Multiuser Detection and Statistical Physics

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Abstract

We present a framework for analyzing multiuser detectors in the context of statistical physics. A multiuser detector is shown to be equivalent to a conditional mean estimator which finds the mean value of the stochastic output of a so-called Bayes retrochannel. The Bayes retrochannel is equivalent to a spin glass in the sense that the distribution of its stochastic output conditioned on the received signal is exactly the distribution of the spin glass at thermal equilibrium. In the large-system limit, the bit-error-rate of the multiuser detector is simply determined by the magnetization of the spin glass, which can be obtained using powerful tools developed in statistical mechanics. In particular, we derive the large-system uncoded bit-error-rate of the matched filter, the MMSE detector, the decorrelator and the optimal detectors, as well as the spectral efficiency of the Gaussian CDMA channel. It is found that all users with different received energies share the same multiuser efficiency, which uniquely determines the performance of a multiuser detector. A universal interpretation of multiuser detection relates the multiuser efficiency to the mean-square error of the conditional mean estimator output in the large-system limit.

Index Terms: Multiuser detection, statistical mechanics, code-division multiple access, spin glass, self-averaging property, free energy, replica method, multiuser efficiency.

1 Introduction

Multiuser detection is central to the fulfillment of the capabilities of code-division multiple access (CDMA), which is becoming the ubiquitous air-interface in future generation communication systems. In a CDMA system, all frequency and time resources are allocated to all users simultaneously. To distinguish between users, each user is assigned a user-specific spreading sequence on which the user's information symbol is modulated before transmission. By selecting mutually orthogonal spreading sequences for all users, each user can be separated completely by matched filtering to one's spreading sequence. It is not very realistic to maintain orthogonality in a mobile environment and hence multiple access interference (MAI) arises. The problem of demodulating in the presence of the MAI therefore becomes vital for a CDMA system.

A variety of multiuser detectors [1] have been proposed to mitigate the MAI. The simplest one is the single-user matched filter, which totally ignores the existence of the MAI. Its performance is not very satisfactory and is particularly limited by the near-far problem. In the other extreme, the individually optimal (IO) and the jointly optimal (JO) detectors achieve the minimum probability of error but entail prohibitive complexity which is exponential in the number of users. A wide spectrum of multiuser detectors offer performance in between the matched filter and the optimal detectors with substantially reduced complexity. The most popular ones include the MMSE detector and the decorrelator. The performance of multiuser detectors has been studied extensively in the literature. A collection of results is found in [1]. In general, the performance is dependent on the spreading factor, the number of users, the received signal-to-noise ratios (SNR), and the instantaneous spreading sequences. The dependence on this many parameters results in very complex expressions for all but the simplest case. Not only are these expressions hard to evaluate, but the complication allows little useful insight into the detection problem. To eliminate the dependency by averaging over all spreading sequences (e.g. [2]) is plausible but usually a prohibitive task.

Recently, it is found that performance analysis can be greatly simplified for random-spread systems the size of which tend to infinity [1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Of special interest is the case where the number of users and the spreading factor both tend to infinity with their ratio fixed. This is referred to as the large-system limit in the literature. As far as linear multiuser detectors are concerned, an immediate advantage of the large-system setting is that the multiple-access interference, as a sum of contributions from all interfering users, becomes Gaussian-like in distribution under mild conditions in the many-user limit [13]. This allows the output signal-to-interference ratio (SIR) and the uncoded bit-error-rate (BER) to be easily characterized for linear detectors such as the matched filter, the MMSE detector and the decorrelator. The large-system treatment also finds its success in deriving the capacity (or the spectral efficiency) of CDMA channels when certain multiuser detectors are used. Somewhat surprisingly, in all the above instances, the SIR, the BER and the spectral efficiency are independent of the spreading sequence assignment with probability 1. The underlying theory is that dependency on the spreading sequences vanishes in the large-system domain. In particular, the empirical eigenvalue distribution of a large random sequence correlation matrix converges to a deterministic distribution with probability 1 [14, 15, 16]. One may compare this to the concept of typical sequences in information theory [17]. We can say that for a sufficiently large system, almost all spreading sequence assignments are "typical" and lead to the (same) average performance.

Unlike in the above, a more recent but quite different view of large CDMA systems is inspired by the successful analysis of certain error-control codes using methods developed in statistical mechanics [18, 19, 20, 21, 22, 23, 24]. In [25] a CDMA system finds an equivalent spin glass similar to the Hopfield model. Here a spin glass is a statistical mechanical system consisting of a large number of interacting spins. In [26, 27], well-known multiuser detectors are expressed as Marginal-Posterior-Mode detectors, which can be embedded in a spin glass. The large-system performance of a multiuser detector is then found as a thermodynamic limit of a certain macroscopic property of the corresponding spin glass, which can be obtained by powerful techniques sharpened in statistical mechanics. In [28, 24], Tanaka analyzed the IO and the JO detectors, the decorrelator and the MMSE detector under the assumption that all users are received at the same energy (perfect power control). In addition to the rederivation of some previously known results, Tanaka found for the first time, the large-system BER of the optimal detectors at finite SNRs, assuming perfect power control.

This new statistical mechanics approach to large systems emerges to be more fundamental. In fact, the convergence of the empirical eigenvalue distribution, which underlies many above-mentioned large-system results, can be proved in statistical mechanics [29, Chapter 1]. Deeply rooted in statistical physics, the new approach brings a fresh look into the decades-old multiuser detection problem. In this work, we present a systematic treatment of multiuser detection in the context of statistical mechanics based on [30, 31]. We introduce the concept of *Bayes retrochannel*, which takes the multiaccess channel output as the input and generates a stochastic estimate of the originally transmitted data. The characteristic of the Bayes retrochannel is the posterior probability distribution under some postulated prior and conditional probability distributions. A multiuser detector is equivalent to a *conditional mean estimator* which finds the expected value of the stochastic output of the Bayes retrochannel. By carefully choosing the postulated prior and conditional probability distributions of the Bayes retrochannel, we can arrive at different multiuser detection optimality criteria. Importantly, the Bayes retrochannel is found to be equivalent to a spin glass with the spreading sequence assignment and the received signal as quenched randomness. That is, the conditional output distribution of the Bayes retrochannel is exactly the same as the distribution of the spin glass system at thermal equilibrium. Thus, in the large-system limit, the performance of the detector finds its counterpart as a certain macroscopic property of the thermodynamic system, which can be obtained using the replica method developed in statistical mechanics.

In particular, we present analytical results for the large-system BER of the matched filter, the decorrelator, the MMSE detector and the optimal detectors. We also find the spectral efficiency of the Gaussian CDMA channel, both with and without the constraint of binary inputs. Unlike in [28], we do not assume equal-energy users in any case. It is found that the all users share the same multiuser efficiency, which is the solution to a fixed-point equation similar to the Tse-Hanly equation in [4]. A universal interpretation of multiuser detection relates multiuser efficiency to the ouput mean-square error of the corresponding conditional mean estimator in the many-user limit. Finally, we present a canonical interference canceller that approximates a general multiuser detector.

This paper is organized as follows. In section 2 the multiuser CDMA is introduced, and known and new large-system results are presented. In section 3 we relate multiuser detection to spin glass in statistical mechanics. Section 4 carries a detailed analysis of linear multiuser detectors. Results for the optimal detectors are obtained in section 5. A general interpretation of uncoded multiuser detection is discussed in 6. A statistical mechanics look at the information theoretic spectral efficiency is presented in section 7.

2 Multiuser Detection: Known and New Results

2.1 The CDMA Channel

We study a K-user symbol-synchronous CDMA system with a spreading factor of N. It suffices to consider one symbol interval. Let the spreading sequence of user k be denoted as $\mathbf{s}_k = \frac{1}{\sqrt{N}}[s_{1k}, s_{2k}, \dots, s_{Nk}]^\top$, where the s_{nk}'s are independently and randomly chosen ±1's. Let $\mathbf{d} = [d_1, \dots, d_K]^\top$ be a vector consisting of the K users' transmitted symbols, each symbol being equally likely to be ±1. The prior probability distribution is simply

$$p_0(\mathbf{d}) = 2^{-K}, \quad \forall\, \mathbf{d} \in \{-1, 1\}^K. \tag{1}$$

Let $P_1, \dots, P_K$ be the K users' respective received energies per symbol. The received signal in the n^{th} chip interval is then expressed as

$$r_n = \frac{1}{\sqrt{N}} \sum_{k=1}^{K} \sqrt{P_k}\, s_{nk} d_k + \sigma_0 \nu_n, \quad n = 1, \dots, N \tag{2}$$

where $\{\nu_n\}$ are independent standard Gaussian random variables, and σ_0^2 the noise variance. Note that the spreading sequences are randomly chosen for each user and not dependent on the received energies.

We can normalize the averaged transmitted energy by absorbing a common factor into the noise variance. Without loss of generality, we assume

$$\frac{1}{K} \sum_{k=1}^{K} P_k = 1. \tag{3}$$

The SIR[1] of user k under matched filtering in absence of interfering users is P_k/σ_0^2 and the average SIR of all users is $1/\sigma_0^2$. We assume that the energies are known deterministic numbers, and as $K \to \infty$, the empirical distributions of $\{P_k\}$ converge to a known distribution, hereafter referred to as the *energy distribution*. For convenience we assume that the energy of every user is bounded, $0 < P_{\text{min}} \le P_k \le P_{\text{max}} < \infty$. The mean value of this distribution must be 1.

The characteristic of the Gaussian CDMA channel can be described as

$$p_0(r_n|\mathbf{d}, \mathbf{S}) = (2\pi\sigma_0^2)^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma_0^2} \left(r_n - \frac{1}{\sqrt{N}} \sum_{k=1}^{K} \sqrt{P_k}\, s_{nk} d_k \right)^2 \right] \tag{4}$$

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[1] The SIR is defined as the energy ratio of the useful signal to the noise in the output. In contrast, the SNR of user k is usually defined as $P_k/(2\sigma_0^2)$.

where the $N \times K$ matrix $\mathbf{S} = [\mathbf{s}_1, \dots, \mathbf{s}_K]$. Let $\mathbf{r} = [r_1, \dots, r_N]^\top$, $\mathbf{A} = \mathrm{diag}(\sqrt{P_1}, \dots, \sqrt{P_K})$ and $\boldsymbol{\nu} = [\nu_1, \dots, \nu_N]^\top$, we have a compact form for (2) and (4)

$$\mathbf{r} = \mathbf{SAd} + \sigma_0 \boldsymbol{\nu} \tag{5}$$

and

$$p_0(\mathbf{r}|\mathbf{d}, \mathbf{S}) = (2\pi\sigma_0^2)^{-\frac{N}{2}} \exp\left[-\frac{1}{2\sigma_0^2}\|\mathbf{r} - \mathbf{SAd}\|^2\right] \tag{6}$$

where $\|\cdot\|$ denotes the Euclidean norm of a vector.

2.2 Multiuser Detection: Known Results

Assume that all received energies and the noise variance are fixed and known. A multiuser detector observes a received signal vector $\mathbf{r}$ in each symbol interval and tries to recover the transmitted symbols using knowledge of the instantaneous spreading sequences $\mathbf{S}$. In general, the detector outputs a soft decision statistic for each user of interest, which is a function of $(\mathbf{r}, \mathbf{S})$,

$$\tilde{d}_k = f_k(\mathbf{r}, \mathbf{S}), \quad k \in \{1, \dots, K\}. \tag{7}$$

Whenever the soft output can be separated as a useful signal component and an interference, their energy ratio gives the SIR. Usually, a hard decision is made according to the sign of the soft output,

$$\hat{d}_k = \mathrm{sgn}(\tilde{d}_k). \tag{8}$$

Assuming binary symmetric priors, the bit-error-rate (or, the probability of error) for user k is

$$\mathsf{P}_k = \mathsf{P}\left(\hat{d}_k \neq d_k\right) = \mathsf{P}\left(\tilde{d}_k < 0 | d_k = 1\right). \tag{9}$$

Another important performance index is the *multiuser efficiency*, which is the ratio between the energy that a user would require to achieve the same BER in absence of interfering users and the actual energy [1],

$$\eta_k = \frac{\left[\sigma_0 \cdot Q^{-1}(\mathsf{P}_k)\right]^2}{P_k}. \tag{10}$$

Immediately, the BER can be expressed in the multiuser efficiency as

$$\mathsf{P}_k = Q\left(\frac{\sqrt{\eta_k \cdot P_k}}{\sigma_0}\right). \tag{11}$$

In this paper, we study the matched filter, the decorrelator, the MMSE detector and the optimal detectors. The BER and the SIR performance of these CDMA detectors have received considerable attention in the literature. In general, the performance is dependent on the system size (K, N) as well as the instantaneous spreading sequences $\mathbf{S}$, and is therefore very hard to quantify. It turns out that this dependency vanishes in the so-called *large-system limit*, i.e., the user number K and the spreading factor N both tend to infinity but with their ratio K/N converging to a constant β. In the following we briefly describe each of these detectors and present previously known large-system results.

2.2.1 The Single-user Matched Filter

The most innocent detection is achieved by matched filtering using the desired user's spreading sequence. A soft decision is obtained for user k,

$$\tilde{d}_k^{(\mathrm{mf})} = \mathbf{s}_k^{\mathrm{H}}\,\mathbf{r} \tag{12}$$

$$= \sqrt{P_k}\,d_k + \sum_{i\neq k}(\mathbf{s}_k^{\mathrm{H}}\mathbf{s}_i)\sqrt{P_i}\,d_i + \sigma_0 w_k \tag{13}$$

where w_k is a standard Gaussian random variable. The second term, the MAI has a variance of β as $K = \beta N \to \infty$. Hence the large-system SIR is simply

$$\mathsf{SIR}_k^{(\mathrm{mf})} = \frac{P_k}{\sigma_0^2 + \beta}. \tag{14}$$

It can be shown using the central limit theorem that the MAI converges to a Gaussian random variable in the large-system limit. Thus the BER[2] is

$$\mathsf{P}_k^{(\mathrm{mf})} = Q\left(\sqrt{\mathsf{SIR}_k^{(\mathrm{mf})}}\right). \tag{15}$$

The multiuser efficiency is the same for all users

$$\eta^{(\mathrm{mf})} = \frac{1}{1 + \frac{\beta}{\sigma_0^2}}. \tag{16}$$

It is worth noting that a single multiuser efficiency determines the matched filter performance for all users, since the SIR can be obtained as

$$\mathsf{SIR}_k^{(\mathrm{mf})} = \frac{P_k}{\sigma_0^2} \cdot \eta^{(\mathrm{mf})} \tag{17}$$

and then the BER by (15). This is a result of the inherent symmetry of the multiuser game. Indeed, from every user's point of view, the total interference from the rest of the users is statistically the same in the large-system limit. The only difference among the decision statistics is their own energies. By normalizing with respect to one's own energy, the multiuser efficiency is the same for every user.

