introduction

A wide variety of digital communication systems (particularly in wireless, satellite, deep-space, and sensor networks) operate in the power-limited region where both spectral efficiency (b/s/Hz) and energy-per-bit are relatively low. The information theoretic analysis of those channels, in addition to to leading to the most efficient bandwidth utilization, reveals design insights on good signaling strategies.

The following conclusions about signaling and capacity in the wideband regime have been drawn in the literature:

- On-off signaling approaches capacity as the duty cycle vanishes.
- The derivative at zero signal-to-noise ratio of the Shannon capacity determines the wideband fundamental limits.
- Capacity is not affected by fading.
- Receiver knowledge of channel fade coefficients is useless.

In this paper we show that these conclusions, originally obtained under the assumption of infinite bandwidth do not carry over to the more important case in which bandwidth is large but finite.

II. Spectral Efficiency vs. E_b/N_0

In this paper, we deal with additive-noise channels in a general setting which allows other random channel impairments such as fading. Consider the following discrete-time channel with m complex dimensions:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}, \tag{1}$$

where the real and imaginary parts of the noise components are independent and satisfy

$$E[||\mathbf{n}||^2] = mN_0, \tag{2}$$

$\mathbf{H}$ is an $m \times n$ complex matrix whose random coefficients have finite second moments, independent real and imaginary parts, and are independent of $\mathbf{x}$ and $\mathbf{n}$.

Our approach is to analyze the first-order behavior of the spectral efficiency vs $\frac{E_b}{N_0}$ function in the wideband limit:

$$\begin{aligned} 10\log_{10}\frac{E_b}{N_0}(\mathsf{C}) &= 10\log_{10}\frac{E_b}{N_0}_{\min} \\ &+ \frac{\mathsf{C}}{\mathcal{S}_0}10\log_{10}2 \\ &+ o(\mathsf{C}), \quad \mathsf{C}\to 0 \end{aligned} \tag{3}$$

where $\frac{E_b}{N_0}_{\min}$ denotes the minimum $\frac{E_b}{N_0}$ required for reliable communication, and $\mathcal{S}_0$ denotes the slope of spectral efficiency in b/s/Hz/3 dB at the point $\frac{E_b}{N_0}_{\min}$:

$$\mathcal{S}_0 \stackrel{\text{def}}{=} \lim_{\frac{E_b}{N_0}\downarrow\frac{E_b}{N_0}_{\min}} \frac{\mathsf{C}(\frac{E_b}{N_0})}{10\log_{10}\frac{E_b}{N_0} - 10\log_{10}\frac{E_b}{N_0}_{\min}} 10\log_{10}2 \tag{4}$$

III. Minimum $\frac{E_b}{N_0}$

Since $C(\mathsf{SNR})$ is a monotonically increasing function, and

$$\mathsf{SNR} = \frac{E_b}{N_0}C(\mathsf{SNR}) \tag{5}$$

we have

$$\frac{E_b}{N_0}_{\min} = \lim_{\mathsf{SNR}\downarrow 0}\frac{\mathsf{SNR}}{C(\mathsf{SNR})} = \frac{\log_e 2}{\dot{C}(0)} \tag{6}$$

where

$\dot{C}(0)$ = derivative at 0 of $C(\mathsf{SNR})$ computed in nats/dimension.

Theorem 1. Consider the m-dimensional complex channel

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \tag{7}$$

where the complex Gaussian vector $\mathbf{n}$ has independent identically distributed components and satisfies (2). Then, the required received energy per bit for reliable communication satisfies

$$\frac{E_b^{\mathrm{r}}}{N_0}_{\min} = \log_e 2 = -1.59 \text{ dB}, \tag{8}$$

regardless of whether $\mathbf{H}$ is known at the transmitter and/or receiver.

One caveat that should be observed regarding Theorem 1, is that it was obtained enforcing the same second-moment constraint over all transmitted symbols. However, if the channel matrix $\mathbf{H}$ changes from symbol to symbol and its values are known instantaneously at the transmitter it is often feasible to do *power control* at the transmitter whereby the instantaneous transmitted power depends on the actual realization of $\mathbf{H}$. Then, the optimum power control strategy only spends energy when the channel conditions are particularly favorable. If the magnitude of the entries in $\mathbf{H}$ follows a distribution with unbounded support, then the transmitted $\frac{E_b}{N_0}$ can be made as small as desired.

IV. Wideband Slope b/s/Hz/3 dB

Theorem 2. At $\frac{E_b}{N_0}_{\min}$, the slope of the spectral efficiency vs. $\frac{E_b}{N_0}$ in b/s/Hz per 3 dB is given by

$$\mathcal{S}_0 = \frac{2\left[\dot{C}(0)\right]^2}{-\ddot{C}(0)} \tag{9}$$

with $\dot{C}$ and $\ddot{C}$, the first and second derivative, respectively, of the function $C(\mathsf{SNR})$ computed in nats.

