Stability Properties of Slotted Aloha with Multipacket Reception Capability

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Abstract—**The stability of the Aloha random access algorithm in an infinite-user slotted channel with multipacket reception capability is considered. This channel is a generalization of the usual collision channel, in that it allows the correct reception of one or more packets involved in a collision. The number of successfully received packets in each slot is modeled as a random variable which depends exclusively on the number of simultaneous attempted transmissions. This general model includes as special cases channels with capture, noise, and code division multiplexing. It is shown by means of drift analysis that the channel backlog Markov chain is ergodic if the packet arrival rate is less than the expected number of packets successfully received in a collision of n as n goes to infinity. Finally, the properties of the backlog in the nonergodicity region are examined.**

I. Introduction

One of the main problems in random access communications is the determination of the maximum stable throughput. In particular, an important result is that the Aloha protocol is unstable [1]–[3] in an infinite-user slotted collision channel where a transmission is successful only if no other users attempt transmissions simultaneously. Several strategies have been designed to stabilize this channel, such as collision resolution algorithms (see [4], for example) where transmissions are deferred until the current conflict is solved, and more recently, Aloha-type strategies using decentralized control, where the retransmission probability is updated according to previous channel outcomes. It has been shown [5]–[7] that the maximum stable throughput achievable by such Aloha-type strategies with decentralized control is e^{-1}.

However, the collision channel model does not hold in many important practical multiuser communication systems [8]–[21] because simultaneous transmission of several packets does not necessarily result in the destruction of all the transmitted information. For instance, the capture phenomenon is common in local area radio networks [12]–[15]; if the power of one of the received packets is sufficiently large compared to the power of the other packets involved in a collision, then the strongest packet can be correctly decoded, while the other packets are lost. Other examples are multiple-access channels where several users transmit simultaneously in the same frequency band, and a multiuser detector demodulates the information transmitted by all active users (e.g., [8]–[11]). Although those systems do not necessarily require a random access protocol, it is sometimes useful to exercise some flow control through such a protocol so as to limit the maximum number of simultaneous transmitters, in order to bound the multiuser receiver complexity and guarantee lower bit-error rates.

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Previous studies of some of the aforementioned systems [9], [12]–[18] where some of the packets involved in a collision may be correctly received have shown that the performances are noticeably improved with respect to slotted Aloha. However, even in those special cases, no precise stability result is available, either because finite population networks with no buffer space were considered, or because the Poisson approximation of channel traffic was used for infinite population networks. In [19] (see also [20]), upper and lower bounds are derived for the capacity of a multiple access channel where all packets are correctly received if the collision size does not exceed a fixed threshold and otherwise all packets are destroyed.

In this paper, we consider a generalization of the collision channel, where the receiver can demodulate several packets simultaneously. It is assumed that the number of correctly demodulated packets is a random variable, which, given the number of packets simultaneously transmitted, is independent of the backlog and of the number of previous retransmission attempts. This random variable can take any integer value between zero and the collision size. Thus, the channel is described by a matrix of conditional probabilities (ϵ_{nk}) where ϵ_{nk} is the probability that k packets are correctly demodulated given that there were n simultaneous transmissions. We analyze the usual Aloha algorithm with the multipacket reception capability just described. Users are synchronized so that transmissions take place within one slot, and at the end of each slot, stations that did transmit a packet learn whether or not their transmission was successful. Unsuccessful or backlogged packets are retransmitted in each subsequent slot with probability p, $0 < p \leq 1$. It turns out that multipacket reception capability can stabilize Aloha. Our main result states that the maximum stable throughput is equal to the limit of the average number of packets correctly received in collisions of size n when n goes to infinity. To show this, we model the channel backlog as a Markov chain, and then study its properties by using some simple drift analysis techniques.

The last part of this paper is a study of the properties of the backlog in the nonergodicity region. Unlike the backlog Markov chain for slotted Aloha which is always transient [1], the backlog for our model does in general have a null recurrence region of positive length, which depends on the matrix (ϵ_{nk}) and on the retransmission probability p. However, transience in the nonergodicity region can be ensured for a large class of systems, and in particular for channels where the number of successful simultaneous transmissions is bounded.

II. Multipacket Reception Model

Let A_k be the number of new packets arriving during time slot k. Assume that $(A_k)_{k\geq 0}$ are i.i.d. random variables with probability distribution:

$$P[A_k = n] = \lambda_n \qquad (n \geq 0)$$

such that the mean arrival rate $\lambda = \sum_{n=1}^{\infty} n\lambda_n$ is finite. New packets are transmitted with probability one at the beginning of the first slot following their arrival.

Given that n packets are being transmitted in one slot, we define for $n \geq 1, 0 \leq k \leq n$

$$\epsilon_{nk} = P[k \text{ packets are correctly received} \mid n \text{ are transmitted}].$$

The multipacket reception properties of the channel are summarized by the stochastic matrix

$$E = \begin{bmatrix} \epsilon_{10} & \epsilon_{11} & & & \\ \epsilon_{20} & \epsilon_{21} & \epsilon_{22} & 0 & \\ \cdot & \cdot & & & \\ \epsilon_{n0} & \epsilon_{n1} & & \epsilon_{nn} & \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & & \cdot \end{bmatrix}$$

which we refer to as the *reception matrix* of the channel. For instance, the reception matrix for the usual collision channel is

$$\begin{bmatrix} 0 & 1 & \\ 1 & 0 & 0 \\ 1 & 0 & \\ 1 & \cdot & \\ \cdot & \cdot & \\ \cdot & \cdot & \end{bmatrix}$$

while for a system with capture it has the form

$$\begin{bmatrix} 0 & 1 & \\ 1-x_2 & x_2 & 0 \\ \cdot & \cdot & \\ 1-x_n & x_n & \\ \cdot & \cdot & \\ \cdot & \cdot & \end{bmatrix}$$

where x_n is the probability of capture given that the collision size is n. The model studied in [19], [20] can be described by a reception matrix of the form

$$\begin{bmatrix} 0 & 1 & & \\ 0 & 0 & 1 & 0 \\ \cdot & \cdot & & \cdot \\ 0 & 0 & & 1 \\ 1 & 0 & & \\ 1 & 0 & & 0 \\ \cdot & \cdot & & \end{bmatrix}.$$

Note that by letting $\epsilon_{10} \neq 0$ our model allows not only collisions but also background noise to be a source of errors.

Denote by X_n the number of backlogged packets in the system at the beginning of slot n. The discrete-time process $(X_n)_{n \geq 0}$ is easily seen to be a homogeneous Markov chain. We define the system to be stable if $(X_n)_{n \geq 0}$ is ergodic and unstable otherwise. The average number of packets correctly received in collisions of size n is denoted by $C_n = \sum_{k=1}^{n} k\epsilon_{nk}$. We can now state the main result.