2.2.2 The MMSE Detector

The MMSE detector is a linear filter which minimizes the mean-square error between the original data and its outputs:

$$\tilde{\mathbf{d}}^{(\mathrm{mmse})} = \mathbf{A}^{-1}\left[\mathbf{S}^{\mathrm{H}}\mathbf{S} + \sigma^2\mathbf{A}^{-2}\right]^{-1}\mathbf{S}^{\mathrm{H}}\mathbf{r}. \tag{18}$$

The decision statistic for user k can be described as

$$\tilde{d}_k^{(\mathrm{mmse})} = H_{kk}d_k + \sum_{i\neq k} H_{ik}d_i + \sigma_0 w_k \tag{19}$$

where H_{ik} is the element of $\mathbf{H} = \mathbf{A}^{-1}\left[\mathbf{S}^{\mathrm{H}}\mathbf{S} + \sigma^2\mathbf{A}^{-2}\right]^{-1}\mathbf{S}^{\mathrm{H}}\mathbf{S}\mathbf{A}$ on the i^{th} row and the k^{th} column, and w_k is a Gaussian random variable.

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[2] Precisely, we refer to the BER in the large-system limit. Unless otherwise stated, all performance indexes such as BER, SIR, multiuser efficiency and spectral efficiency refer to large-system performance hereafter.

In the case of equal-energy users, the large-system SIR of the MMSE detector was first obtained in [1] as

$$\mathsf{SIR}^{\text{(mmse)}} = \frac{1}{\sigma_0^2} - \frac{1}{4\sigma_0^2}\left(\sqrt{\left(1+\sqrt{\beta}\right)^2+\sigma_0^2} - \sqrt{\left(1-\sqrt{\beta}\right)^2+\sigma_0^2}\right)^2 \tag{20}$$

which is the unique positive solution to

$$\mathsf{SIR} = \frac{1}{\sigma_0^2 + \frac{\beta}{\mathsf{SIR}+1}} \tag{21}$$

A generalization of (21) to the case of arbitrary energy distribution using random matrix theory results in the so-called Tse-Hanly equation [4]. In the large-system limit, with probability 1, the output decision statistic given by (19) converges in distribution to a Gaussian random variable [9, 13]. Hence the BER is determined by the SIR by a simple expression similar to (15). Using (17), the Tse-Hanly equation is distilled to the following fixed-point equation for the multiuser efficiency in [10]

$$\eta + \beta \underset{P}{\mathsf{E}}\left\{\frac{P\,\eta}{P\,\eta + \sigma_0^2}\right\} = 1 \tag{22}$$

where $\underset{P}{\mathsf{E}}\{\cdot\}$ denotes the expectation taken over the subscript random variable P, drawn according to the energy distribution here. Note that the subscript is often omitted if no ambiguity arises as for on which random variables the expectation is taken. Again, due to the fact that the output MAI seen by each user has the same asymptotic distribution, the multiuser efficiency is the same for all users.

2.2.3 The Decorrelator

The decorrelating detector, or, the decorrelator, removes the MAI in the expense of enhanced thermal noise. Its output is

$$\tilde{\mathbf{d}}^{\text{(dec)}} = \mathbf{A}^{-1}[\mathbf{S}^{\mathrm{H}}\mathbf{S}]^{+}\mathbf{S}^{\mathrm{H}}\mathbf{r} \tag{23}$$

where $[\cdot]^+$ denotes the Moore-Penrose psudo-inverse, which reduces to the normal matrix inverse for non-singular square matrices. In case of $\beta \le 1$, the large-system multiuser efficiency is [1]

$$\eta^{\text{(dec)}} = 1 - \beta. \tag{24}$$

It is incorrectly claimed in [28] that the multiuser efficiency is 0 if $\beta > 1$. We find the correct answer in section 4.

2.2.4 The Optimal Detectors

The jointly optimal detector maximizes the joint posterior probability and result in

$$\hat{\mathbf{d}}^{\text{(jo)}} = \arg \max_{\mathbf{d}\in\{-1,1\}^K} p_0(\mathbf{d}|\mathbf{r},\mathbf{S}) \tag{25}$$

where

$$p_0(\mathbf{d}|\mathbf{r},\mathbf{S}) \triangleq \frac{p_0(\mathbf{r}|\mathbf{d},\mathbf{S})}{\sum_{\mathbf{e}\in\{-1,1\}^K} p_0(\mathbf{r}|\mathbf{e},\mathbf{S})p_0(\mathbf{e})}. \tag{26}$$

The individually optimal detector maximizes the marginal posterior probability and results in

$$\hat{d}_k^{(\text{io})} = \arg \max_{d_k \in \{-1,1\}} p_0(d_k|\mathbf{r}, \mathbf{S}) \tag{27}$$

where

$$p_0(d_k|\mathbf{r}, \mathbf{S}) \overset{\triangle}{=} \sum_{\mathbf{d}_{\bar{k}} \in \{-1,1\}^{K-1}} p_0(\mathbf{d}|\mathbf{r}, \mathbf{S}) \tag{28}$$

where $\mathbf{d}_{\bar{k}}$ denotes the vector $\mathbf{d}$ with the k^{th} element stricken out. The IO detector achieves the minimum possible BER among all multiuser detectors.

The multiuser efficiency of the optimal detector is shown to converge to 1 if the zero-noise limit is first taken and then the large-system limit [6]. For finite SNRs, the large-system performance of the optimal detectors has been solved for the case where all users' energies are the same [26, 28]. In parallel with the format of the result in (22), the multiuser efficiency of the IO detector is the solution to a fixed-point equation

$$\eta + \frac{\beta\eta}{\sigma_0^2}\left[1 - \frac{1}{\sqrt{2\pi}} \int e^{-\frac{z^2}{2}} \tanh\left(\sqrt{\frac{\eta}{\sigma_0^2}}\, z + \frac{\eta}{\sigma_0^2}\right) \mathrm{d}z\right] = 1. \tag{29}$$

The efficiency of the JO detector is also found in [26] but omitted here. We find the multiuser efficiency for an arbitrary energy distribution for both the IO and the JO detectors in section 5.

2.2.5 Spectral Efficiency

Of fundamental importance about a CDMA channel is its spectral efficiency, defined as the total number of bits per dimension that can be transmitted reliably. In [5] the large-system spectral efficiency of the Gaussian CDMA channel for the detectors of interest in this paper is obtained for equal-energy case.

If a linear detector discussed in section 2.2 is used, the spectral efficiency for user k is

$$\mathsf{C}_k^{(\text{l})} = \frac{\beta}{2} \log\left(1 + \mathsf{SIR}_k^{(\text{l})}\right) \tag{30}$$

where $\mathsf{SIR}_k^{(\text{l})}$ is the user's output SIR.[3] It can be easily justified by noticing that these linear detectors output asymptotically Gaussian decision statistics [13].

Without any constraint on the type of detector, the spectral efficiency of a fading CDMA channel is [10]

$$\mathsf{C} = \frac{\beta}{2} \mathop{\mathsf{E}}_{P}\left\{\log\left(1 + \frac{\eta^{(\text{mmse})} P}{\sigma_0^2}\right)\right\} + \frac{1}{2}\left(\eta^{(\text{mmse})} - 1 - \log \eta^{(\text{mmse})}\right) \tag{31}$$

where the expectation is taken over the received energy distribution due to fading. If the inputs to the CDMA channel are constrained to be binary, the spectral efficiency is obtained for equal-energy case [28]

$$\mathsf{C} = -\frac{\beta}{\sqrt{2\pi}} \int e^{-\frac{z^2}{2}} \log\cosh\left(\frac{\eta^{(\text{io})}}{\sigma_0^2} + \sqrt{\frac{\eta^{(\text{io})}}{\sigma_0^2}}\, z\right) \mathrm{d}z + \frac{\beta\eta^{(\text{io})}}{\sigma_0^2} + \frac{1}{2}\left(\eta^{(\text{io})} - 1 - \log \eta^{(\text{io})}\right) \tag{32}$$

where $\eta^{(\text{io})}$ is the efficiency of the individually optimal detector, i.e., the solution to (29).

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[3] The unit of spectral efficiency is nat per dimension per channel use throughout this paper unless otherwise stated.

2.3 Summary of New Results

In this paper, we study multiuser systems from a statistical mechanics perspective. In a unified framework, all the known results in Section 2.2 are rederived, and the following new results are found.

In Section 4 and 5, we show that for every multiuser detector of our interest, the BER of a user with received energy P is

$$\mathsf{P}(P) = Q\left(\frac{\sqrt{\eta \cdot P}}{\sigma_0}\right) \tag{33}$$

where η is the common multiuser efficiency for all users,

$$\eta = \frac{1}{1 + \frac{\beta}{\sigma_0^2}(1 - 2m + q)} \tag{34}$$

where m and q are some quantities dependent on the choice of the detector and the energy distribution.

For the matched filter and the MMSE detector, equation (34) is consistent to the previous multiuser efficiency results given by (16) and (22) respectively. For the decorrelator, if $\beta < 1$, the efficiency follows (24); otherwise, the efficiency is the solution to

$$\eta + \frac{\eta\beta}{\sigma_0^2}\left[1 - 2m + \mathsf{E}\left\{\frac{P^3 + P^2\sigma_0^2/\eta}{[P + \beta(1-m)]^2}\right\}\right] = 1 \tag{35}$$

where m satisfies

$$\mathsf{E}\left\{\frac{\beta P}{P + \beta(1-m)}\right\} = 1. \tag{36}$$

For the optimal detectors, the parameters m and q in the efficiency expression (34) satisfy joint equations

$$E = \frac{1}{\sigma^2 + \beta(1-q)} \tag{37a}$$

$$F = \frac{\sigma_0^2 + \beta(1 - 2m + q)}{[\sigma^2 + \beta(1-q)]^2} \tag{37b}$$

$$m = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{z^2}{2}} \mathop{\mathsf{E}}_{P}\left\{P \tanh\left(PE + \sqrt{PF}\, z\right)\right\} \mathrm{d}z \tag{37c}$$

$$q = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{z^2}{2}} \mathop{\mathsf{E}}_{P}\left\{P \tanh^2\left(PE + \sqrt{PF}\, z\right)\right\} \mathrm{d}z. \tag{37d}$$

In case of the JO detector, $\sigma = 0$ in (37). In case of the IO detector, $\sigma = \sigma_0$ and (34) can be simplified to a fixed-point equation

$$\eta + \frac{\beta\eta}{\sigma_0^2}\left[1 - \frac{1}{\sqrt{2\pi}} \int e^{-\frac{z^2}{2}} \mathop{\mathsf{E}}_{P}\left\{P \tanh\left(\frac{\eta P}{\sigma_0^2} + \sqrt{\frac{\eta P}{\sigma_0^2}}\, z\right)\right\} \mathrm{d}z\right] = 1. \tag{38}$$

It is straight forward to show that the optimal efficiency converges to 1 if the large-system limit is taken before the zero-noise limit.

The new findings in the performance of uncoded transmission are summarized using a general and original interpretation of multiuser detection from the statistical mechanics perspective in Section 6.

Moreover, we show in Section 7 that the spectral efficiency of a multiuser CDMA channel without power control is exactly (31) where the expectation is taken over the energy distribution. The spectral efficiency of

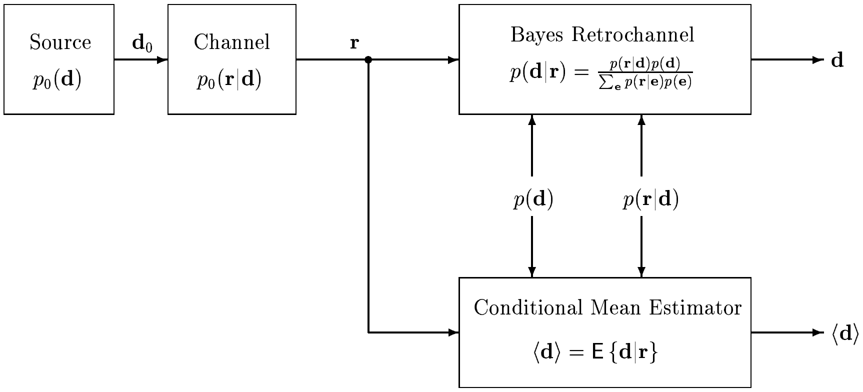

Figure 1: The Bayes retrochannel and the conditional mean estimator.

the same channel with binary input constraint is found as

$$\mathsf{C} = -\frac{\beta}{\sqrt{2\pi}} \int e^{-\frac{z^2}{2}} \underset{P}{\mathsf{E}} \left\{ \log\,\cosh \left(\frac{\eta^{(\mathrm{io})} P}{\sigma_0^2} + \sqrt{\frac{\eta^{(\mathrm{io})} P}{\sigma_0^2}}\, z \right) \right\} \mathrm{d}z + \frac{\beta \eta^{(\mathrm{io})}}{\sigma_0^2} + \frac{1}{2} \left(\eta^{(\mathrm{io})} - 1 - \log \eta^{(\mathrm{io})} \right) \tag{39}$$

where $\eta^{(\mathrm{io})}$ is the efficiency of the individually optimal detector, i.e., the solution to (38). This is a generalization of (32).

3 Conditional Mean Estimator and Spin Glass

3.1 Bayes Retrochannel and Conditional Mean Estimator

We study a general estimation problem as depicted in Fig. 1. A (vector) source symbol $\mathbf{d}_0$ is drawn according to a prior distribution $p_0(\mathbf{d}_0)$. The channel, upon an input $\mathbf{d}_0$, generates an output $\mathbf{r}$ according to a conditional probability distribution $p_0(\mathbf{r}|\mathbf{d})$. We want to find an estimator that, upon receipt of $\mathbf{r}$, gives a good estimate of the originally transmitted data $\mathbf{d}_0$.

One interesting candidate is an adjunct channel with a conditional distribution $p(\mathbf{d}|\mathbf{r})$, which is induced by a postulated prior distribution $p(\mathbf{d})$ and a postulated conditional distribution $p(\mathbf{r}|\mathbf{d})$ using the Bayes formula

$$p(\mathbf{d}|\mathbf{r}) = \frac{p(\mathbf{r}|\mathbf{d})p(\mathbf{d})}{\sum_{\mathbf{e}} p(\mathbf{r}|\mathbf{e})p(\mathbf{e})}. \tag{40}$$

We call this channel a *Bayes retrochannel*. If the postulated prior and conditional distributions are the same as the true ones, i.e., $p(\mathbf{d}) \equiv p_0(\mathbf{d})$ and $p(\mathbf{r}|\mathbf{d}) \equiv p_0(\mathbf{r}|\mathbf{d})$, then $p(\mathbf{d}|\mathbf{r})$ is exactly the posterior probability distribution corresponding to the true source and the true channel. In general, the postulated prior as well as the conditional distribution can be different to the true ones. In fact, $p(\mathbf{d})$ and $p_0(\mathbf{d})$ may even have different allowed symbol sets. In case the support of $p(\mathbf{d})$ is the Euclidean space instead of a discrete set, the sum in (40) shall be replaced by an integral. Clearly, the retrochannel output is different to the usual notion of an estimate of $\mathbf{d}_0$ since it is a random variable instead of a deterministic function of $\mathbf{r}$. Hence, the retrochannel can be regarded as a "stochastic estimator".