Theorem 3. Consider the m-dimensional complex channel

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \tag{10}$$

where the complex Gaussian vector $\mathbf{n}$ has independent identically distributed components. Suppose that both the transmitter and the receiver know $\mathbf{H}$, then (without transmitter power control),

$$\mathcal{S}_0 = \frac{2\ell}{m\,\kappa(\sigma_{\max}(\mathbf{H}))} \tag{11}$$

with the *kurtosis* of a random variable Z defined as

$$\kappa(Z) = \frac{E[Z^4]}{E^2[Z^2]}, \tag{12}$$

$\sigma_{\max}(\mathbf{H})$ denotes the maximal singular value of $\mathbf{H}$, and ℓ is equal to the multiplicity of $\sigma_{\max}(\mathbf{H})$.

Kurtosis is a measure of the randomness of a random variable; its minimum value is 1, achieved uniquely by a deterministic variable. The fading penalty on capacity is due to the concavity of the $\log(1+x)$ function. The larger the "spread" of the fading distribution, the larger is the penalty. Theorem 3 states that in the low spectral efficiency region, the required bandwidth is proportional to the kurtosis of the maximal singular value of the channel. If the number of rows and columns of $\mathbf{H}$ grow, while keeping a constant ratio, and its coefficients are independent identically distributed, then the maximal singular value converges to a deterministic constant, and its multiplicity goes to 1. Accordingly, if m represents the number of receiving antennas, in the limit the slope is 2 b/s/Hz/(3 dB), i.e., the same value obtained with one antenna but without fading.

Theorem 4. Consider the m-dimensional complex channel

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \tag{13}$$

where the complex Gaussian vector $\mathbf{n}$ has independent identically distributed components. Suppose that the receiver knows the $m \times n$ matrix $\mathbf{H}$, but the transmitter has no knowledge of the channel matrix. Then

$$\mathcal{S}_0 = \frac{2}{m}\frac{\left(\mathrm{trace}(E[\mathbf{H}^\dagger\mathbf{H}])\right)^2}{\sum_{i=1}^{n}\sum_{j=1}^{n} E[|(\mathbf{H}^\dagger\mathbf{H})_{ij}|^2]}. \tag{14}$$

In the multiantenna literature it is common to model the entries of $\mathbf{H}$ as independent zero-mean zero-mean complex Gaussian random variables. Under those assumptions, in the wideband regime, the spectral efficiency is a multiple of the harmonic mean of the number of receive and transmit antennas:

$$\mathcal{S}_0 = \frac{2nm}{m+n}\text{b/s/Hz/(3 dB)}. \tag{15}$$

While receiver side information of the channel fading does not improve $\frac{E_b}{N_0}_{\min}$, it has a drastic effect on the required bandwidth in the wideband regime as the following result shows.

Theorem 5. Consider the $m = n = 1$ Ricean fading channel

$$\mathsf{y} = (\bar{\mathsf{h}} + \mathsf{g})\mathsf{x} + \mathsf{n} \tag{16}$$

where $\bar{\mathsf{h}}$ is deterministic, g is zero-mean complex Gaussian with variance γ^2, and the additive noise is also Gaussian. Then, the wideband slope is equal to (regardless of transmitter side information)

$$\mathcal{S}_0 = \frac{1}{1 - \frac{1}{2}\left(1 + \frac{\gamma^2}{|\bar{\mathsf{h}}|^2}\right)^{-2}}, \tag{17}$$

if the receiver knows the Rayleigh-distributed channel coefficients, and

$$\mathcal{S}_0 = 0, \tag{18}$$

for all $\bar{\mathsf{h}}$, and $\gamma > 0$, if the receiver does not know the channel coefficients.

Theorem 5 points out that it is very demanding in terms of bandwidth to achieve $\frac{E_b^r}{N_0}_{\min}$ close to -1.59dB in the Rician channel, regardless of the relative strengths of the specular and Rayleigh components. In many practical cases in which the specular component is not negligible, QPSK is an attractive suboptimal alternative as the following result shows.

Theorem 6. Consider the Rician channel (16) with $\bar{\mathsf{h}} \neq 0$ and a receiver that does not know the Rayleigh coefficients. Then QPSK achieves

$$\frac{E_b}{N_0}_{\min} = \frac{\log_2 e}{|\bar{\mathsf{h}}|^2} \tag{19}$$

and

$$\mathcal{S}_0 = 2, \tag{20}$$

i.e., the same wideband performance as if the Rayleigh component were absent.