Theorem 1: If C_n has a limit $C = \lim_{n \to \infty} C_n$, then[1] the system is stable for all arrival distributions such that $\lambda < C$ and is unstable for $\lambda > C$. This also holds if C is infinite: if $\lim_{n \to \infty} C_n = +\infty$, then the system is always stable.

The proof is given in Section III. In the remainder of this section, we use Theorem 1 to analyze several simple random access channels that fall within the scope of the multipacket reception channel.

1) Mobile Users with Pairwise Transmissions: Consider an infinite number of transmitters $T_1, T_2, \cdots$, and an infinite number of receivers $R_1, R_2, \cdots$, whose positions in the plane are i.i.d. random variables. Suppose that transmissions are pairwise

[1]This result holds under the assumption that the Markov chain of the number of backlogged packets is irreducible and aperiodic (for details and sufficient conditions, see Section III).

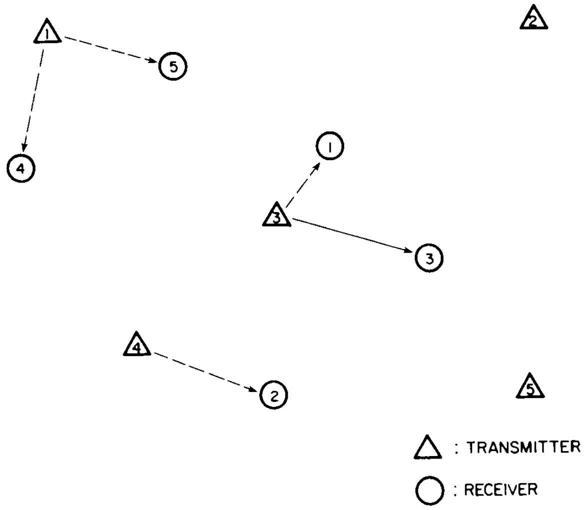

Fig. 1. Pairwise transmissions with only one success (3-3).

in the sense that transmitter T_n sends packets only to receiver R_n, and R_n is only interested in the packets sent by T_n (see Fig. 1). Assume also that each receiver can only detect correctly the packet sent by the closest transmitter (in particular, this is the case if there is perfect capture, see Example 3 below). The successes of transmissions occurring at the same time are independent, so that for $n \geq 2$

$$\epsilon_{nk} = \binom{n}{k} p(n)^k (1-p(n))^{n-k}$$

where $p(n)$ is the probability that any given transmitter is successful in a collision of size n, which is equal to $1/n$ if we assume that all locations are memoryless, i.e., independent from slot to slot. It follows that

$$C_n = np(n) = 1$$

and the maximum throughput is 1. More generally, if because of channel noise, the message of the closest transmitter is received correctly with probability α (in other words $\epsilon_{11} = \alpha$), then the throughput is equal to α. The assumption that the locations of the stations are memoryless is equivalent to assuming that they move infinitely fast. If this simplifying assumption is dropped, then the number of successes depends not only on the current number of retransmissions, but also on the previous history of retransmissions, and thus the problem is no longer encompassed by our multipacket reception model. In Fig. 2, the result of a simulation shows that for moderate speeds, the actual throughput is well approximated by the foregoing analysis.

2) Frequency Hopping Random Access Channel: Consider a finite population of N users transmitting by frequency hopping, as in [11], [22]. For each packet he wants to transmit, a user selects with equal probability one frequency in a fixed set of q frequencies. A packet is correctly received iff no other packet is transmitted on the same frequency during the same slot. We compute $(\epsilon_{Nk})_{1 \leq k \leq N}$, and $C = \lim_{N \to \infty} C_N$. If the users have infinite buffer space, then C can be taken as a good approximation for large N of the maximum stable throughput of the system, which is unknown. If the users have no buffer space, as is often assumed, the backlog Markov chain is always ergodic, but even then, one should expect reasonable delays in large population problems only for arrival rates below C. The computation of the reception matrix of this channel is a simple combinatorial problem of random assignment of objects to cells (e.g., see [23, App. A]).

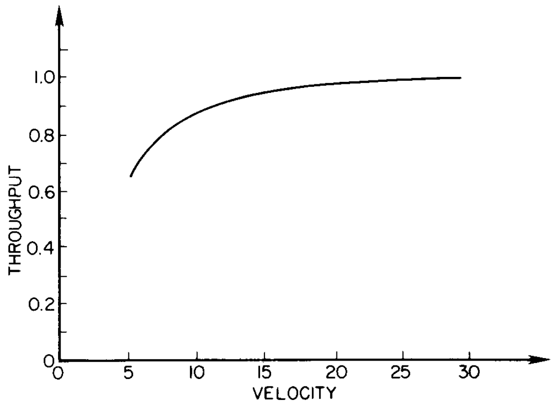

Fig. 2. Throughput as a function of velocity for mobile users with pairwise transmissions. Stations moving in a square region; velocity units: percentage of square side traveled in one slot. Retransmission probability set to 0.1.

Denote by $T_1, T_2, \cdots, T_N$ the users, all involved in the collision, and also denote by S the set of users whose packets are correctly received. Two cases need to be considered.

a) $2 \leq N \leq q$: We have, for $1 \leq j \leq N$

$$\epsilon_{Nj}=\binom{N}{j} P[S=\{T_1, T_2, \cdots, T_j\}] \tag{1}$$

and the following decomposition:

$$P[\{T_1, T_2, \cdots, T_j\} \subseteq S]=P[S=\{T_1, T_2, \cdots, T_j\}] + P\left[\bigcup_{k=j+1}^{N} \{\{T_1, \cdots, T_j, T_k\} \subseteq S\}\right]$$

easily yields the desired expression

$$P[S=\{T_1, T_2, \cdots, T_j\}]=\sum_{k=0}^{N-j} (-1)^k \cdot \binom{N-j}{k} P[\{T_1, T_2, \cdots, T_{k+j}\} \subseteq S] \tag{2}$$

where only one term is left to compute

$$P[\{T_1, T_2, \cdots, T_{k+j}\} \subseteq S] = \frac{q(q-1) \cdots (q-j-k+1)(q-j-k)^{N-j-k}}{q^N} \tag{3}$$

for $1 \leq j \leq N, 0 \leq k \leq N-j$. Putting (1), (2), and (3) together gives the result

$$\epsilon_{Nj}=\binom{N}{j} \sum_{k=0}^{N-j} (-1)^k \binom{N-j}{k} \cdot \frac{q(q-1) \cdots (q-j-k+1)(q-j-k)^{N-j-k}}{q^N} \tag{4}$$