We also consider a so called *conditional mean estimator*, which gives a deterministic output upon an input

$\mathbf{r}$ as

$$\hat{\mathbf{d}} = \langle \mathbf{d} \rangle \overset{\triangle}{=} \mathsf{E}\left\{\mathbf{d}|\mathbf{r}\right\} \tag{41}$$

where, by definition, the operator $\langle \cdot \rangle$ gives the expectation taken over a distribution $p(\mathbf{d}|\mathbf{r})$, which depends on the postulated prior and conditional distributions assumed for the Bayes retrochannel. In this case, the output of the estimator is exactly the mean value of the stochastic estimate generated by the Bayes retrochannel.

Essentially, the conditional mean estimator is a soft decision detector with the freedom of choosing the postulated prior and conditional distributions. Interestingly, tuning the postulated distributions allows us to realize arbitrary detectors. In particular, if the source and channel symbols are both scalar, the prior is symmetric binary, $p_0(d = 1) = p_0(d = -1) = \frac{1}{2}$, and the postulated distributions are the same as the true ones, then the conditional mean estimate is

$$\langle d \rangle = p(d = 1|r) - p(d = -1|r) \tag{42}$$

whose sign is the maximum a posteriori detector output for the scalar channel.

We now study the conditional mean estimator in the Gaussian CDMA channel setting. Recall that the spreading sequence matrix is $\mathbf{S}$. Let the conditional distribution be

$$p(\mathbf{r}|\mathbf{d}, \mathbf{S}) = (2\pi\sigma^2)^{-\frac{N}{2}} \exp\left[-\frac{1}{2\sigma^2}\|\mathbf{r} - \mathbf{SAd}\|^2\right] \tag{43}$$

which differs from the true channel law $p_0(\mathbf{r}|\mathbf{d}, \mathbf{S})$ by a positive control parameter σ, where in case $\sigma = \sigma_0$, $p_0(\mathbf{r}|\mathbf{d}, \mathbf{S}) \equiv p(\mathbf{r}|\mathbf{d}, \mathbf{S})$. Using the Bayes formula, the posterior probability distribution is then

$$p(\mathbf{d}|\mathbf{r}, \mathbf{S}) = Z^{-1}(\mathbf{r}, \mathbf{S})\, p(\mathbf{d}) \exp\left[-\frac{1}{2\sigma^2}\|\mathbf{r} - \mathbf{SAd}\|^2\right] \tag{44}$$

where

$$Z(\mathbf{r}, \mathbf{S}) = \sum_{\mathbf{d}} p(\mathbf{d}) \exp\left[-\frac{1}{2\sigma^2}\|\mathbf{r} - \mathbf{SAd}\|^2\right]. \tag{45}$$

Note that $Z(\mathbf{r}, \mathbf{S})$ is in fact proportional to the conditional density of $p(\mathbf{r}|\mathbf{S})$. The conditional mean estimator outputs the mean value of the posterior probability distribution,

$$\tilde{d}_k = \langle d_k \rangle = \mathsf{E}\left\{d_k|\mathbf{r}, \mathbf{S}\right\} \tag{46}$$

where the expectation is taken over $p(\mathbf{d}|\mathbf{r}, \mathbf{S})$. We identify a few choices for the prior distribution $p(\mathbf{d})$ and the control parameter σ for the conditional distribution under which the conditional mean estimator becomes equivalent to each of the multiuser detectors discussed in section 2.2.

3.1.1 Linear Detectors

We assume standard Gaussian priors,

$$p^{(l)}(\mathbf{d}) = \prod_{k=1}^{K}\left[\frac{1}{\sqrt{2\pi}} e^{-\frac{d_k^2}{2}}\right] = (2\pi)^{-\frac{K}{2}} \exp\left[-\frac{1}{2}\|\mathbf{d}\|^2\right]. \tag{47}$$

The posterior probability distribution is then

$$p^{(l)}(\mathbf{d}|\mathbf{r}, \mathbf{S}) = [Z^{(l)}(\mathbf{r}, \mathbf{S})]^{-1} \exp\left[-\frac{1}{2}\|\mathbf{d}\|^2 - \frac{1}{2\sigma^2}\|\mathbf{r} - \mathbf{SAd}\|^2\right] \tag{48}$$

where

$$Z^{(\mathrm{l})}(\mathbf{r},\mathbf{S}) = \int \mathrm{d}\mathbf{d}\, \exp\left[-\frac{1}{2}\|\mathbf{d}\|^2\right] \exp\left[-\frac{1}{2\sigma^2}\|\mathbf{r}-\mathbf{SAd}\|^2\right]. \tag{49}$$

Here we use the superscript (l) to denote linear detectors.

Since the probability given by (48) is exponential in a quadratic function in $\mathbf{d}$, hence $\mathbf{d}$ is a Gaussian random vector conditioned on $\mathbf{r}$ and $\mathbf{S}$, i.e., given $\mathbf{r}$ and $\mathbf{S}$, the components of $\mathbf{d}$, $d_1, \dots, d_K$, are jointly Gaussian. The conditional mean $\mathsf{E}\{d_k|\mathbf{r},\mathbf{S}\}$ is therefore consistent with $\mathsf{E}\{\mathbf{d}|\mathbf{r},\mathbf{S}\}$, which maximizes the posterior probability distribution $p^{(\mathrm{l})}(\mathbf{d}|\mathbf{r},\mathbf{S})$ as a property of Gaussian distributions. The extremum is the solution to

$$\frac{\partial p^{(\mathrm{l})}(\mathbf{d}|\mathbf{r},\mathbf{S})}{\partial \mathbf{d}^{\mathrm{H}}} = 0, \tag{50}$$

which yields

$$\langle \mathbf{d}\rangle_\sigma^{(\mathrm{l})} = \mathbf{A}^{-1}\left[\mathbf{S}^{\mathrm{H}}\mathbf{S} + \sigma^2\mathbf{A}^{-2}\right]^{-1}\mathbf{S}^{\mathrm{H}}\mathbf{r} \tag{51}$$

where we use the subscript σ to denote an average over the posterior probability distribution with the control parameter σ.

By choosing different values for σ, we arrive at different linear detectors. If $\sigma \to \infty$, then

$$\sigma^2 \cdot \langle d_k\rangle_\sigma^{(\mathrm{l})} \longrightarrow \sqrt{P_k}\ \mathbf{s}_k^{\mathrm{H}}\,\mathbf{r}. \tag{52}$$

Hence the conditional mean estimate is consistent in sign with the matched filter output. If $\sigma = \sigma_0$, equation (52) gives exactly the soft output of the MMSE receiver as in (19). If $\sigma \to 0$, we approach the soft output of the decorrelator as given by (23). Indeed, control parameter can be used to tune a parameterized conditional mean estimator to a desired one in a set of detectors.

3.1.2 The Optimal Detectors

Let $p(\mathbf{d})$ be the true binary symmetric priors $p_0(\mathbf{d})$. The posterior probability distribution is then

$$p^{(\mathrm{o})}(\mathbf{d}|\mathbf{r},\mathbf{S}) = \left[Z^{(\mathrm{o})}(\mathbf{r},\mathbf{S})\right]^{-1}\exp\left[-\frac{1}{2\sigma^2}\|\mathbf{r}-\mathbf{SAd}\|^2\right],\ \ \mathbf{d}\in\{-1,1\}^K \tag{53}$$

where

$$Z^{(\mathrm{o})}(\mathbf{r},\mathbf{S}) = \sum_{\mathbf{d}\in\{-1,1\}^K}\exp\left[-\frac{1}{2\sigma^2}\|\mathbf{r}-\mathbf{SAd}\|^2\right]. \tag{54}$$

We use the superscript (o) for the optimal detectors.

Suppose that the control parameter takes the value of σ_0, then the conditional probability distribution is the true channel law. The conditional mean estimate is

$$\langle d_k\rangle_{\sigma=\sigma_0}^{(\mathrm{o})} = p_0(d_k=1|\mathbf{r},\mathbf{S}) - p_0(d_k=-1|\mathbf{r},\mathbf{S}). \tag{55}$$

Clearly, this soft output is consistent in sign with the hard decision of the IO detector as given by (27).

Alternatively, if $\sigma \to 0$, all probability mass of the distribution $p^{(\mathrm{o})}(\mathbf{d}|\mathbf{r},\mathbf{S})$ is eventually concentrated on a vector $\hat{\mathbf{d}}$ that achieves the minimum of $\|\mathbf{r}-\mathbf{SAd}\|$, which also maximizes the posterior probability $p_0(\mathbf{d}|\mathbf{r},\mathbf{S})$. The conditional mean estimator output

$$\lim_{\sigma\to 0}\langle d_k\rangle_\sigma^{(\mathrm{o})} \tag{56}$$

then singles out the k^{th} component of this $\hat{\mathbf{d}}$ at the minimum. Therefore by letting $\sigma \to 0$ the conditional mean estimator is equivalent to the JO detector as given by (25).

Worth mentioning here is that if $\sigma \to \infty$, the conditional mean estimator reduces to the matched filter. This can be verified by noticing that

$$p^{(\mathrm{o})}(\mathbf{d}|\mathbf{r},\mathbf{S}) \propto 1 - \frac{1}{2\sigma^2}\|\mathbf{r}-\mathbf{S}\mathbf{A}\mathbf{d}\|^2 + o\left(\frac{1}{\sigma^2}\right) \tag{57}$$

and hence

$$\lim_{\sigma\to\infty} \sigma^2 \cdot \langle d_k\rangle_\sigma^{(\mathrm{o})} \propto 2\sqrt{P_k}\,\mathbf{s}_k^\top \mathbf{r}. \tag{58}$$

So far, we have expressed each multiuser detector of our interest as a conditional mean estimator. In the remaining part of this section we construct equivalent thermodynamic systems and relate the conditional mean estimator performance to their macroscopic properties.

3.2 Preliminaries of Statistical Mechanics

A principal goal of statistical mechanics is to study the macroscopic properties of physical systems containing a large number of particles starting from the knowledge of microscopic interactions between the particles. Let the microscopic state of a system be described by the configuration of some K variables, $\{d_k\}_{k=1}^K$. A configuration of the system is $\mathbf{d} = [d_1,\ldots,d_K]^\top$. The basic quantity characterizing the microscopic states is the energy (also known as the *Hamiltonian*), which is a function of the state variables, denoted by $H(\mathbf{d})$. The configuration of the system evolves over time according to some physical laws. After long enough time the system will be in thermal equilibrium. An observable quantity of the system, which is the average value of the quantity over time, can be obtained by averaging over the ensemble of the states. In particular, the energy of the system is

$$\mathcal{E} = \sum_{\mathbf{d}} p(\mathbf{d})H(\mathbf{d}) \tag{59}$$

where $p(\mathbf{d})$ is the probability of the system being found in configuration $\mathbf{d}$. In other words, as far as the macroscopic properties are concerned, it suffices to describe the system statistically instead of solving the exact dynamic trajectories. Another fundamental quantity is the entropy, defined as

$$-\sum_{\mathbf{d}} p(\mathbf{d})\log p(\mathbf{d}). \tag{60}$$

It is assumed that the system is not completely isolated. At thermal equilibrium, due to interactions with the surrounding world, the energy of the system is conserved, and the entropy of the system is the maximum possible.

Consider now, what must the probability distribution be, which would maximize the entropy. One can use the Langrange multiplier method and consider the following cost function

$$f = -\sum_{\mathbf{d}} p(\mathbf{d})\log p(\mathbf{d}) - \beta\left(\sum_{\mathbf{d}} p(\mathbf{d})H(\mathbf{d}) - \mathcal{E}\right) - \gamma\left(\sum_{\mathbf{d}} p(\mathbf{d}) - 1\right) \tag{61}$$

where β and γ are the multipliers. Variation with respect to $p(\cdot)$ gives the Boltzmann distribution

$$p(\mathbf{d}) = Z^{-1}\exp\left[-\beta H(\mathbf{d})\right] \tag{62}$$

where

$$Z = \sum_{\mathbf{d}} \exp\left[-\beta H(\mathbf{d})\right] \tag{63}$$

is a normalizing factor called the *partition function*, and the parameter β is the *inverse temperature* (not to be confused with the limit of K/N for the multiuser systems), which is determined by the energy constraint

$$Z^{-1} \sum_{\mathbf{d}} H(\mathbf{d}) \exp\left[-\beta H(\mathbf{d})\right] = \mathcal{E}. \tag{64}$$

An important quantity in statistical mechanics is the free energy, defined as the energy minus the entropy divided by the inverse temperature

$$\mathcal{F} = \mathcal{E} + \frac{1}{\beta} \sum_{\mathbf{d}} p(\mathbf{d}) \log p(\mathbf{d}), \tag{65}$$

which can also be expressed as

$$\begin{aligned} \mathcal{F} &= \sum_{\mathbf{d}} p(\mathbf{d}) H(\mathbf{d}) + \frac{1}{\beta} \sum_{\mathbf{d}} p(\mathbf{d}) \log\left(Z^{-1} \exp\left[-\beta H(\mathbf{d})\right]\right) && (66) \\ &= -\frac{1}{\beta} \log Z. && (67) \end{aligned}$$

In all, at thermal equilibrium, the entropy is maximized and the free energy minimized. The system is found in a configuration with probability that is negative exponential in the associated energy of the configuration. The most probable configuration is the ground state which has the minimum associated energy.

3.3 The Bayes Retrochannel and Spin Glass

A spin glass is a magnetic system consisting of many directional spins, in which the interaction of the spins is determined by the so-called *quenched random variables*, i.e., whose values are determined by the realization of the spin glass [32].[4] The energy, the entropy and the free energy of a spin glass are similarly defined as in Section 3.2 with $H(\mathbf{d})$ replaced by a Hamiltonian $H_{\mathbf{r}}(\mathbf{d})$ parameterized by some quenched random variable $\mathbf{r}$.