for $1 \leq j \leq N$. Notice in particular that $\epsilon_{N,N-1} = 0$. Let us now compute the average number of packets correctly received in collisions of size N, $C_N = \Sigma_{j=1}^{N} j\epsilon_{Nj}$. By using (4) and summing at $j + k$ constant, we get

$$q^N C_N=\sum_{i=1}^{N} q(q-1) \cdots (q-i+1)(q-i)^{N-i} \cdot \sum_{n=0}^{i-1} (-1)^n \frac{N!}{n!(i-n-1)!(N-i)!}$$

which can be simplified as

$$q^N C_N=\sum_{i=1}^{N} \frac{N!}{(i-1)!(N-i)!} \cdot q(q-1) \cdots (q-i+1)(q-i)^{N-i}(1-1)^{i-1}$$

to get the final result

$$C_N=N\left(1-\frac{1}{q}\right)^{N-1}.$$

b) $N > q$: In this case, there can be at best $q-1$ successes in a collision of size N. The same method applies to get the following probabilities:

$$\epsilon_{Nj}=\binom{N}{j} \sum_{k=0}^{q-j-1} \binom{N-j}{k} (-1)^k \cdot \frac{q(q-1) \cdots (q-j-k+1)(q-j-k)^{N-j-k}}{q^N} \qquad (1 \leq j \leq q-1)$$

$$\epsilon_{Nj}=0 \qquad (q \leq j \leq N)$$

resulting in the same expected number of successes as before

$$C_N=N\left(1-\frac{1}{q}\right)^{N-1}.$$

Now we let the population size N go to infinity and we apply our result. If we let N grow to infinity while keeping q constant, we have $\lim_{N\to\infty} C_N = 0$, so the system is always unstable. On the other hand, if we let N go to infinity while keeping q equal to a fixed percentage of the population size, i.e., N/q constant, then $\lim_{N\to\infty} C_N = +\infty$, and the system is always stable. It is easily shown that to get a finite maximum stable throughput, q has to grow as $N/\ln N$.

3) Mobile Radio Network with Capture: Consider an infinite number of users independently and uniformly distributed in a circle of radius R, whose positions are independent from slot to slot. Users transmit packets to a common receiver located at the center of the network. Denote by P_1 and P_2 the received powers of the strongest and the next to strongest packets involved in a collision. Assume, as in [12]–[14], that the strongest packet is correctly received iff $P_1/P_2 > K$ (K being a system dependent constant), and that all the other packets involved in the collision are not received successfully. Assume, moreover, that the received power of a packet only depends on the distance r between the sender and the receiver

$$P=\frac{\text{constant}}{r^\alpha} \qquad (\alpha \geq 2).$$

Then there will be capture iff

$$r_2 > \beta r_1$$

where $\beta = K^{1/\alpha}$ is the capture parameter, and r_1, r_2 are the distances of the closest and the next to closest senders from the receiver.

Denote by D the distance between a given user and the receiver. It is easily shown that the pdf of D is given by

$$p_D(r) = 2\frac{r}{R^2} \qquad (0 \le r \le R).$$

Given N users, denote by U_N the closest from the receiver, and by D_N its distance from the receiver. Computing the cdf of D_N and taking its derivative, we obtain

$$p_{D_N}(r) = 2N\frac{r}{R^2}\left[1-\left(\frac{r}{R}\right)^2\right]^{N-1} \qquad (0 \le r \le R). \tag{5}$$

Given $D_N = r$, the other $N - 1$ users are uniformly distributed in the annular region (r, R). So if N users collide and $D_N = r$, U_N will be correctly received iff all the other users are in the annular region $(\beta r, R)$, which is empty if $\beta r > R$. Therefore, if we denote by

$$P_N(r) = P[\text{capture} \mid N \text{ collide, } D_N = r] \qquad (N \ge 2)$$

we have

$$P_N(r) = \begin{cases} \left[\dfrac{R^2-\beta^2 r^2}{R^2 - r^2}\right]^{N-1} & \text{if } r \le \dfrac{R}{\beta} \\ 0 & \text{if } r \ge \dfrac{R}{\beta} \end{cases}. \tag{6}$$

Thus, the probability of capture in a collision of N ($N \ge 2$) is

$$\epsilon_{N1} = \int_0^{R/\beta} P_N(r) p_{D_N}(r)\, dr.$$

Using (5) and (6), and with the change of variable $x = \beta/R$, this is easily computed

$$\epsilon_{N1} = \int_0^{1/\beta} 2Nx(1-\beta^2x^2)^{N-1}\, dx = \frac{1}{\beta^2}.$$

It follows that $C = 1/\beta^2$ is the maximum stable throughput. Notice, in particular, that for $\beta = 1$ (perfect capture), we have $C = 1$ and for $\beta \to \infty$ (no capture), we have $C \to 0$.

Under certain conditions, the performances of Aloha in the multipacket channel can be improved by varying the retransmission probability as a function of the channel history, and a maximum stable throughput of $\sup_{x \ge 0} e^{-x}\Sigma_{n=1}^{\infty} C_n/n!x^n$ can be reached (see [31]).

III. Ergodicity Region

The number of backlogged packets in the system at time n, $(X_n)_{n\ge 0}$, is a homogeneous Markov chain whose one-step transition probability matrix can be computed as a function of p, $(\lambda_k)_{k\ge 0}$, and E. Denoting by $B_i(j)$ the probability of having j retransmissions out of i backlogged packets

$$B_i(j) = \binom{i}{j} p^j (1-p)^{i-j} \tag{7}$$

we get

$$P_{00} = \lambda_0 + \sum_{n=1}^{\infty} \lambda_n \epsilon_{nn}$$

$$P_{0k} = \sum_{n=0}^{\infty} \lambda_{k+n}\epsilon_{k+n,n} \qquad (k \ge 1)$$

and for $i \ge 1$

$$P_{i,i-k} = \sum_{n=0}^{\infty} \lambda_n \sum_{j=k}^{i} B_i(j)\epsilon_{n+j,n+k} \qquad (1 \le k \le i)$$

$$P_{ii} = \lambda_0\left[B_i(0) + \sum_{j=1}^{i} B_i(j)\epsilon_{j0}\right] + \sum_{n=1}^{\infty} \lambda_n \sum_{j=0}^{i} B_i(j)\epsilon_{n+j,n}$$

$$P_{i,i+k} = \sum_{n=0}^{\infty} \lambda_{k+n} \sum_{j=0}^{i} B_i(j)\epsilon_{j+k+n,n} \qquad (k \ge 1). \tag{8}$$

Sufficient conditions for $(X_n)_{n\ge 0}$ to be irreducible and aperiodic are as follows:

- if $0 < p < 1$:

$$\lambda_0 \ne 0 \tag{9a}$$

$$\lambda_0 + \sum_{n=1}^{\infty} \lambda_n \epsilon_{nn} < 1 \tag{9b}$$

$$\epsilon_{10} \ne 1 \tag{9c}$$

- if $p = 1$:

$$\lambda_0 \ne 0 \tag{9a}$$

$$\lambda_0 + \sum_{n=1}^{\infty} \lambda_n \epsilon_{nn} < 1 \tag{9b}$$

$$\text{for all } i \ge 1, \ \epsilon_{i0} \ne 1. \tag{9d}$$

These are only sufficient conditions, but they hold for almost all nontrivial systems. For example, if (9b) does not hold, then zero is an absorbing state, since the left-hand side of (9b) is equal to P_{00}. Also, (9c) simply means that the successful reception of a single packet in the absence of other active users is possible. Assume, for instance, that $0 < p < 1$ and that the arrivals are Poisson distributed. Then we only have to assume (9c), and (9b) is true unless there is perfect reception, that is $\epsilon_{nn} = 1$ for all $n \ge 1$, in which case the system would of course always be stable. The case $p = 1$ gives rise to a number of pathological situations, hence, the much stronger condition (9d). It generally turns out that either (9d) is not necessary or the stability region of the system is obvious. For instance, it is clear from the transition probabilities that slotted Aloha with $p = 1$ is always unstable. In any case, it is assumed in what follows that $(X_n)_{n\ge 0}$ is irreducible and aperiodic.

Proof of Theorem 1: The proof is based on drift analysis. Recall that in general, the drift at state i ($i \ge 0$) is defined by

$$d_i = E[X_{t+1} - X_t \mid X_t = i].$$

If we denote by Σ_t the number of successful transmissions in slot t, we have

$$X_{t+1} - X_t = A_t - \Sigma_t$$

and therefore

$$d_i = \lambda - E[\Sigma_t \mid X_t = i]. \tag{10}$$

Now if R_t is the number of retransmissions in slot t, we get

$$P[\Sigma_t = k \mid X_t = i,\ A_t = n,\ R_t = j] = \epsilon_{n+j,k}$$

for $0 \le j \le i$, $0 \le k \le n + j$ and with the convention that $\epsilon_{00} =$

$C_0 = 0$. Thus,

$$E[\Sigma_t | X_t = i,\ A_t = n,\ R_t = j] = C_{n+j}$$

and

$$E[\Sigma_t | X_t = i] = \sum_{n=0}^{\infty} \lambda_n \sum_{j=0}^{i} B_i(j) C_{n+j}. \tag{11}$$

The value of the drifts for our model follows from (10) and (11)

$$d_i = \lambda - \sum_{n=0}^{\infty} \lambda_n \sum_{j=0}^{i} B_i(j) C_{n+j}. \tag{12}$$

The idea of the proof is to compute $\lim_{i\to\infty} d_i$ which will turn out to be a very simple expression, and then apply the results of [3] and [24] to determine the ergodicity region of $(X_n)_{n\geq 0}$. Let us first recall the two results that will be used in the sequel.

Lemma A (Pakes [24]): Let $(X_n)_{n\geq 0}$ be an irreducible and aperiodic Markov chain having as state space the nonnegative integers, denote by (P_{ij}) its transition probability matrix, and by d_i its drift at state i. Then if for all $i\ |d_i| < \infty$, and if $\limsup_{i\to\infty} d_i < 0$, $(X_n)_{n\geq 0}$ is ergodic.

Lemma B (Kaplan [3]): Under the assumptions of Lemma A, if for some integer $N \geq 0$ and some constants $B \geq 0$, $c \in [0, 1]$ the following two conditions hold, then $(X_n)_{n\geq 0}$ is not ergodic:

i) for all $i \geq N$, $d_i > 0$

ii) for all $i \geq N$, all $\theta \in [c, 1]$, $\theta^i - \Sigma_j P_{ij}\theta^j \geq -B(1-\theta)$.

From (12), it can be seen that $|d_i|$ is finite since

$$|d_i| \leq \lambda + \sum_{n=0}^{\infty} \lambda_n \sum_{j=0}^{i} B_i(j) C_{n+j} \leq 2\lambda + ip.$$

Next, the drift limit is given by the following lemma.

Lemma 1: If C_n has a limit C, finite or not, then $\lim_{i\to\infty} \Sigma_{n=0}^{\infty} \lambda_n \Sigma_{j=0}^{i} B_i(j) C_{n+j} = C$.

Proof of Lemma 1: We consider two separate cases depending on whether C is finite.

1) $C = +\infty$.

Fix $A > 0$ and pick $r \geq 0$ such that $\lambda_r \neq 0$. There exists an integer M such that for all $n \geq M$, $C_n > A$. Fix such an M. Then we have for $i \geq M$

$$\sum_{n=0}^{\infty} \lambda_n \sum_{j=0}^{i} B_i(j) C_{n+j} > \lambda_r \sum_{j=0}^{i} B_i(j) C_{j+r} > \lambda_r A \sum_{j=M}^{i} B_i(j)$$

which terminates the proof, since for any fixed $M \geq 0$

$$\lim_{i\to\infty} \sum_{j=M}^{i} B_i(j) = 1. \tag{13}$$

2) $C < +\infty$.

We have for $i > M$

$$\left| \sum_{n=0}^{\infty} \lambda_n \sum_{j=0}^{i} B_i(j) C_{n+j} - C \right| \leq \sum_{j=0}^{M} B_i(j) \sum_{n=0}^{\infty} \lambda_n |C_{n+j} - C| + \sum_{j=M+1}^{i} B_i(j) \sum_{n=0}^{\infty} \lambda_n |C_{n+j} - C|. \tag{14}$$

Fix $\epsilon > 0$. There exists M such that for all $n > M$, $|C_n - C| < \epsilon/2$. Fix such an M. Then

$$\sum_{j=M+1}^{i} B_i(j) \sum_{n=0}^{\infty} \lambda_n |C_{n+j} - C| < \frac{\epsilon}{2}.$$

Also, if L is an upper bound for C_n

$$\sum_{j=0}^{M} B_i(j) \sum_{n=0}^{\infty} \lambda_n |C_{n+j} - C| \leq 2L \sum_{j=0}^{M} B_i(j) < \frac{\epsilon}{2}$$

for i big enough because (13) holds, which takes care of the first term in (14) and ends the proof of Lemma 1.