In the multiuser detection context, we study an artificial spin glass system which has a Hamiltonian $H_{\mathbf{r},\mathbf{S}}(\mathbf{d})$, in which the received signal $\mathbf{r}$ and the spreading sequences $\mathbf{S}$ are regarded as quenched random variables. Let the energy $\mathcal{E}$ be such that the inverse temperature is $\beta = 1$. Then, at thermal equilibrium, the probability distribution of the spin glass system configuration is

$$p(\mathbf{d}|\mathbf{r},\mathbf{S}) = Z^{-1}(\mathbf{r},\mathbf{S}) \exp\left[-H_{\mathbf{r},\mathbf{S}}(\mathbf{d})\right] \tag{68}$$

where

$$Z(\mathbf{r},\mathbf{S}) = \sum_{\mathbf{d}} \exp\left[-H_{\mathbf{r},\mathbf{S}}(\mathbf{d})\right]. \tag{69}$$

Compare (68) to the postulated posterior probability distributions associated with the corresponding Bayes retrochannels, $p^{(\mathrm{l})}(\mathbf{d}|\mathbf{r},\mathbf{S})$ and $p^{(\mathrm{o})}(\mathbf{d}|\mathbf{r},\mathbf{S})$, given by (48) and (53) respectively. The configuration distribution $p(\mathbf{d}|\mathbf{r},\mathbf{S})$ can be made to take each of the two posterior probability distributions if we choose the

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[4] Imagine a system consisting molecules with random magnetic spins that evolve over time, while the random positions of the molecules are fixed for each concrete instance as in a piece of glass.

corresponding Hamiltonian as

$$H^{(\mathrm{l})}_{\mathbf{r},\mathbf{S}}(\mathbf{d}) = \frac{1}{2}\|\mathbf{d}\|^2 + \frac{1}{2\sigma^2}\|\mathbf{r} - \mathbf{S}\mathbf{A}\mathbf{d}\|^2, \ \ \mathbf{d} \in \mathcal{R}^K \tag{70}$$

and

$$H^{(\mathrm{o})}_{\mathbf{r},\mathbf{S}}(\mathbf{d}) = \frac{1}{2\sigma^2}\|\mathbf{r} - \mathbf{S}\mathbf{A}\mathbf{d}\|^2, \ \ \mathbf{d} \in \{-1, 1\}^K. \tag{71}$$

In other words, by defining an appropriate Hamiltonian, the distribution of the thermodynamic system at equilibrium is exactly the same as the posterior probability distribution associated with a multiuser detector which assumes certain prior and conditional distributions. In this case, the probability that the transmitted symbols are $\mathbf{d}$, given the observation $\mathbf{r}$ and the spreading sequences $\mathbf{S}$, is the same as the probability that the thermodynamic system is at configuration $\mathbf{d}$, given quenched random variables $\mathbf{r}$ and $\mathbf{S}$. The common exponential structure of the probabilities results in simple expressions for the energies. Indeed, the Gaussian distribution is also a Boltzmann distribution with the squared Euclidean norm as the Hamiltonian.

A first look at the above does not yield any new ideas in analyzing the multiuser system. However, the spin glass model allows many concepts and methodologies developed in physics to be brought into the study of multiuser systems. Central to our detection problem is the self-averaging principle and the replica method.

3.4 Self-averaging Principle and Replica Method

3.4.1 Self-averaging Principle

A macroscopic quantity is often the expected value of a function of the stochastic estimate conditioned on the quenched randomness, i.e., $\langle f(\mathbf{d})\rangle$, for some $f : \mathcal{R}^K \to \mathcal{R}$. It is not difficult to see that a macroscopic quantity is an empirical average under certain realization of the quenched random variables. The self-averaging principle states that, with probability 1, the empirical average of a macroscopic quantity converges to its ensemble average with respect to the quenched randomness in the large-system (thermodynamic) limit. Precisely, as $K \to \infty$,

$$\langle f(\mathbf{d})\rangle \to \lim_{K\to\infty} \langle\!\langle \langle f(\mathbf{d})\rangle \rangle\!\rangle \ \ \text{with probability 1} \tag{72}$$

where $\langle\!\langle\cdot\rangle\!\rangle$ denotes the ensemble average with respect to the quenched random variables. A rigorous justification of the self-averaging principle is beyond the scope of this work but we believe it is a variation of the law of large numbers.

It is often easier to quantify the ensemble average since it is no longer dependent on the many quenched random variables in a large system. Due to the merit of the self-averaging principle, the fluctuation of a macroscopic property of a physical system vanishes as the system size increases. Therefore, the average is often highly representative of the property of an arbitrary realization of the system.

An example of macroscopic quantities in the multiuser detection context is the free energy per user (briefly referred to as the free energy hereafter) defined as

$$-\frac{1}{K}\log Z(\mathbf{r}, \mathbf{S}). \tag{73}$$

In the large-system limit, the free energy converges to

$$\mathcal{F} = -\lim_{K\to\infty} \left\langle\!\!\left\langle \frac{1}{K}\log Z(\mathbf{r}, \mathbf{S}) \right\rangle\!\!\right\rangle, \tag{74}$$

where $\langle\!\langle \cdot \rangle\!\rangle$ is defined explicitly as

$$\langle\!\langle f(\mathbf{r}, \mathbf{S}) \rangle\!\rangle \triangleq \underset{\mathbf{r},\mathbf{S}}{\mathsf{E}} \left\{ f(\mathbf{r}, \mathbf{S}) \right\}, \quad \forall f(\cdot, \cdot), \tag{75}$$

i.e., the expectation taken over the quenched random variables $\mathbf{r}$ and $\mathbf{S}$, which are the received signal and the spreading sequences respectively in this case.

3.4.2 Replica Method

The replica method is vital in the evaluation of the macroscopic quantities. Take the free energy for example. To circumvent the difficulty of taking the average of the logarithm in (74), we first write the free energy in an alternative expression

$$\mathcal{F} = - \lim_{K \to \infty} \frac{1}{K} \lim_{u \to 0} \frac{\partial}{\partial u} \log \langle\!\langle Z^u(\mathbf{r}, \mathbf{S}) \rangle\!\rangle, \tag{76}$$

which follows from

$$\lim_{u \to 0} \frac{\mathrm{d}}{\mathrm{d}u} \log \mathsf{E} \left\{ Z^u \right\} = \mathsf{E} \left\{ \log Z \right\}, \quad \forall Z > 0. \tag{77}$$

We then evaluate the average in (76) only for integer u and then take the formal limit as $u \to 0$ of the resulting expression to obtain the free energy. This trick is known as the “replica method” in statistical mechanics. There are intensive ongoing efforts in the mathematics and physics community to find a rigorous proof for the replica method which we shall avoid in this work.

A slight modification of the above method yields general macroscopic properties in the form of (72). We can define a modified posterior probability distribution

$$p(\mathbf{d}|\mathbf{r}, \mathbf{S}; h) = [Z(\mathbf{r}, \mathbf{S}; h)]^{-1} e^{h f(\mathbf{d})} p(\mathbf{d}|\mathbf{r}, \mathbf{S}) \tag{78}$$

where $p(\mathbf{d}|\mathbf{r}, \mathbf{S})$ is the original posterior probability distribution and

$$Z(\mathbf{r}, \mathbf{S}; h) = \sum_{\mathbf{d}} e^{h f(\mathbf{d})} p(\mathbf{d}|\mathbf{r}, \mathbf{S}) \tag{79}$$

is the modified partition function. We can then find

$$\begin{aligned}
& \lim_{h \to 0} \frac{\partial}{\partial h} \langle\!\langle \log Z(\mathbf{r}, \mathbf{S}; h) \rangle\!\rangle \\
= \ & \lim_{h \to 0} \left\langle\!\!\left\langle [Z(\mathbf{r}, \mathbf{S}; h)]^{-1} \sum_{\mathbf{d}} f(\mathbf{d}) e^{h f(\mathbf{d})} p(\mathbf{d}|\mathbf{r}, \mathbf{S}) \right\rangle\!\!\right\rangle \qquad (80) \\
= \ & \left\langle\!\!\left\langle [Z(\mathbf{r}, \mathbf{S})]^{-1} \sum_{\mathbf{d}} f(\mathbf{d}) p(\mathbf{d}|\mathbf{r}, \mathbf{S}) \right\rangle\!\!\right\rangle \qquad (81) \\
= \ & \langle\!\langle \langle f(\mathbf{d}) \rangle \rangle\!\rangle. \qquad (82)
\end{aligned}$$

Clearly, a macroscopic property may be obtained by way of calculating the free energy of the modified system

$$\langle\!\langle \log Z(\mathbf{r}, \mathbf{S}; h) \rangle\!\rangle \tag{83}$$

using the replica method.

3.5 BER

3.5.1 Correlation

In the multiuser detection setting, we study the most important performance measure, the bit-error-rate. We first consider equal-energy users, i.e., $P_k = 1$ for all k. Conditioned on $(\mathbf{r}, \mathbf{S})$, the percentage of erroneously detected bits in the current symbol interval is $\frac{1}{2}[1 - M(\mathbf{r}, \mathbf{S})]$ where

$$M(\mathbf{r}, \mathbf{S}) = \frac{1}{K}\hat{\mathbf{d}}^{\top}\mathbf{d}_0 = \frac{1}{K}\sum_{k=1}^{K} \mathsf{sgn}\left(\langle d_k \rangle\right) d_{0k} \tag{84}$$

is the *correlation* of the detector output and the original transmitted symbols. Although not in the form of $\langle f(\mathbf{d}) \rangle$, the correlation is a macroscopic quantity and hence also satisfies the self-averaging principle. Hence, $M(\mathbf{r}, \mathbf{S}) \to \mathsf{M}$ with probability 1 as $K \to \infty$ where the limit

$$\mathsf{M} = \lim_{K\to\infty} \langle\!\langle M(\mathbf{r}, \mathbf{S}) \rangle\!\rangle \tag{85}$$

is independent of the spreading sequences and the noise realization. The BER of every user converges in the large-system limit to

$$\mathsf{P} = \frac{1}{2}(1 - \mathsf{M}). \tag{86}$$

Due to symmetry of the spreading sequences, as far as the BER is concerned, we can assume that all transmitted symbols are $+1$, i.e., $d_{0k} = 1$ for all k. The average correlation is now

$$\left\langle\!\!\left\langle \frac{1}{K}\sum_{k=1}^{K} \mathsf{sgn}\left(\langle d_k \rangle\right) \right\rangle\!\!\right\rangle. \tag{87}$$

If d_k represents the local directional magnetization of a spin glass, then the average correlation is the overall magnetization of the system.

3.5.2 Arbitrary Energy Distribution

For an arbitrary energy distribution, without loss of generality, we study the BER of user 1 only. We assume first a degenerate energy distribution in which the first K_1 users are received at the same energy P_1, such that $\alpha_1 = K_1/K$ is a fixed positive number. The first K_1 users share the same BER, which can be written as

$$\mathsf{P}_1 = \frac{1}{2}(1 - \mathsf{M}_1) \tag{88}$$

where

$$\mathsf{M}_1 = \lim_{K\to\infty} \left\langle\!\!\left\langle \frac{1}{K_1}\sum_{k=1}^{K_1} \mathsf{sgn}\left(\langle d_k \rangle\right) \right\rangle\!\!\right\rangle. \tag{89}$$

is well-defined since the correlation inside $\langle\!\langle\cdot\rangle\!\rangle$ is also self-averaging as $K_1 = \alpha_1 K \to \infty$. It suffices then to evaluate (89).

In case of an arbitrary energy distribution F, we construct a sequence of degenerate distributions, $F^{(l)}, l = 1, 2, \dots$, that converge to F. For each L, we assume that the first $K_1^{(l)} = \alpha_1^{(l)} K$ users in the system take the same energy P_1. In general $\alpha^{(l)}$ diminishes but is always positive. For each l, we find a BER for user 1, and taking the limit yields the BER of user 1 under energy distribution F. This strategy also applies to calculating the free energy.

3.5.3 Replica Analysis

The replica method introduced in Section 3.4.2 can be a basis for evaluating the correlations. It is hard to calculate M_1 directly. Consider instead the quantity

$$\lim_{K\to\infty} \left\langle\!\!\left\langle \frac{1}{K_1} \sum_{k=1}^{K_1} \langle d_k \rangle^i \right\rangle\!\!\right\rangle. \tag{90}$$

If we can solve this for every i, then we can obtain the correlation (89) by considering a series of polynomials converging to sgn$(\cdot)$.

However, the quantity (90) can not be written in the form of (72). We circumvent this difficulty by resorting to a variation of the replica method. Without loss of generality, we consider group 1 users, whose BER is determined by the correlation M_1. Consider a replicated system with a posterior probability distribution as

$$p(\{\mathbf{d}_a\}_{a=1}^u|\mathbf{r},\mathbf{S};h) = Z^{(-u)}(\mathbf{r},\mathbf{S};h)\cdot \exp\left[h\sum_{k=1}^{K_1}\prod_{m=1}^{i} d_{a_m k}\right] p(\{\mathbf{d}_a\}_{a=1}^u|\mathbf{r},\mathbf{S}) \tag{91}$$

where $Z^{(-u)}(\mathbf{r},\mathbf{S};h)$ is the inverse of

$$Z^{(u)}(\mathbf{r},\mathbf{S};h) = \sum_{\{d_{ak}\}} \exp\left[h\sum_{k=1}^{K_1}\prod_{m=1}^{i} d_{a_m k}\right] p(\{\mathbf{d}_a\}_{a=1}^u|\mathbf{r},\mathbf{S}) \tag{92}$$

where $\sum_{\{d_{ak}\}}$ denotes a sum over all $d_{ak}\in\{-1,1\}$, $a=1,\dots,u$, $k=1,\dots,K$. Note that $1\le a_1 < \cdots < a_i \le u$ are arbitrary indexes selected from the u replicas, and $Z^{(u)}(\mathbf{r},\mathbf{S};h)$ is not a trivial power of $Z(\mathbf{r},\mathbf{S};h)$. Nevertheless, by differentiating the free energy with respect to h, we get

$$\lim_{u\to 0}\frac{\partial}{\partial h}\left\langle\!\!\left\langle \log Z^{(u)}(\mathbf{r},\mathbf{S};h)\right\rangle\!\!\right\rangle\Big|_{h=0} = \left\langle\!\!\left\langle \sum_{k=1}^{K_1} \langle d_k\rangle^i \right\rangle\!\!\right\rangle. \tag{93}$$

Equation (93) can be verified by noticing that its left hand side can be evaluated as

$$\lim_{u\to 0}\left\langle\!\!\left\langle Z^{-u}(\mathbf{r},\mathbf{S}) \sum_{\{d_{ak}\}} \left(\sum_{k=1}^{K_1}\prod_{m=1}^{i} d_{a_m k}\right) p(\{\mathbf{d}_a\}_{a=1}^u|\mathbf{r},\mathbf{S}) \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \sum_{k=1}^{K_1}\prod_{m=1}^{i} \langle d_{a_m k}\rangle \right\rangle\!\!\right\rangle, \tag{94}$$

which is equal to the right hand side of (93) since the average in (94) is on a system of u independent and statistically identical replicas, where $\langle d_{ak}\rangle = \langle d_k\rangle$ for all $a=1,\dots,u$. Therefore, the free energy of the modified replicated system is the key to the evaluation of the correlation.