Putting together (12) and Lemmas A and 1, we get that 1) if $\lim_{n\to\infty} C_n = +\infty$, then $\lim_{i\to\infty} d_i = -\infty$, and $(X_n)_{n\geq 0}$ is ergodic; and 2) if $\lim_{n\to\infty} C_n = C < +\infty$, then $\lim_{i\to\infty} d_i = \lambda - C$, and $(X_n)_{n\geq 0}$ is ergodic for $\lambda < C$. If $\lambda > C$, we can apply Lemma B and conclude that $(X_n)_{n\geq 0}$ is not ergodic provided that Kaplan's condition ii) holds. This is the purpose of Lemma 2, which is the last step in the proof of Theorem 1.

Lemma 2: If for all $n \geq 1$, $C_n < L$ for some $L \in (0, \infty)$, then Kaplan's condition holds: there exists a constant B, an integer N, and a real $c \in [0, 1]$ such that

$$\theta^i - \sum_j P_{ij}\theta^j \geq -B(1-\theta) \qquad \text{all } i \geq N,\ \theta \in [c, 1].$$

Proof of Lemma 2: According to [25], it is enough to show that the downward part of the drift, defined as

$$D(i) = -\sum_{k=1}^{i} k P_{i,i-k}$$

is bounded below. From the transition probabilities (8), we get

$$D(i) = -\sum_{k=1}^{i} k \sum_{n=0}^{\infty} \lambda_n \sum_{j=k}^{i} B_i(j) \epsilon_{n+j,n+k}$$

which can also be put in the form

$$D(i) = -\sum_{j=1}^{i} B_i(j) \sum_{n=0}^{\infty} \lambda_n \sum_{k=1}^{j} k \epsilon_{n+j,n+k}$$

from which it follows that

$$D(i) \geq -\sum_{j=1}^{i} B_i(j) \sum_{n=0}^{\infty} \lambda_n C_{n+j} \geq -L.$$

□

Notice that in the proof of Theorem 1 (and this also holds for Theorem 2 below), the exact expression (7) for $B_i(j)$ is never used. The only requirements are that $(B_i(j))_{0\leq j\leq i}$ is a probability distribution, and that (13) holds. Therefore, our results are valid for a larger class of retransmission policies than was first assumed. For example, there could be K priority groups, each with a different retransmission probability.

Although Theorem 1 is quite general, in many practical cases, the reception matrix has a very simple structure and the stability region can be obtained with virtually no computations. This happens for instance in radio networks with capture where all is needed is the limit of the second column of the matrix, or also in the simple case where above a certain collision size N, the transmission is too garbled for the receiver to be able to decode anything correctly, so that $C_n = 0$ for $n > N$.

This last example is a particular case of a noteworthy feature of Theorem 1, namely that the stability region does not depend on any finite number of rows of the reception matrix. In fact, any number of modifications of the matrix that leaves $\lim_{n\to\infty} C_n$ unchanged does not affect the stability region. Although this may be surprising at first sight, it can be intuitively explained by the fundamental instability of the collision channel: unless the receiver is perfect (all ϵ_{nn} equal to 1), the backlog will eventually exceed any prefixed value with probability one, thus it is the limit of C_n that determines the stability region.

The stability region is also unchanged if the first transmission of packets is delayed. If new packets are backlogged, that is, transmitted for the first time with probability p in each slot following their arrival (this transmission rule appears in the literature as controlled-access or delayed first transmission), the drifts become $d_i = \lambda - \sum_{j=1}^{i} B_i(j)C_j$ for $i \geq 1$, and from Lemmas 1 and 2 the ergodicity region remains the same.

If C_n does not have a limit, Theorem 1 does not give the stable throughput of the system. Even though in almost all practical cases, and indeed in all the examples of Section II, C_n does have a limit, it is conceptually interesting to examine the case when $\liminf_{n\to\infty} C_n \neq \limsup_{n\to\infty} C_n$. It is worth pointing out that adding constraints as strong as the following on the reception matrix still does not imply that C_n has a limit:

i) $(\epsilon_{n0})_{n\geq 1}$ is nondecreasing

ii) $(\epsilon_{nk})_{n\geq k}$ is nonincreasing for all $k \geq 1$

iii) $\epsilon_{nk} \geq \epsilon_{n,k+1}$ for $n \geq 2,\ 1 \leq k \leq n-1$

although the counterexamples we have been able to build are somewhat contrived. Notice that conditions i) and ii) above imply that each column has a limit $\alpha_k = \lim_{n\to\infty} \epsilon_{nk} (k \geq 0)$, which is very likely to happen in practice. In any case, Theorem 2 below still gives some information on the stability region, although the exact result requires in general the complete knowledge of the sequence $(C_n)_{n\geq 1}$. In fact, given any nonnegative numbers $\alpha < \gamma < \beta$, one can construct a reception matrix with nth row average C_n such that:

i) $\liminf_{n\to\infty} C_n = \alpha$

ii) $\limsup_{n\to\infty} C_n = \beta$

and such that the maximum stable throughput is γ.

Theorem 2: The system is stable for $\lambda < \liminf_{n\to\infty} C_n$ and unstable for $\lambda > \limsup_{n\to\infty} C_n$.

Proof:

a) If $\lambda < \liminf_{n\to\infty} C_n$, then $(X_n)_{n\geq 0}$ is ergodic.

If $\liminf_{n\to\infty} C_n = +\infty$, then $\lim_{n\to\infty} C_n = +\infty$, and the result has already been proved, so assume that $\liminf_{n\to\infty} C_n$ is finite. From Lemma A, it is enough to prove that for all $\epsilon > 0$, there exists N such that

$$d_i < \lambda - \liminf_{n\to\infty} C_n + \epsilon \qquad \text{all } i \geq N.$$

Recall from (12) that we have

$$d_i = \lambda - \sum_{n=0}^{\infty} \lambda_n \sum_{j=0}^{i} B_i(j) C_{n+j}. \tag{15}$$

So it is only needed to prove that for all $\epsilon > 0$ there exists N such that

$$\sum_{n=0}^{\infty} \lambda_n \sum_{j=0}^{i} B_i(j) C_{n+j} > \liminf_{n\to\infty} C_n - \epsilon \qquad \text{all } i \geq N.$$

Now by definition there exists M such that for all $k \geq M$:

$$C_k > \liminf_{n\to\infty} C_n - \epsilon$$

and therefore for all $i > M$:

$$\sum_{n=0}^{\infty} \lambda_n \sum_{j=0}^{i} B_i(j) C_{n+j} > (\liminf_{n\to\infty} C_n - \epsilon) \sum_{j=M}^{i} B_i(j)$$

which completes the proof since (13) holds.

b) If $\lambda > \limsup_{n\to\infty} C_n$, then $(X_n)_{n\geq 0}$ is not ergodic.