4 Performance Analysis for Linear Detectors

We are now ready to analyze the BER performance of multiuser detectors using the replica method as described in Section 3.4. We study linear detectors in this section and then the optimal detectors in Section 5. The free energy of the linear system is first evaluated. We then calculate the correlation through evaluating the more involved free energy of a modified posterior probability distribution.

4.1 Free Energy

We assume the standard Gaussian priors introduced in (47),

$$p(\mathbf{d}) = p^{(\mathrm{l})}(\mathbf{d}) = (2\pi)^{-\frac{K}{2}} \exp\left[-\frac{1}{2}\|\mathbf{d}\|^2\right], \tag{95}$$

and let the conditional distribution be defined as (43)

$$p(\mathbf{r}|\mathbf{d},\mathbf{S}) = (2\pi\sigma^2)^{-\frac{N}{2}} \exp\left[-\frac{1}{2\sigma^2}\|\mathbf{r}-\mathbf{SAd}\|^2\right], \tag{96}$$

then the posterior probability distribution (44)

$$p(\mathbf{d}|\mathbf{r},\mathbf{S}) = p^{(\mathrm{l})}(\mathbf{d}|\mathbf{r},\mathbf{S}) \tag{97}$$

corresponds to the conditional mean estimator which can be tuned to the matched filter, the MMSE detector, or the decorrelator by a control parameter σ.

Consider u replicas of the same Bayes retrochannel with the same input $\mathbf{r}$ and $\mathbf{S}$. They can also be regarded as u identical and independent spin glasses with the same quenched randomness $(\mathbf{r},\mathbf{S})$. Statistically there is no difference among these replicas. Let $\mathbf{d}_a = [d_{a1},\dots,d_{aK}]^\top$ denote the channel outputs (or the spin glass configuration) of the a^{th} replica. The posterior probability distribution for the replicated system is

$$p(\{\mathbf{d}_a\}_{a=1}^u|\mathbf{r},\mathbf{S}) = Z^{-u}(\mathbf{r},\mathbf{S}) \prod_{a=1}^u \left\{p(\mathbf{d}_a)\exp\left[-\frac{1}{2\sigma^2}\|\mathbf{r}-\mathbf{SAd}_a\|^2\right]\right\} \tag{98}$$

where $Z^{-u}(\mathbf{r},\mathbf{S})$ is the inverse of

$$Z^u(\mathbf{r},\mathbf{S}) = \prod_{a=1}^u \int p(\mathbf{d}_a)\exp\left[-\frac{1}{2\sigma^2}\|\mathbf{r}-\mathbf{SAd}_a\|^2\right] \mathrm{d}\mathbf{d}_a. \tag{99}$$

Note that (99) can be regarded as an expectation over the postulated prior distribution

$$Z^u(\mathbf{r},\mathbf{S}) = \mathsf{E}_{\{d_{ak}\}}\left\{\prod_{a=1}^u \exp\left[-\frac{1}{2\sigma^2}\|\mathbf{r}-\mathbf{SAd}_a\|^2\right]\right\} \tag{100}$$

where $\mathsf{E}_{\{d_{ak}\}}\{\cdot\}$ denotes the expectation taken over the symbols of the replicated system, $d_{ak} \in \{-1,1\}$, $a = 1,\dots,u$, $k = 1,\dots,K$.

We assume that, in the free energy expression (76), the order of the limit in K and the limit of the derivative in u can be exchanged. No rigorous justification of this assumption is available in the literature except for certain cases such as the S-K model [32]. We follow the assumption so that the free energy can then be obtained as

$$\mathcal{F} = -\lim_{u\to 0}\frac{\partial}{\partial u}\lim_{K\to\infty}\frac{1}{K}\log\langle\!\langle Z^u(\mathbf{r},\mathbf{S})\rangle\!\rangle. \tag{101}$$

The average in (101) can be evaluated as an integral

$$\langle\!\langle Z^u(\mathbf{r},\mathbf{S})\rangle\!\rangle = \int \mathsf{E}_{\mathbf{S}}\{p_0(\mathbf{r}|\mathbf{d}_0,\mathbf{S})Z^u(\mathbf{r},\mathbf{S})\}\ \mathrm{d}\mathbf{r} \tag{102}$$

where the expectation is taken over the distribution of the spreading sequences. Note that in order to take the average with respect to $\mathbf{r}$, we need to use the true data prior which in this case puts unit mass on the vector

$\mathbf{d}_0 = [1, \dots, 1]^\top$. For ease of notation, we let $\sigma_a = \sigma$ for $a = 1, \dots, u$. Then (102) becomes

$$\begin{aligned}
&\langle\!\langle Z^u(\mathbf{r}, \mathbf{S}) \rangle\!\rangle \\
= \quad & \int \mathop{\mathsf{E}}_{\mathbf{S}} \left\{ (2\pi\sigma_0^2)^{-\frac{N}{2}} \exp\left[-\frac{1}{2\sigma_0^2} \|\mathbf{r} - \mathbf{SAd}_0\|^2 \right] \mathop{\mathsf{E}}_{\{d_{ak}\}} \left\{ \prod_{a=1}^{u} \exp\left[-\frac{1}{2\sigma^2} \|\mathbf{r} - \mathbf{SAd}_a\|^2 \right] \right\} \right\} \mathrm{d}\mathbf{r} \quad (103) \\
= \quad & \int \mathop{\mathsf{E}}_{\mathbf{S}} \left\{ (2\pi\sigma_0^2)^{-\frac{N}{2}} \mathop{\mathsf{E}}_{\{d_{ak}\}} \left\{ \prod_{n=1}^{N} \prod_{a=0}^{u} \exp\left[-\frac{1}{2\sigma_a^2} \left(r_n - \frac{1}{\sqrt{N}} \sum_{k=1}^{K} \sqrt{P_k}\, s_{nk} d_{ak} \right)^2 \right] \right\} \right\} \left(\prod_{n=1}^{N} \mathrm{d}r_n \right) \quad (104) \\
= \quad & \mathop{\mathsf{E}}_{\{d_{ak}\}} \left\{ \int (2\pi\sigma_0^2)^{-\frac{1}{2}} \mathop{\mathsf{E}}_{\mathbf{s}} \left\{ \prod_{a=0}^{u} \exp\left[-\frac{1}{2\sigma_a^2} \left(r - \frac{1}{\sqrt{N}} \sum_{k=1}^{K} \sqrt{P_k}\, s_k d_{ak} \right)^2 \right] \right\} \mathrm{d}r \right\}^N \quad (105)
\end{aligned}$$

where the inner expectation in (105) is now taken over $\mathbf{s} = [s_1, \dots, s_K]$, a vector of i.i.d. random chips taking the same distribution as the s_{nk}'s. For $a = 0, 1, \dots, u$, we define a set of random variables

$$v_a = \frac{1}{\sqrt{K}} \sum_{k=1}^{K} \sqrt{P_k}\, s_k d_{ak}. \tag{106}$$

The expectation over $\mathbf{s}$ in (105) can be replaced by an expectation over the random vector $\mathbf{v} = [v_0, v_1, \dots, v_u]^\top$, which is dependent on d_{ak}'s. Hence (105) can be rewritten as

$$\langle\!\langle Z^u(\mathbf{r}, \mathbf{S}) \rangle\!\rangle = \mathop{\mathsf{E}}_{\{d_{ak}\}} \left\{ \exp\left[\beta^{-1} \cdot K \cdot G_K^{(u)} \left(\{d_{ak}\} \right) \right] \right\} \tag{107}$$

where

$$G_K^{(u)} \left(\{d_{ak}\} \right) = \log \int (2\pi\sigma_0^2)^{-\frac{1}{2}} \mathop{\mathsf{E}}_{\mathbf{v}} \left\{ \prod_{a=0}^{u} \exp\left[-\frac{1}{2\sigma_a^2} \left(r - \sqrt{\beta}\, v_a \right)^2 \right] \right\} \mathrm{d}r. \tag{108}$$

In order to find the free energy, we first evaluate $G_K^{(u)}$ in the large-system limit. Note that the s_k's are i.i.d. random chips. For fixed d_{ak}'s, each v_a is a sum of K weighted independent random chips, and hence converges to a Gaussian random variable as $K \to \infty$. In fact, due to a generalization of the central limit theorem, the variables $\{v_a\}_{a=0}^{u}$ converge to a set of jointly Gaussian random variables, with zero mean and covariance matrix $\mathbf{Q}$, where

$$Q_{ab} = \mathop{\mathsf{E}}_{\mathbf{s}} \{v_a v_b\} = \frac{1}{K} \sum_{k=1}^{K} P_k d_{ak} d_{bk}, \quad a, b = 0, \dots, u. \tag{109}$$

Note that although not made explicit in the notation, Q_{ab} is a function of $\{d_{ak}, d_{bk}\}_{k=1}^{K}$. Trivially, $Q_{00} = 1$. The reader is referred to Appendix 9 for a justification of the asymptotic normality through the Edgeworth expansion [33]. As a result, we have

$$\exp\left[G_K^{(u)} \left(\{d_{ak}\} \right) \right] = \exp\left[G^{(u)}(\mathbf{Q}) \right] + \mathcal{O}(K^{-1}) \tag{110}$$

where

$$G^{(u)}(\mathbf{Q}) = \log \int (2\pi\sigma_0^2)^{-\frac{1}{2}} \mathop{\mathsf{E}}_{\mathbf{v}(\mathbf{Q})} \left\{ \prod_{a=0}^{u} \exp\left[-\frac{1}{2\sigma_a^2} \left(r - \sqrt{\beta}\, v_a \right)^2 \right] \right\} \mathrm{d}r \tag{111}$$

assuming that $\mathbf{v}(\mathbf{Q})$ is a Gaussian random vector with covariance matrix $\mathbf{Q}$. In Appendix 10 we evaluate (111) and obtain

$$G^{(u)}(\mathbf{Q}) = -\frac{1}{2}\log\det(\mathbf{I}+\mathbf{\Sigma Q}) - \frac{1}{2}\log\left(1+u\frac{\sigma_0^2}{\sigma^2}\right) \tag{112}$$

where $\mathbf{\Sigma}$ is a $(u+1)\times(u+1)$ matrix with the (a,b) entry

$$\mathbf{\Sigma}_{ab} = \frac{\beta}{\sigma_a^2}\delta(a,b) - \frac{\beta\sigma_0^2\sigma^2}{\sigma_a^2\sigma_b^2(\sigma^2+u\sigma_0^2)} \tag{113}$$

where $\delta(a,b)$ is 1 if $a=b$ and 0 otherwise.

By (107) and (110), we have

$$\begin{aligned} \frac{1}{K}\log\langle\!\langle Z^u(\mathbf{r},\mathbf{S})\rangle\!\rangle &= \frac{1}{K}\log \mathop{\mathsf{E}}_{\{d_{ak}\}}\left\{\exp\left[\beta^{-1}\cdot K\cdot G^{(u)}(\mathbf{Q})\right]+\mathcal{O}(1)\right\} && (114)\\ &= \frac{1}{K}\log\int\exp\left[\beta^{-1}\cdot K\cdot G^{(u)}(\mathbf{Q})\right]\,\mathrm{d}\mu_K^{(u)}(\mathbf{Q})+\mathcal{O}(K^{-1}) && (115)\end{aligned}$$

where the expectation over $\{d_{ak}\}$ is reduced to an integral over the probability measure of $\mathbf{Q}$, expressed as

$$\mu_K^{(u)}(\mathbf{Q}) = \mathop{\mathsf{E}}_{\{d_{ak}\}}\left\{\prod_{a\le b}^{u}{}'\,\delta\left(\sum_{k=1}^{K}P_k d_{ak}d_{bk} - KQ_{ab}\right)\right\} \tag{116}$$

where $\delta(\cdot)$ is the Dirac function. From Cramér's theorem in the large deviations theory, we know that, the probability measure of the empirical means $Q_{ab} = \frac{1}{K}\sum_{k=1}^{K}P_k d_{ak}d_{bk}$ satisfies, as $K\to\infty$, the large deviation property with some rate function $I^{(u)}(\mathbf{Q})$ [34]. From (115), we have by Varadhan's theorem

$$\lim_{K\to\infty}\frac{1}{K}\log\langle\!\langle Z^u(\mathbf{r},\mathbf{S})\rangle\!\rangle = \sup_{\mathbf{Q}}[\beta^{-1}\cdot G^{(u)}(\mathbf{Q}) - I^{(u)}(\mathbf{Q})] \tag{117}$$

where the supreme is over all possible $\mathbf{Q}$ that can be resulted from varying d_{ak}'s in (109).

The problem is now to evaluate $I^{(u)}$ and then find the extremum in (117). Using the Fourier transform representation of the Dirac function and through some algebra as shown in Appendix 11, we rewrite the measure as

$$\mu_K^{(u)}(\mathbf{Q}) = \int\exp\left[K\left(\mathop{\mathsf{E}}_{P}\{\log X(u,P)\} - \sum_{a\le b}^{u}{}'\,\tilde{Q}_{ab}Q_{ab}\right)\right]\left(\prod_{a\le b}^{u}{}'\,\frac{\mathrm{d}\tilde{Q}_{ab}}{2\pi j}\right) \tag{118}$$

where

$$X(u,P) = \mathop{\mathsf{E}}_{\{d_a\}}\left\{\exp\left[P\sum_{a\le b}^{u}{}'\,\tilde{Q}_{ab}d_a d_b\right]\right\} \tag{119}$$

where the $\mathop{\mathsf{E}}_{\{d_a\}}\{\cdot\}$ is taken over i.i.d. random symbols $d_1,\dots,d_u$ taking the same distribution as d_{ak}. In (118) and (119), $\prod_{a\le b}^{u}{}'$ is the product and $\sum_{a\le b}^{u}{}'$ the sum running over all pairs of (a,b) that $0\le a\le b\le u$ but $b>0$. Note the only occurrence of K as an exponential factor in (118). The rate $I^{(u)}$ can be identified as

$$I^{(u)}(\mathbf{Q}) = -\sup_{\tilde{\mathbf{Q}}}\left[\mathop{\mathsf{E}}_{P}\{\log X(u,P)\} - \sum_{a\le b}^{u}{}'\,\tilde{Q}_{ab}Q_{ab}\right]. \tag{120}$$

Together with (117), we have

$$\begin{aligned}&\lim_{K\to\infty}\frac{1}{K}\log\langle\!\langle Z^u(\mathbf{r},\mathbf{S})\rangle\!\rangle\\ &=\sup_{\mathbf{Q}}\left\{-\frac{1}{2\beta}\log\det(\mathbf{I}+\mathbf{\Sigma}\mathbf{Q})+\sup_{\tilde{\mathbf{Q}}}\left[\mathsf{E}_{P}\left\{\log X(u,P)\right\}-{\sum_{a\le b}^{u}}'\tilde{Q}_{ab}Q_{ab}\right]\right\}-\frac{1}{2\beta}\log\left(1+u\frac{\sigma_0^2}{\sigma^2}\right).\end{aligned} \tag{121}$$

To find the extremum in (121) with respect to $\mathbf{Q}$ and $\tilde{\mathbf{Q}}$ directly is a prohibitive task. Instead, we assume replica symmetry, i.e., for $a,b=1,\dots,u$, $a\neq b$,

$$Q_{0a} = \mathsf{E}\left\{v_0v_a\right\}=m \tag{122a}$$
$$Q_{ab} = \mathsf{E}\left\{v_av_b\right\}=q \tag{122b}$$
$$Q_{aa} = \mathsf{E}\left\{v_a^2\right\}=p \tag{122c}$$

and

$$\tilde{Q}_{0a} = E \tag{123a}$$
$$\tilde{Q}_{ab} = F \tag{123b}$$
$$\tilde{Q}_{aa} = G/2. \tag{123c}$$

We seek the extremum with the above replica symmetry constraint. The replica symmetry is a convenient choice over all possible ansatz. It is valid in most cases of interest in the large-system limit due to symmetry over replicas. The reader is referred to [32, 28] for a discussion on the validity of replica symmetry. Although no rigorous justification of replica symmetry is available in the literature, its successful use in physics and other areas gives strong evidence that replica symmetry should also be sensible in the CDMA multiuser detection problem. In this paper, we follow this assumption and rederive many known results and obtain some new results. In case replica symmetry is not true, the resulting BER is an upper bound of the true system performance.

In Appendix 10, we evaluate (111) under replica symmetry and obtain

$$G^{(u)}(m,q,p)=-\frac{u-1}{2}\log\left(1+\frac{\beta}{\sigma^2}(p-q)\right)-\frac{1}{2}\log\left[1+\frac{\beta}{\sigma^2}(p-q)+\frac{u}{\sigma^2}\left(\sigma_0^2+\beta(1-2m+q)\right)\right]. \tag{124}$$

In Appendix 11, we find the rate of (118) and obtain

$$\begin{aligned}&I^{(u)}(E,F,G,m,q,p)\\ &=-\mathsf{E}_{P}\left\{-\frac{u-1}{2}\log(1+P(F-G))-\frac{1}{2}\log(1+(1-u)PF-PG)+\frac{uP^2E^2}{2(1+(1-u)PF-PG)}\right\}\\ &\quad+uEm+\frac{u(u-1)}{2}Fq+\frac{u}{2}Gp.\end{aligned} \tag{125}$$

By (101) and (117), the free energy is therefore

$$\begin{aligned}
\mathcal{F} &= -\lim_{u\to 0}\frac{\partial}{\partial u}\left[\beta^{-1}\cdot G^{(u)} - I^{(u)}\right] && (126)\\
&= \frac{1}{2}\mathsf{E}_P\left\{\log(1+P(F-G)) - \frac{P^2E^2+PF}{1+P(F-G)}\right\} + Em - \frac{1}{2}Fq + \frac{1}{2}Gp \\
&\quad + \frac{1}{2\beta}\log\left(1+\frac{\beta}{\sigma^2}(p-q)\right) + \frac{1}{2\beta}\frac{\sigma_0^2+\beta(1-2m+q)}{\sigma^2+\beta(p-q)} && (127)
\end{aligned}$$

where the parameters (m,q,p,E,F,G) are such that $G^{(u)}$ and $I^{(u)}$ achieve their respective saddle-point so that the free energy achieves its extremum. It is not difficult to see by equating the partial derivatives of the free energy to zero that the parameters are the solution to the following set of joint equations

$$\begin{aligned}
E &= \frac{1}{\sigma^2+\beta(p-q)} && (128a)\\
F &= \frac{\sigma_0^2+\beta(1-2m+q)}{[\sigma^2+\beta(p-q)]^2} && (128b)\\
G &= \frac{\beta(1-2m+2q-p)+(\sigma_0^2-\sigma^2)}{[\sigma^2+\beta(p-q)]^2} && (128c)\\
m &= \mathsf{E}_P\left\{\frac{P^2E}{1+P(F-G)}\right\} && (128d)\\
q &= \mathsf{E}_P\left\{\frac{P^3E^2+P^2F}{[1+P(F-G)]^2}\right\} && (128e)\\
p &= \mathsf{E}_P\left\{P\frac{1+P(2F-G)+P^2E^2}{[1+P(F-G)]^2}\right\}. && (128f)
\end{aligned}$$

We call (128) the saddle-point equations. Immediately,

$$\begin{aligned}
G &= F-E && (129a)\\
p &= 1-m+q. && (129b)
\end{aligned}$$

The free energy is then simplified as

$$\begin{aligned}
\mathcal{F}^{(\mathrm{l})} =& \frac{1}{2}\mathsf{E}\left\{\log(1+PE) - \frac{P^2E^2+PF}{1+PE}\right\} + \frac{E}{2}(3m-1-q) + \frac{F}{2}(1-m)\\
&+\frac{1}{2\beta}\log\left(1+\frac{\beta}{\sigma^2}(1-m)\right) + \frac{1}{2\beta}\frac{\sigma_0^2+\beta(1-2m+q)}{\sigma^2+\beta(1-m)}
\end{aligned}$$

where the simplified saddle-point equations are

$$\begin{aligned}
E &= \frac{1}{\sigma^2+\beta(1-m)} && (130a)\\
m &= \mathsf{E}_P\left\{\frac{P^2E}{1+PE}\right\} && (130b)\\
F &= \frac{\sigma_0^2+\beta(1-2m+q)}{[\sigma^2+\beta(1-m)]^2} && (130c)\\
q &= \mathsf{E}_P\left\{\frac{P^3E^2+P^2F}{(1+PE)^2}\right\}. && (130d)
\end{aligned}$$

Clearly, (130a)–(130b) can be used to solve E and m independent of the other variables. We can then solve F and q from (130c)–(130d).

4.2 The Correlation

As described in Section 3.5, we calculate the correlation by studying a modified replicated system the partition function of which is given by (92) and can be rewritten as

$$Z^{(u)}(\mathbf{r}, \mathbf{S}; h) = \mathop{\mathsf{E}}_{\{d_{ak}\}} \left\{ \exp\left[h \sum_{k=1}^{K_1} \prod_{m=1}^{i} d_{a_m k} \right] \prod_{a=1}^{u} \exp\left[-\frac{1}{2\sigma^2} \|\mathbf{r} - \mathbf{S}\mathbf{A}\mathbf{d}_a\|^2 \right] \right\}, \tag{131}$$

where we also assume that the first K_1 users take the same energy P_1. In Appendix 12 we evaluate the following quantity

$$\lim_{u \to 0} K_1^{-1} \frac{\partial}{\partial h} \log \langle\!\langle Z^u(\mathbf{r}, \mathbf{S}; h) \rangle\!\rangle \bigg|_{h=0} \tag{132}$$

and plugging into (93) yields

$$\lim_{K \to \infty} \left\langle\!\!\left\langle \frac{1}{K_1} \sum_{k=1}^{K_1} \langle d_k \rangle^i \right\rangle\!\!\right\rangle = \mathop{\mathsf{E}}_{z} \left\{ \left(\frac{P_1 E + \sqrt{P_1 F}\, z}{1 + P_1 E} \right)^i \right\}. \tag{133}$$

By considering a series of polynomials converging to sgn$(\cdot)$, we have by (89) and (133)

$$\mathsf{M}_1 = \mathop{\mathsf{E}}_{z} \left\{ \mathrm{sgn}\left(\frac{P_1 E + \sqrt{P_1 F}\, z}{1 + \bar{P}_1 E} \right) \right\}. \tag{134}$$

Also noting that $E > 0$, we have

$$\begin{aligned} \mathsf{M}_1 &= \mathop{\mathsf{E}}_{z} \left\{ \mathrm{sgn}\left(P_1 E + \sqrt{P_1 F}\, z \right) \right\} & (135) \\ &= \mathsf{P}\left(z > -\sqrt{P_1 \frac{E^2}{F}} \right) - \mathsf{P}\left(z < -\sqrt{P_1 \frac{E^2}{F}} \right) & (136) \\ &= 1 - 2\, Q\left(\sqrt{P_1 \frac{E^2}{F}} \right) & (137) \end{aligned}$$

By (88), we can easily conclude that user 1 takes the BER

$$\mathsf{P}_1 = Q\left(\sqrt{P_1 \frac{E^2}{F}} \right) \tag{138}$$

where E and F are obtained through saddle-point equations (130).

The above results can be easily generalized to the case of an arbitrary energy distribution $\mathcal{F}$. Noting that E and F in (138) depend only on the distribution $\mathcal{F}$, the BER of user k with energy P_k converges to

$$\mathsf{P}_k = Q\left(\sqrt{P_k \frac{E^2}{F}} \right). \tag{139}$$

The multiuser efficiency

$$\eta = \frac{E^2}{F}\sigma_0^2 = \frac{1}{1+\frac{\beta}{\sigma_0^2}(1-2m+q)} \tag{140}$$

is independent of the user number. The effective energy of user k, $\sigma_0^2 P_k E^2/F$, is proportional to one's own transmit energy. The output SIR for user k is clearly $P_k E^2/F$.

4.3 Replica Symmetry Stability

The validity of the replica symmetry can be checked against symmetry breaking. The replica symmetry solution is stable if the Hessian of the free energy with respect to replica symmetry breaking perturbation is positive definite. The boundary between the replica symmetry stable region and the unstable region in the parameter space is called the AT line.

Following [28], we find that for linear detectors, stability is guaranteed if

$$1-\beta m^2 > 0. \tag{141}$$

Note that even if the free energy is stable against symmetry breaking perturbation, replica symmetry may not be true since the minimum of the free energy may be achieved at a point that is not symmetric in the replicas. In case replica symmetry is true, the multiuser efficiency result (140) is exact. Otherwise it is a lower bound of the true efficiency, since the true free energy of the system is less than that achieved with the replica symmetry constraint.

4.4 Linear Multiuser Detectors

4.4.1 The Matched Filter

Let $\sigma \to \infty$ so that the conditional mean estimator studied in this section becomes equivalent to the matched filter. The saddle-point equations give $E, F \to \infty$ and $m, p \to 0$ and therefore by (140), the multiuser efficiency is exactly $\eta^{(\mathrm{mf})}$ given by (16) in section 2.

4.4.2 The MMSE Detector

Let $\sigma = \sigma_0$ so that the conditional mean estimator becomes the MMSE detector. Equation (130c) reduces to

$$F = E + \frac{\beta(q-m)}{[\sigma^2+\beta(1-m)]^2}. \tag{142}$$

Interestingly, equations (130b) and (130d) lead to

$$q - m = \mathsf{E}\left\{\frac{P^2(F-E)}{(1+PE)^2}\right\}. \tag{143}$$

Plugging into (142), we see that

$$F - E = (F-E)\cdot\beta\,\mathsf{E}\left\{\frac{P^2E^2}{(1+PE)^2}\right\}. \tag{144}$$

It is not difficult to check that

$$\beta\,\mathsf{E}\left\{\frac{P^2E^2}{(1+PE)^2}\right\} \tag{145}$$

cannot take the value of 1, otherwise it contradicts (130a) and (130b). Therefore,

$$F = E \tag{146}$$

and as a consequence,

$$q = m. \tag{147}$$

Plugging (146)–(147) into the saddle-point equations (130) and eliminating all variables but E, we get

$$E = \frac{1}{\sigma_0^2 + \beta \,\mathsf{E}\left\{\frac{P}{1+PE}\right\}}. \tag{148}$$

Note that $\eta = \sigma_0^2 E$. A rearrangement of the above is exactly the Tse-Hanly equation given as (22) in section 2. Interestingly, E has the physical meaning of the average value of the output SIR of the MMSE detector.

It is not difficult to show that (141) always holds, hence replica symmetry is always stable for the MMSE detector.

4.4.3 The Decorrelator

Consider now the case $\sigma \to 0$, where the conditional mean estimator converges to the decorrelator. Equations (130a)–(130b) give

$$E\sigma^2 + \beta \,\mathsf{E}\left\{\frac{PE}{1+PE}\right\} = 1. \tag{149}$$

Suppose $\beta < 1$. As $\sigma \to 0$ we must have $E \to \infty$. By (130b), $m \to 1$. Plugging (130c) into (130d) and taking the limit $E \to \infty$ and $m \to 1$, we get

$$q = 1 + \frac{\sigma_0^2}{1-\beta}. \tag{150}$$

Therefore, by (140),

$$\eta^{(\text{dec})} = 1 - \beta, \quad \beta < 1. \tag{151}$$

If $\beta > 1$, letting $\sigma \to 0$ in (149) yields a finite solution for E as the unique solution to

$$\beta \,\mathsf{E}\left\{\frac{PE}{1+PE}\right\} = 1. \tag{152}$$

The parameter m is then determined by (130b) and q solved from (130c)–(130d). The multiuser efficiency is found as a positive number by (140). In the special case of equal-energy users, $E = 1/\beta - 1$ and some algebra gives

$$\eta^{(\text{dec})} = \frac{\beta - 1}{(\beta-1)^2 + \beta\sigma_0^2}, \quad \beta > 1, \tag{153}$$

which was also obtained through matrix eigenvalue analysis in [35]. Somewhat counterintuitively, by letting $K > N$ the decorrelator gets out of the poor performance at K less than but close to N. The reason is that when $K < N$ but $K \approx N$, the decorrelator may smother the desired user's signal while trying to tune out interferences. When $K > N$, however, there is no hope of tuning out all interferences and the decorrelator finds a least-square solution instead, which gives non-zero multiuser efficiency. Nevertheless, since the BER has an error floor in this case, the asymptotic (low noise) multiuser efficiency is 0 for $\beta > 1$.