Since λ is finite, in this case $\limsup_{n\to\infty} C_n$ is necessarily finite. Therefore, $(C_n)_{n\geq 1}$ is bounded and from Lemma 2, Kaplan's condition holds. Thus, it is enough to show that for all $\epsilon > 0$, there exists N such that

$$d_i > \lambda - \limsup_{n\to\infty} C_n - \epsilon \qquad \text{all } i \geq N.$$

From (15), we only need to show

$$\sum_{n=0}^{\infty} \lambda_n \sum_{j=0}^{i} B_i(j) C_{n+j} < \limsup_{n\to\infty} C_n + \epsilon \qquad \text{all } i \geq N.$$

Since there exists M such that for all $k \geq M$

$$C_k < \limsup_{n\to\infty} C_n + \epsilon$$

then if L is an upper bound for C_n, we have for $i \geq M$

$$\sum_{n=0}^{\infty} \lambda_n \sum_{j=0}^{i} B_i(j) C_{n+j} < L \sum_{j=0}^{M-1} B_i(j) + \limsup_{n\to\infty} C_n + \epsilon$$

from which the result follows, using (13). □

IV. Behavior of the Backlog Markov Chain in the Nonergodicity Region

In this section, we further investigate the properties of $(X_n)_{n\geq 0}$ in the case $\lambda > C$, assuming of course that $(C_n)_{n\geq 1}$ has a finite limit. It has been proved in [1] that the backlog Markov chain for the usual slotted Aloha algorithm is transient, but this result cannot be generalized to our model when $\lambda > C$. We give below an example showing that $(X_n)_{n\geq 0}$ can be null recurrent when the mean arrival rate λ belongs to an interval of positive length. The boundary between the null recurrence and the transience regions generally depends in a rather complicated manner on both the reception matrix and the retransmission probability p. We give a sufficient condition for $(X_n)_{n\geq 0}$ to be transient when $\lambda > C$, as well as bounds on the recurrence region.

Consider the reception matrix defined by

$$\epsilon_{nk} = \frac{1}{n^2} \qquad (1 \leq k \leq n)$$

$$\epsilon_{n0} = 1 - \frac{1}{n}$$

for $n \geq 1$. Then $C_n = \sum_{k=1}^{n} k/n^2 = (n+1)/2n$, and $C = 1/2$. Using Lemmas C and D below, we show in [26] that X_n is recurrent for $\lambda < R(p)$ and transient for $\lambda > R(p)$, where $R(p)$ is a function of the retransmission probability p and is given by

$$R(p) = \frac{1}{p} + \frac{(1-p)}{p^2} \ln(1-p) \qquad (0 < p < 1)$$

$$R(1) = 1.$$

It is easily seen that $R(p)$ is an increasing function of p for $p \in]0, 1[$ with extrema $\lim_{p\to 0} R(p) = 1/2$ and $\lim_{p\to 1} R(p) = 1$. Fig. 3 summarizes the behavior of X_n for this example.

It is somehow surprising to see that in this case, as well as in all

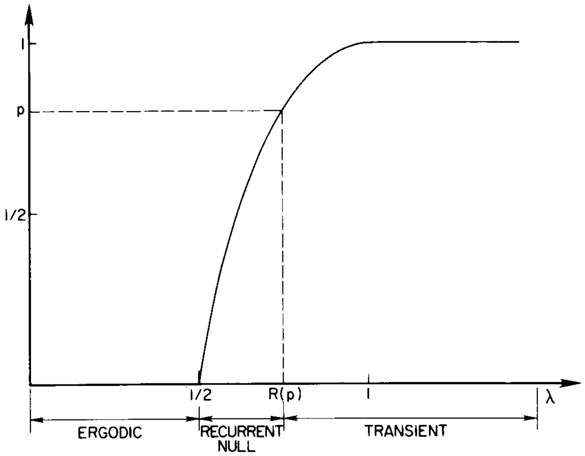

Fig. 3. Transience and ergodicity regions as a function of the retransmission probability when $\epsilon_{nk} = 1/n^2$.

the other examples we have computed, the recurrence region becomes larger as p increases. Intuitively, the recurrence of X_n when $\lambda > C$ seems to be due to the fact that transitions from any state i to 0 (or to some fixed integer k_0) are possible and that the probability of such an event, P_{i0} (or P_{ik_0}), goes to zero slowly with i. It can be checked that these probabilities are increasing functions of p when i is large enough.

Transience is ensured for $\lambda > C$ if the supremum of the elements of the kth-column goes to zero faster than k^2. This condition holds for all the examples in Section II, as well as for many real life cases, due to the practical limitations on the receiver capabilities. In particular, it is always verified if the reception matrix has only a finite number of nonzero columns (or equivalently, if the backlog Markov chain has uniformly bounded downwards transitions, as defined in [3]) which happens, for instance, if there is capture. Note that the proof of Theorem 3 below is of course valid for the conventional collision channel, and in this case becomes somewhat simpler than the proof in [1].

Theorem 3: If $\lim_{k\to\infty} k^2 \sup_{n\geq k} \epsilon_{nk} = 0$, then $(X_n)_{n\geq 0}$ is transient for $\lambda > C$.

Because of the complexity and lack of structure of the one-step transition probabilities (8), few results on the recurrence and transience of Markov chains can be applied to our model. Before proving Theorem 3, let us introduce the following two criteria from [27].

Lemma C: Let $(X_n)_{n\geq 0}$ be an irreducible and aperiodic Markov chain, having as state space the set of nonnegative integers, and with one-step transition probability matrix $P = (P_{ij})$. $(X_n)_{n\geq 0}$ is recurrent if and only if there exists a sequence $(y_n)_{n\geq 0}$ such that

1) $\lim_{n\to\infty} y_n = +\infty$

2) for some integer $N>0$ $\sum_{j=0}^{\infty} y_j P_{ij} \leq y_i$ all $i \geq N$.

We will only use the sufficiency part, which has also been proved in [24].

Lemma D: With the same assumptions as in Lemma C, $(X_n)_{n\geq 0}$ is transient if and only if there exists a sequence $(y_n)_{n\geq 0}$ such that

1) $(y_n)_{n\geq 0}$ is bounded

2) for some integer $N>0$ $\sum_{j=0}^{\infty} y_j P_{ij} \leq y_i$ all $i \geq N$

3) for some $k \geq N$ $y_k < y_0, \cdots, y_{N-1}$.

Sufficiency under the additional constraints $y_i > 0$ and $\lim_{i\to\infty} y_i = 0$ has also been proved in [28]. Also, the sufficiency parts of both lemmas are an immediate consequence of [29, Theorems 5 and 6] together with the results in [30].

Proof of Theorem 3: We use Lemma D with $y_n = 1/(n+1)^\theta$, $\theta \in]0, 1[$. We have

$$\sum_j P_{ij} y_j \leq y_i \Leftrightarrow \sum_{k=1}^{i} (y_{i-k} - y_i) P_{i,i-k} + \sum_{k=1}^{\infty} (y_{i+k} - y_i) P_{i,i+k} \leq 0 \quad (16)$$

and

$$(i+1)^{1+\theta} \sum_{k=1}^{i} (y_{i-k} - y_i) P_{i,i-k} + (i+1)^{1+\theta} \cdot \sum_{k=1}^{\infty} (y_{i+k} - y_i) P_{i,i+k} = D'(i) + U'(i) \quad (17)$$

where we have defined

$$D'(i) = (i+1)^{1+\theta} \sum_{k=1}^{i} \left[\frac{1}{(i+1-k)^\theta} - \frac{1}{(i+1)^\theta} \right] \cdot \sum_{n=0}^{\infty} \lambda_n \sum_{j=k}^{i} B_i(j) \epsilon_{n+j,n+k}$$

$$U'(i) = (i+1)^{1+\theta} \sum_{k=1}^{\infty} \left[\frac{1}{(i+1+k)^\theta} - \frac{1}{(i+1)^\theta} \right] \cdot \sum_{n=0}^{\infty} \lambda_{k+n} \sum_{j=0}^{i} B_i(j) \epsilon_{n+k+j,n}. \quad (18)$$

The drift of X_n at state i can be computed from the transition probabilities (8)

$$d_i = -\sum_{k=1}^{i} k P_{i,i-k} + \sum_{k=1}^{\infty} k P_{i,i+k} = D(i) + U(i) \quad (19)$$

where we have defined

$$D(i) = -\sum_{k=1}^{i} k \sum_{n=0}^{\infty} \lambda_n \sum_{j=k}^{i} B_i(j) \epsilon_{n+j,n+k}$$

$$U(i) = \sum_{k=1}^{\infty} k \sum_{n=0}^{\infty} \lambda_{n+k} \sum_{j=0}^{i} B_i(j) \epsilon_{j+k+n,n}. \quad (20)$$

The idea of the proof is to show that

$$\lim_{i\to\infty} [D'(i) + U'(i)] = -\theta \lim_{i\to\infty} d_i \quad (21)$$

and since it has been proved in Section III that $\lim_{i\to\infty} d_i = \lambda - C$, we will be able to conclude that $(X_n)_{n\geq 0}$ is transient for $\lambda > C$.

1) $\lim_{i\to\infty} [D'(i) + \theta D(i)] = 0.$

From (18) and (20)

$$D'(i)+\theta D(i)=(i+1)\sum_{k=1}^{i}\left[\left(\frac{i+1}{i+1-k}\right)^{\theta}-1-\frac{\theta k}{i+1}\right]\cdot\sum_{n=0}^{\infty}\lambda_n\sum_{j=k}^{i}B_i(j)\epsilon_{n+j,n+k}$$

which is more conveniently written as

$$D'(i)+\theta D(i)=(i+1)\sum_{n=0}^{\infty}\lambda_n\sum_{j=1}^{i}B_i(j)\sum_{k=1}^{j}\cdot\left[\left(\frac{i+1}{i+1-k}\right)^{\theta}-1-\frac{\theta k}{i+1}\right]\epsilon_{n+j,n+k}.$$

This expression is nonnegative since

$$\left(\frac{i+1}{i+1-k}\right)^{\theta}-1-\frac{\theta k}{i+1}>0 \qquad (1\le k\le i).$$

Define $\gamma_k = \sup_{n\ge k}\epsilon_{nk}$. Then

$$0\le D'(i)+\theta D(i)\le(i+1)\sum_{n=0}^{\infty}\lambda_n\sum_{j=1}^{i}B_i(j)\sum_{k=1}^{j}\cdot\left[\left(\frac{i+1}{i+1-k}\right)^{\theta}-1-\frac{\theta k}{i+1}\right]\gamma_{n+k}$$
$$\le(i+1)\sum_{n=0}^{\infty}\lambda_n\sum_{k=1}^{i}\left[\left(\frac{i+1}{i+1-k}\right)^{\theta}-1-\frac{\theta k}{i+1}\right]\gamma_{n+k}.$$

That is

$$D'(i)+\theta D(i)\le x_1(i)+x_2(i) \qquad (22)$$

with, assuming for instance that i is odd

$$x_1(i)=(i+1)\sum_{n=0}^{\infty}\lambda_n\sum_{k=1}^{(i+1)/2}\left[\left(\frac{i+1}{i+1-k}\right)^{\theta}-1-\frac{\theta k}{i+1}\right]\gamma_{n+k}$$
$$x_2(i)=(i+1)\sum_{n=0}^{\infty}\lambda_n\sum_{k=(i+3)/2}^{i}\left[\left(\frac{i+1}{i+1-k}\right)^{\theta}-1-\frac{\theta k}{i+1}\right]\gamma_{n+k}. \qquad (23)$$

We show that $x_1(i)$ and $x_2(i)$ go to zero independently. Fix $\epsilon > 0$. Define for $0 < x \le i$ the function

$$p_i(x)=\frac{i+1}{x^2}\left[\left(\frac{i+1}{i+1-x}\right)^{\theta}-1\right]-\frac{\theta}{x}.$$

It is easily proved that for each $i \ge 1$, $p_i(x)$ is a positive nondecreasing function of x. Also

$$p\left(\frac{i+1}{2}\right)=\frac{1}{i+1}\,[4(2^{\theta}-1)-2\theta]=\frac{A}{i+1}$$

where A is a positive constant depending only on θ. From (23)

$$x_1(i)=\sum_{n=0}^{\infty}\lambda_n\sum_{k=1}^{(i+1)/2}k^2p_i(k)\gamma_{n+k}\le\frac{A}{i+1}\sum_{n=0}^{\infty}\lambda_n\sum_{k=1}^{n+(i+1)/2}k^2\gamma_{n+k}.$$

If $\lim_{k\to\infty}k^2\gamma_k = 0$, then $\lim_{n\to\infty}1/n\sum_{k=1}^{n}k^2\gamma_k = 0$. So we can choose i large enough so that for $n \ge (i+1)/2$, $\sum_{k=1}^{n}k^2\gamma_k < n\epsilon$. Then

$$x_1(i)\le\epsilon\frac{A}{i+1}\sum_{n=0}^{\infty}\lambda_n\left(n+\frac{i+1}{2}\right)=\epsilon A\left(\frac{\lambda}{i+1}+\frac{1}{2}\right).$$

Now if we choose i big enough so that for $k > (i+3)/2$, we have $\gamma_k < \epsilon/k^2$, then

$$x_2(i)\le\epsilon\sum_{n=0}^{\infty}\lambda_n\sum_{k=(i+3)/2}^{i}(i+1)\cdot\left[\left(\frac{i+1}{i+1-k}\right)^{\theta}-1-\frac{\theta k}{i+1}\right]\frac{1}{(n+k)^2}$$
$$\le\frac{4\epsilon}{i+3}\sum_{k=(i+3)/2}^{i}\left[\left(\frac{i+1}{i+1-k}\right)^{\theta}-1-\frac{\theta k}{i+1}\right].$$

By bounding the sum in the last equation by integrals, it can be seen that it is upper bounded by a linear function of i.