5 The Optimal Detectors

The method developed in section 4 can also be applied to analyze the performance of the optimal detectors. Let the postulated prior distribution be the true priors (1), and the postulated conditional distribution be (43) with a control parameter σ. Then the posterior probability distribution is given as by (53)

$$p(\mathbf{d}|\mathbf{r},\mathbf{S}) = p^{(\mathrm{o})}(\mathbf{d}|\mathbf{r},\mathbf{S}). \tag{154}$$

We consider u replicas of the retrochannel. The partition function is

$$Z^u(\mathbf{r},\mathbf{S}) = 2^{-uK} \sum_{\{d_{ak}\}} \prod_{a=1}^{u} \exp\left[-\frac{1}{2\sigma^2}\|\mathbf{r}-\mathbf{S}\mathbf{A}\mathbf{d}_a\|^2\right]. \tag{155}$$

It is then straightforward to evaluate the free energy by definition (74) using the techniques developed in Section 4. Details are relegated to Appendix 13 and the result is

$$\begin{aligned}\mathcal{F}^{(\mathrm{o})} = &-\mathsf{E}_{P,z}\left\{\log\cosh\left(PE+\sqrt{PF}\,z\right)\right\} + Em + \frac{1}{2}F(1-q)\\ &+\frac{1}{2\beta}\log\left(1+\frac{\beta}{\sigma^2}(1-q)\right) + \frac{1}{2\beta}\frac{\sigma_0^2+\beta(1-2m+q)}{\sigma^2+\beta(1-q)},\end{aligned} \tag{156}$$

where the parameters $\{E, F, m, q\}$ satisfy the saddle-point equations

$$E = \frac{1}{\sigma^2+\beta(1-q)} \tag{157a}$$

$$m = \mathsf{E}_{P,z}\left\{P\tanh\left(PE+\sqrt{PF}\,z\right)\right\} \tag{157b}$$

$$F = \frac{\sigma_0^2+\beta(1-2m+q)}{[\sigma^2+\beta(1-q)]^2} \tag{157c}$$

$$q = \mathsf{E}_{P,z}\left\{P\tanh^2\left(PE+\sqrt{PF}\,z\right)\right\}. \tag{157d}$$

Following the same procedure as that for the linear detectors, in Appendix 13 we obtain the correlation for a group of equal energy users as

$$\lim_{K\to\infty}\left\langle\!\!\left\langle \frac{1}{K_1}\sum_{k=1}^{K_1}\langle d_k\rangle^i \right\rangle\!\!\right\rangle = \mathsf{E}_z\left\{\tanh^i\left(P_1E+\sqrt{P_1F}\,z\right)\right\}. \tag{158}$$

Hence we have the same expression for the correlation as (134). The multiuser efficiency is given by the same (140) where the parameters m and q are the solution to (157).

Following [28], the replica symmetry solution is stable against symmetry breaking if

$$1-\beta E^2\mathsf{E}\left\{P\cosh^{-4}\left(PE+\sqrt{PF}\,z\right)\right\} > 0. \tag{159}$$

In case replica symmetry is true, the multiuser efficiency result (140) is exact; otherwise it is a lower bound of the true efficiency.

5.1 The IO and the JO Detector

As shown in Section 3.1.2, various types of detectors can be achieved by tuning the control parameter σ. In particular, the conditional mean estimator reduces to the matched filter if $\sigma\to\infty$. In this case $m, q\to 0$ and we get the matched filter efficiency (16) by (140).

In case of $\sigma = \sigma_0$, the conditional mean estimator is exactly the IO detector. Notice that for all x,

$$\begin{aligned} & \underset{z}{\mathsf{E}}\left\{\tanh(x+\sqrt{x}\,z) - \tanh^2(x+\sqrt{x}\,z)\right\} \\ = \; & \frac{1}{\sqrt{2\pi x}} \int e^{-\frac{(y-x)^2}{2x}} \left[\tanh(y) - \tanh^2(y)\right] \mathrm{d}y \qquad (160) \\ = \; & \frac{1}{\sqrt{2\pi x}} e^{-\frac{1}{2}x} \int e^{-\frac{y^2}{2x}} \frac{e^y - e^{-y}}{(e^y + e^{-y})^2} \, \mathrm{d}y \qquad (161) \\ = \; & 0 \qquad (162) \end{aligned}$$

since the integrand in (161) is an odd function. It can be shown that the solution to the saddle-point equations satisfies $F = E$ and $q = m$. Therefore the multiuser efficiency, $\sigma_0^2 E$, can be found as the solution to the fixed-point equation (38), which is repeated here in a slightly different form:

$$\eta + \frac{\beta\eta}{\sigma_0^2}\left[1 - \underset{P,z}{\mathsf{E}}\left\{P \tanh\left(\frac{\eta P}{\sigma_0^2} + \sqrt{\frac{\eta P}{\sigma_0^2}}\, z\right)\right\}\right] = 1. \qquad (163)$$

It is not difficult to show that the replica symmetry is always stable for the IO detector. The result (163) covers the previous result (29) as a special case of perfect power control.

In case $\sigma \to 0$, saddle-point equations (157) jointly determine the performance of the JO detector. The multiuser efficiency can be obtained by (140).

There may be multiple solutions to the saddle-point equations (157). For certain system load β, noise level and input energies, three solutions coexist, namely, the unequivocal solution with the smallest BER, the indecisive solution with larger BER, and a physically unstable solution. We take the solution that minimizes the free energy as the valid solution. The reason is that for every realization of the spreading sequences and the noise, the detector makes its decision by minimizing the Hamiltonian, and henceforth the free energy achieves its minimum. Numerical examples show that physically unstable solution is always ruled out. The resulting BER vs. SNR performance has a water-fall effect.

5.2 Asymptotic Multiuser Efficiency

Prior to this work, the asymptotic multiuser efficiency, defined as the multiuser efficiency in the zero-noise limit [1], has been studied for the optimal detectors. Tse and Verdú reported that single-user performance is essentially achieved if the zero-noise limit is taken first and then the large-system limit, and posed the question whether the two limits commute [6]. Tanaka cautions that when unequivocal and indecisive solutions to the saddle-point equations coexist, the multiuser efficiency is 0 if the large-system limit is taken first due to ergodicity breaking [28].

We note, however, in the case of the optimal detectors, the performance is monotonic in the signal-to-noise ratio for any finite size system. Monotonicity is still true in the large-system limit. Therefore, the large-system multiuser efficiency is monotonically increasing in the input signal-to-noise ratio. In case multiple solutions to the saddle-point equations coexist, a water-fall is observable in the efficiency vs. SNR performance. In the following we establish from the newly obtained large-system efficiency result that, in the zero-noise limit, the large-system efficiency converges to 1, and thereby gives an explicit mathematical treatment of this problem.

By (157), the multiuser efficiency, $\eta = \frac{E^2}{F}\sigma_0^2$ can be expressed as

$$\eta = \frac{1}{1 + \frac{\beta}{\sigma_0^2}\mathsf{E}\left\{P\left[1 - \tanh\left(PE + \sqrt{PF}\,z\right)\right]^2\right\}}. \qquad (164)$$

Note that

$$\frac{1}{\sigma_0^2}\mathsf{E}\left\{P\left[1-\tanh\left(PE+\sqrt{PF}\,z\right)\right]^2\right\}=\frac{1}{\eta}\frac{E^2}{F}\mathsf{E}\left\{4P\left[e^{2\left(PE+\sqrt{PF}\,z\right)}+1\right]^{-2}\right\}. \tag{165}$$

To prove that $\eta \to 1$, it suffices to show that the right hand side of (165), which can be explicitly written as an integral

$$\frac{4}{\sqrt{2\pi}}\int\int P\frac{E^2}{F}\left[e^{2\left(PE+\sqrt{PF}\,z\right)}+1\right]^{-2}e^{-\frac{z^2}{2}}\,\mathrm{d}z\,\mathrm{d}F(P) \tag{166}$$

diminishes in the zero-noise limit. It is not difficult to verify from (157) that in the limit of $\sigma < \sigma_0 \to 0$, the parameters $E, F \to \infty$ and $m, q \to 1$. Clearly, the integrand in (166) diminishes for all z and P in the zero-noise limit. The bounded convergence theorem [36] guarantees that the integral of (166) also converges to 0 as long as the integrand is bounded for all z, P and sufficiently large E, F. Boundedness is straightforward by noticing that the integrand is upper bounded by

$$x^2e^{-\frac{z^2}{2}}e^{-4y(x+z)} \le e^{-7}/4 \tag{167}$$

where $x=\sqrt{PE^2/F}>0$ and $y=\sqrt{PF}>1$ for $F>1/P_{\text{min}}$. Note that the multiuser efficiency (164) is a lower bound of the true multiuser efficiency in case replica symmetry is not valid. In all, the optimal multiuser efficiency is unanimously 1 regardless of the order to take the zero-noise and large-system limits.

6 Discussions

The parameters m and q have interesting physical meanings. Take the linear detector for example. Consider a degenerate energy distribution in which the first K_1 users take equal energy P_1. By letting $i=1$ in (133), we get

$$\lim_{K\to\infty}\left\langle\!\!\left\langle\frac{1}{K_1}\sum_{k=1}^{K_1}\langle d_k\rangle\right\rangle\!\!\right\rangle=\frac{P_1E}{1+P_1E} \tag{168}$$

and by letting $i=2$,

$$\lim_{K\to\infty}\left\langle\!\!\left\langle\frac{1}{K_1}\sum_{k=1}^{K_1}\langle d_k\rangle^2\right\rangle\!\!\right\rangle=\frac{P_1^2E^2+P_1F}{(1+P_1E)^2}. \tag{169}$$

With slight abuse of notation we define

$$\langle\!\langle x\rangle\!\rangle=\lim_{K\to\infty}\left\langle\!\!\left\langle\frac{1}{K}\sum_{k=1}^{K}P_kx_k\right\rangle\!\!\right\rangle. \tag{170}$$

Then we easily find

$$\begin{aligned}\langle\!\langle\langle d\rangle\rangle\!\rangle &= \lim_{K\to\infty}\left\langle\!\!\left\langle\frac{1}{K}\sum_{k=1}^{K}P_k\langle d_k\rangle\right\rangle\!\!\right\rangle && (171)\\ &= \mathop{\mathsf{E}}_{P}\left\{P\frac{PE}{1+PE}\right\} && (172)\\ &= m. && (173)\end{aligned}$$

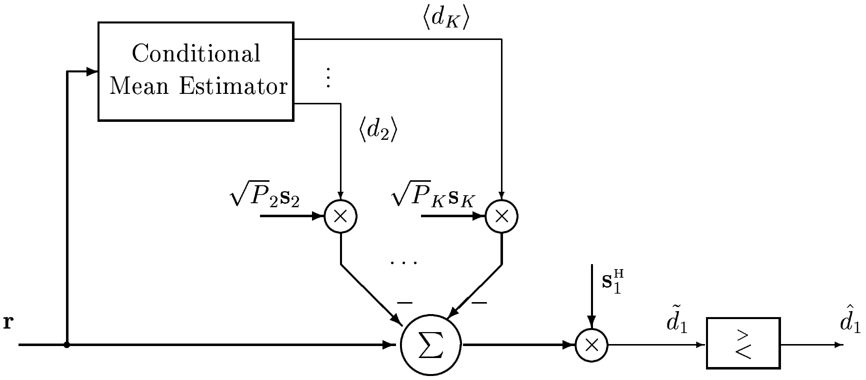

Figure 2: A canonical interference canceller.

Similarly,

$$\left\langle\!\!\left\langle \langle d \rangle^2 \right\rangle\!\!\right\rangle = q. \tag{174}$$

The same expressions are resulted if we start with (158) for the optimal detectors. Indeed, (173)–(174) are true for all multiuser detectors studied in this paper.

Let

$$\zeta = 1 - 2m + q. \tag{175}$$

One finds that

$$\begin{aligned} \zeta &= 1 - 2\langle\!\langle \langle d \rangle \rangle\!\rangle + \left\langle\!\!\left\langle \langle d \rangle^2 \right\rangle\!\!\right\rangle && (176) \\ &= \langle\!\langle (1 - \langle d \rangle)^2 \rangle\!\rangle . && (177) \end{aligned}$$

Recall that the transmitted symbol $d_{0k} = 1$ for each user k. The right hand side of (177) is the mean square error of the conditional mean estimator output. Interestingly, the multiuser efficiency is then expressed in the mean square error as

$$\eta = \frac{1}{1 + \zeta \frac{\beta}{\sigma_0^2}} \tag{178}$$

by (140). Comparing it with the matched filter efficiency (16),

$$\eta^{(\mathrm{mf})} = \frac{1}{1 + \frac{\beta}{\sigma_0^2}}, \tag{179}$$

we see that a multiuser detector behaves as a matched filter with a spreading factor expanded by a factor of $1/\zeta$, or a user population reduced by a factor of ζ, or an enhanced Gaussian noise variance by $\zeta\beta$, where ζ is the mean square error of the soft output of the corresponding conditional mean estimator. The expression also reminds us a simple interpretation of the multiuser efficiency. The “1” in the denominator in (178) is contributed by thermal noise, while the term $\zeta\beta/\sigma_0^2$ is contributed by the output MAI.

We conclude by giving a canonical interference canceller as shown in Fig. 2 which makes use of the conditional mean estimator's output to reconstruct the interference for cancellation before matched filtering by the desired user's spreading codes. The soft output for user 1 is expressed as

$$\begin{aligned}\tilde{d}_1 &= \mathbf{s}_1^{\mathrm{H}} \left(\mathbf{r} - \sum_{k=2}^{K} \sqrt{P_k}\, \mathbf{s}_k \left\langle d_k \right\rangle \right) && (180)\\ &= \sqrt{P_1}\, d_{01} + \sum_{k=2}^{K} \mathbf{s}_1^{\mathrm{H}} \mathbf{s}_k \sqrt{P_k} \left(d_{0k} - \langle d_k \rangle \right) + \sigma_0 w_1 && (181)\end{aligned}$$

where w_1 is a standard Gaussian random variable. An important observation here is that the MAI term is asymptotically Gaussian with a variance $\zeta\beta$, so that the resulting multiuser efficiency as expressed by (178). The canonical structure suggests interference cancellation structure for implementing the multiuser detectors, if the posterior mean output for interfering users can be approximated by interference cancellers of the same structure.

7 Spectral Efficiency

The sum capacity of a multiuser channel is equal to the maximum mutual information between transmitted symbols and the received signal, or, equivalently, the reduction in uncertainty about the source due to the received signal. The free energy turns out to be a very relevant measure of the uncertainty of the system. In this section, the relationship in between the sum capacity and the free energy is revealed. We explicitly find the sum capacity of a multiuser CDMA channel given an arbitrary energy distribution for both binary and Gaussian inputs, the latter of which is the optimal input distribution to the multiuser channel.