2) $\lim_{i\to\infty}[U'(i) + \theta U(i)] = 0$.

From (18) and (20)

$$U'(i)+\theta U(i)=\sum_{k=1}^{\infty}(i+1)\left[\left(\frac{i+1}{i+1+k}\right)^{\theta}-1+\frac{\theta k}{i+1}\right]\cdot\sum_{n=0}^{\infty}\lambda_{k+n}\sum_{j=0}^{i}B_i(j)\epsilon_{j+k+n,n}.$$

With a change of variable

$$U'(i)+\theta U(i)=(i+1)\sum_{j=0}^{i}B_i(j)\sum_{n=1}^{\infty}\lambda_n\sum_{k=1}^{n}\cdot\left[\left(\frac{i+1}{i+1+k}\right)^{\theta}-1+\frac{\theta k}{i+1}\right]\epsilon_{n+j,n-k}.$$

By using the following inequalities:

$$0\le\frac{1}{(1+x)^{\theta}}-1+\theta x\le\theta(1+\theta)\frac{x^2}{2} \qquad (x\ge0,\ 0<\theta<1)\}$$

we get

$$0\le U'(i)+\theta U(i)\le\theta\frac{(1+\theta)}{2}(i+1)\sum_{j=0}^{i}B_i(j)\cdot\sum_{n=1}^{N}\lambda_n\sum_{k=1}^{n}\frac{k^2}{(i+1)^2}\epsilon_{n+j,n-k}$$
$$+\theta(i+1)\sum_{j=0}^{i}B_i(j)\sum_{n=N+1}^{\infty}\lambda_n\sum_{k=1}^{n}\frac{k}{i+1}\epsilon_{n+j,n-k}$$
$$\le\frac{1}{i+1}\sum_{n=1}^{N}n^2\lambda_n+\sum_{n=N+1}^{\infty}n\lambda_n.$$

Fix $\epsilon > 0$. Choose N such that $\sum_{N+1}^{\infty}n\lambda_n < \epsilon/2$, and then, N being fixed, choose i large enough so that $1/(i+1)\sum_{n=1}^{N}n^2\lambda_n < \epsilon/2$. □

It should be clear at this point that unlike the ergodicity region, the recurrence region depends in general on the elements of the reception matrix (instead of only the row averages) and on the retransmission probability p. For this reason, an exact expression for the recurrence region seems rather difficult to obtain; nonetheless, the method (see [26]) that we used to study the example in Fig. 3 can be generalized to obtain the following upper and lower bounds on the recurrence region.

Theorem 4: $(X_n)_{n\geq 0}$ is recurrent for $\lambda < L$ and transient for $\lambda > U$, with $L = \max\{l_1, \sup_{0<\theta<1} l_\theta, \sup_{0<\theta<1} l'_\theta\}$ and $U = \min\{u_1, \inf_{0<\theta<1} u_\theta, \inf_{0<\theta<1} u'_\theta\}$ where

$$l_1 = \lim_{i\to\infty} (i+1) \sum_{n=0}^{\infty} \lambda_n \sum_{j=1}^{i} B_i(j) \sum_{k=1}^{n+j} \ln\left(\frac{i+n+1}{i+n-k+1}\right) \epsilon_{n+j,k}$$

$$l_\theta = \frac{1}{\theta} \lim_{i\to\infty} (i+1)^{1-\theta} \sum_{n=0}^{\infty} \lambda_n \sum_{j=1}^{i} B_i(j) \cdot \sum_{k=1}^{n+j} [(i+n+1)^\theta - (i+n-k+1)^\theta]\epsilon_{n+j,k}$$

$$l'_\theta = \frac{1}{\theta} \lim_{i\to\infty} (i+1)\,[\ln(i+1)]^{1-\theta} \sum_{n=0}^{\infty} \lambda_n \sum_{j=1}^{i} B_i(j) \cdot \sum_{k=1}^{n+j} [[\ln(i+n+1)]^\theta - [\ln(i+n-k+1)]^\theta]\epsilon_{n+j,k}$$

and

$$u_1 = \lim_{i\to\infty} (i+1)\,[\ln(i+1)]^2 \sum_{n=0}^{\infty} \lambda_n \sum_{j=1}^{i} B_i(j) \cdot \sum_{k=1}^{n+j} \left[\frac{1}{\ln(i+n+2-k)} - \frac{1}{\ln(i+n+2)}\right] \epsilon_{n+j,k}$$

$$u_\theta = \frac{1}{\theta} \lim_{i\to\infty} (i+1)^{1+\theta} \sum_{n=0}^{\infty} \lambda_n \sum_{j=1}^{i} B_i(j) \cdot \sum_{k=1}^{n+j} \left[\frac{1}{(i+n+1-k)^\theta} - \frac{1}{(i+n+1)^\theta}\right] \epsilon_{n+j,k}$$

$$u'_\theta = \frac{1}{\theta} (i+1)\,[\ln(i+1)]^{1+\theta} \sum_{n=0}^{\infty} \lambda_n \sum_{j=1}^{i} B_i(j) \cdot \sum_{k=1}^{n+j} \left[\frac{1}{[\ln(i+n+2-k)]^\theta} - \frac{1}{[\ln(i+n+2)]^\theta}\right] \epsilon_{n+j,k}.$$

We are assuming that the limits above exist, which indeed happens in most practical cases. The theorem is valid if any of these limits is infinite. In particular, if $L = +\infty$, then X_n is always recurrent. Note that usually, it is not necessary to carry out all the computations, because one of the three terms in the definition of L is equal to one of the terms in the definition of U. In fact, in most cases, we have $\sup_{0<\theta<1} l_\theta = \inf_{0<\theta<1} u_\theta$ if $0 < p < 1$, and $u_1 = l_1$ if $p = 1$. The proof of Theorem 4 can be found in [26].

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