Assuming the spreading sequences are known and fixed, the mutual information between the symbols and the received signal is

$$\begin{aligned}I(\mathbf{d};\mathbf{r}|\mathbf{S}) &= \mathsf{E}\left\{ \log \frac{p_0(\mathbf{r}|\mathbf{d},\mathbf{S})}{p(\mathbf{r}|\mathbf{S})} \right\} && (182)\\ &= \mathsf{E}\left\{ \log p_0(\mathbf{r}|\mathbf{d},\mathbf{S}) \right\} + \mathsf{E}\left\{ -\log p(\mathbf{r}|\mathbf{S}) \right\}. && (183)\end{aligned}$$

Note that $p_0(\mathbf{r}|\mathbf{d},\mathbf{S})$, given as (6), is a Gaussian density, and we have

$$\begin{aligned}\mathsf{E}\left\{ \log p_0(\mathbf{r}|\mathbf{d},\mathbf{S}) \right\} &= \mathsf{E}\left\{ \log p(\sigma_0^2 \boldsymbol{\nu}) \right\} && (184)\\ &= -\frac{N}{2} \left(1 + \log(2\pi\sigma_0^2) \right). && (185)\end{aligned}$$

In the meantime, $p(\mathbf{r}|\mathbf{S})$ is proportional to the partition function $Z(\mathbf{r},\mathbf{S})$ given in (45)

$$p(\mathbf{r}|\mathbf{S}) = (2\pi\sigma_0^2)^{-\frac{N}{2}} Z(\mathbf{r},\mathbf{S}). \qquad (186)$$

Hence by (185) and (186), the mutual information per dimension is

$$\frac{1}{N} I(\mathbf{d},\mathbf{r}|\mathbf{S}) = -\frac{1}{2} + \beta \frac{1}{K} \mathsf{E}\left\{ -\log Z(\mathbf{r},\mathbf{S}) \right\}, \qquad (187)$$

the maximum of which is the spectral efficiency. By definition of the free energy (74), the capacity is found in the large system limit as

$$\mathsf{C} = \beta\mathcal{F} - \frac{1}{2}, \qquad (188)$$

i.e., β times the free energy minus a half nat.

7.1 Gaussian Inputs

The Gaussian prior is known to give the maximum of the mutual information. Let d_k's be independent standard Gaussian random variables so that the prior distribution is expressed by (47). Then the free energy is obtained similarly as in Section 4,

$$\mathcal{F} = \mathcal{F}^{(\mathrm{l})}|_{\sigma=\sigma_0}. \tag{189}$$

Noticing (130a)–(130b) and that $\eta^{(\mathrm{mmse})} = \sigma_0^2 E$, we have

$$\mathcal{F} = \frac{1}{2}\mathsf{E}\left\{\log\left(1+\frac{\eta^{(\mathrm{mmse})}}{\sigma_0^2}P\right)\right\} + \frac{\eta^{(\mathrm{mmse})} - \log\eta^{(\mathrm{mmse})}}{2\beta} \tag{190}$$

where $\eta^{(\mathrm{mmse})}$ is the multiuser efficiency of an "optimal" detector for Gaussian inputs, i.e., the MMSE receiver. Therefore, the spectral efficiency is exactly (31), which is repeated here,

$$\mathsf{C} = \frac{\beta}{2}\mathsf{E}\left\{\log\left(1+\frac{\eta^{(\mathrm{mmse})}}{\sigma_0^2}P\right)\right\} + \frac{1}{2}(\eta^{(\mathrm{mmse})} - 1 - \log\eta^{(\mathrm{mmse})}). \tag{191}$$

Not surprisingly, independent fading on an equal transmit energy system is a perfect cause of the energy distribution. Therefore we have obtained the spectral efficiency as a function of the multiuser efficiency. In fact, we can identify the first term on the right hand side of (191) to be the spectral efficiency of the linear MMSE receiver, i.e., if single-user decoding is applied to the linear MMSE output.

7.2 Binary Inputs

It is practically very interesting to know the spectral efficiency of the multiuser channel where the input symbols to the channel are constrained to be antipodally modulated. Equally probable ± 1's maximizes the mutual information in this case. The free energy is obtained similarly as in Section 5,

$$\mathcal{F} = \mathcal{F}^{(\mathrm{o})}|_{\sigma=\sigma 0}. \tag{192}$$

The resulting capacity of the multiuser CDMA channel subject to binary inputs is

$$\mathsf{C} = -\beta\mathsf{E}\left\{\log\,\cosh\left(\frac{\eta^{(\mathrm{io})}P}{\sigma_0^2} + \sqrt{\frac{\eta^{(\mathrm{io})}P}{\sigma_0^2}}\,z\right)\right\} + \frac{\beta\eta^{(\mathrm{io})}}{\sigma_0^2} + \frac{1}{2}\left(\eta^{(\mathrm{io})} - 1 - \log\eta^{(\mathrm{io})}\right) \tag{193}$$

where $\eta^{(\mathrm{io})}$ is the efficiency of the individually optimal detector.

We can identify that the first term on the right hand side of (191) is the total capacity of the multiuser CDMA channel if single-user decoding is applied to the output of the MMSE receiver. Similarly, first two terms on the right hand side of (193) add up to be the total capacity of the multiuser CDMA channel if only binary inputs is allowed and single-user decoding is applied to the individually optimal detector output [37]. Striking similarity in the capacity loss due to separation of detection and decoding in the two cases is also pointed out in [37].

The spectral efficiency gives the maximum total number of bits per second per chip that can be transmitted reliably from all users to the receiver over the multiuser channel. Unfortunately it does not easily break down to a rate combination of individual users that achieves it as (191) and (193) may seem to suggest.

8 Conclusion

This paper exploits the connection between large-system multiuser detection and statistical mechanics, and presents a new interpretation of multiuser detection in general.

We first introduced the concept of Bayes retrochannel, which takes the received signal as the input and outputs a stochastic estimate of the transmitted symbols. By assuming an appropriate prior distribution and a channel characteristic that may be different to the true ones, a multiuser detector can be expressed as a conditional mean estimator that outputs the mean value of the stochastic output of the Bayes retrochannel.

The Bayes retrochannel is equivalent to a spin glass in the sense that the distribution of its stochastic output conditioned on the received signal is exactly the distribution of the spin glass at thermal equilibrium. The performance of the multiuser detector is then found as a certain macroscopic property of the spin glass. In particular, the BER can be obtained through calculating the overlap of the spin glass. In the large-system limit, the macroscopic properties as such can be solved by powerful tools developed in statistical mechanics.

In this paper we have solved through a unified analysis the large-system uncoded BER of the matched filter, the MMSE detector, the decorrelator, the individually and jointly optimal detectors. We show that under arbitrary received energy distribution, the large-system BER is uniquely determined by the multiuser efficiency, which has a very simple relationship with the output mean square error of the conditional mean estimator. The relationship also implies that a multiuser detector is in general equivalent in performance to interference subtraction using a conditional mean estimator obtained for certain prior and conditional distribution (depending on the detector), and the remaining interference is always Gaussian distributed in the large-system limit.

By identifying the relationship of the mutual information and the free energy, we have also obtained the spectral efficiency of the multiuser CDMA channel with binary input constraint or not.

9 Asymptotic Jointly Gaussian Distribution of $\{\mathbf{v}\}$

A proof of asymptotic joint normality in [28, Appendix B] using the Edgeworth expansion [33] of the probability density function is flawed since the discrete nature of $\mathbf{v}$ does not allow a density function. A simple remedy is to use the cumulative distribution function (c.d.f.) instead. Odd order cumulants of $\mathbf{v}$ are all zero. The second- and fourth-order cumulants of $\mathbf{v}$ are given by

$$\kappa^{a,b} = \mathsf{E}\left\{v_a v_b\right\} = Q_{ab} \tag{194}$$

and

$$\begin{aligned}
\kappa^{a,b,c,d} &= \mathsf{E}\left\{v_a v_b v_c v_d\right\} - \mathsf{E}\left\{v_a v_b\right\}\mathsf{E}\left\{v_c v_d\right\} - \mathsf{E}\left\{v_a v_c\right\}\mathsf{E}\left\{v_b v_d\right\} - \mathsf{E}\left\{v_a v_d\right\}\mathsf{E}\left\{v_b v_c\right\} && (195)\\
&= K^{-2}\sum_{k=1}^{K} P_k^2 d_{ak} d_{bk} d_{ck} d_{dk} && (196)\\
&= O(K^{-1}) && (197)
\end{aligned}$$

for all $a, b, c, d = 0, \ldots, u$. Higher order cumulants are $O(K^{-2})$. Therefore, the joint c.d.f. of $\mathbf{v}$ allows an Edgeworth expansion,

$$F(\mathbf{v}) = F_0(\mathbf{v}) + \frac{1}{4!\cdot K}\sum_{a,b,c,d=0}^{u}\left(\frac{1}{K}\sum_{k=1}^{K} P_k^2 d_{ak} d_{bk} d_{ck} d_{dk}\right)\frac{\partial^4 F_0(\mathbf{v})}{\partial v_a \partial v_b \partial v_c \partial v_d} + O(K^{-2}) \tag{198}$$

where F_0 is the c.d.f. of joint Gaussian variables with a covariance of $\mathbf{Q}$ [33]. In the limit of $K \to \infty$, the distribution converges to the Gaussian distrubution F_0.

10 Evaluation of $G^{(u)}$

Equation (111) can be evaluated as follows.

$$\begin{aligned}
&\exp\left[G^{(u)}(\mathbf{Q})\right] \\
= \;& \mathop{\mathsf{E}}_{\mathbf{v}(\mathbf{Q})}\left\{\int (2\pi\sigma_0^2)^{-\frac{1}{2}} \prod_{a=0}^{u} \exp\left[-\frac{1}{2\sigma_a^2}\left(r-\sqrt{\beta}\,v_a\right)^2\right] \mathrm{d}r\right\} && (199) \\
= \;& \mathop{\mathsf{E}}_{\mathbf{v}(\mathbf{Q})}\left\{(\sigma_0^2)^{-\frac{1}{2}} \int (2\pi)^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\sum_{a=0}^{u}\frac{1}{\sigma_a^2}\right) r^2 + \sum_{a=0}^{u}\frac{\sqrt{\beta}v_a}{\sigma_a^2} r - \frac{1}{2}\sum_{a=0}^{u}\frac{\beta v_a^2}{\sigma_a^2}\right] \mathrm{d}r\right\} && (200) \\
= \;& \mathop{\mathsf{E}}_{\mathbf{v}(\mathbf{Q})}\left\{\left(\sigma_0^2\sum_{a=0}^{u}\frac{1}{\sigma_a^2}\right)^{-\frac{1}{2}} \exp\left[\frac{1}{2}\frac{\beta\left(\sum_{a=0}^{u}\frac{v_a}{\sigma_a^2}\right)^2}{\sum_{a=0}^{u}\frac{1}{\sigma_a^2}} - \frac{1}{2}\sum_{a=0}^{u}\frac{\beta\mathbf{v}_a^2}{\sigma_a^2}\right]\right\} && (201) \\
= \;& \left(1+u\frac{\sigma_0^2}{\sigma^2}\right)^{-\frac{1}{2}} \mathop{\mathsf{E}}_{\mathbf{v}(\mathbf{Q})}\left\{\exp\left[-\frac{1}{2}\mathbf{v}^\top\mathbf{\Sigma}\mathbf{v}\right]\right\} && (202) \\
= \;& \left(1+u\frac{\sigma_0^2}{\sigma^2}\right)^{-\frac{1}{2}} \int \exp\left[-\frac{1}{2}\mathbf{v}^\top\left(\mathbf{\Sigma}+\mathbf{Q}^{-1}\right)\mathbf{v}\right] \mathbf{d}\mathbf{v} && (203)
\end{aligned}$$

where $\mathbf{\Sigma}$ is a $(u+1)\times(u+1)$ matrix with the (a,b) entry given by

$$\mathbf{\Sigma}_{ab} = \frac{\beta}{\sigma_a^2}\delta(a,b) - \frac{\beta\sigma_0^2\sigma^2}{\sigma_a^2\sigma_b^2(\sigma^2+u\sigma_0^2)}. \tag{204}$$

Regarding the integral in (203) as a scaled total mass of a Gaussian random vector with covariance $(\mathbf{\Sigma}+\mathbf{Q}^{-1})$ yields

$$\exp\left[G^{(u)}(\mathbf{Q})\right] = \left[\left(1+u\frac{\sigma_0^2}{\sigma^2}\right)\det(\mathbf{I}+\mathbf{Q}\mathbf{\Sigma})\right]^{-\frac{1}{2}} \tag{205}$$

and taking logarithm leads to (112).

Under the replica symmetry assumption (122) for linear detectors, we evaluate (112) through careful algebra. Note that both $\mathbf{\Sigma}$ and $\mathbf{Q}$ take the form of

$$\begin{bmatrix}
x & y & \cdots & \cdots & y \\
z & v+w & & & \\
\vdots & & \ddots & v & \\
\vdots & & v & \ddots & \\
z & & & & v+w
\end{bmatrix}, \tag{206}$$

the matrix $(\mathbf{I}+\mathbf{Q}\mathbf{\Sigma})$ takes the same form where

$$\begin{aligned} x &= 1+\frac{u\beta(1-m)}{\sigma^2+u\sigma_0^2} && \text{(207a)} \\ y &= -\frac{\beta(1-m)}{\sigma^2+u\sigma_0^2} && \text{(207b)} \\ z &= \frac{\beta(um-uq+q-p)}{\sigma^2+u\sigma_0^2} && \text{(207c)} \\ v &= \frac{\beta}{\sigma^2+u\sigma_0^2}\left[q-m+\frac{\sigma_0^2}{\sigma^2}(q-p)\right] && \text{(207d)} \\ w &= 1+\frac{\beta}{\sigma^2}(p-q). && \text{(207e)} \end{aligned}$$

Using

$$\det(\mathbf{I}+\mathbf{Q}\mathbf{\Sigma}) = [x(uv+w)-uyz]\cdot w^{u-1}, \tag{208}$$

it is straightforward to show that

$$\exp\left[G^{(u)}(\mathbf{Q})\right] = \left(1+\frac{\beta}{\sigma^2}(p-q)\right)^{-\frac{u-1}{2}}\left[1+\frac{\beta}{\sigma^2}(p-q)+\frac{u}{\sigma^2}\left(\sigma_0^2+\beta(1-2m+q)\right)\right]^{-\frac{1}{2}}. \tag{209}$$

Taking logarithm yields (124).

11 Evaluation of $I^{(u)}$

We first use the Fourier transform representation of Dirac function

$$\delta(x) = \frac{1}{2\pi j}\int_{-j\infty+t}^{j\infty+t}\exp\left(\tilde{Q}\cdot x\right)\,\mathrm{d}\tilde{Q}, \quad \forall\, t\in\mathcal{R} \tag{210}$$

to rewrite

$$\delta\left(\sum_{k=1}^{K}P_k d_{ak}d_{bk}-KQ_{ab}\right) = \frac{1}{2\pi j}\int\exp\left[\tilde{Q}_{ab}\left(\sum_{k=1}^{K}P_k d_{ak}d_{bk}-KQ_{ab}\right)\right]\mathrm{d}\tilde{Q}_{ab}. \tag{211}$$

From (116),

$$
\begin{aligned}
&\mu_K^{(u)}(\mathbf{Q}) \\
=\;& \mathop{\mathsf{E}}_{\{d_{ak}\}}\left\{\int \sideset{}{'}\prod_{a\leq b}^{u} \exp\left[\tilde{Q}_{ab}\left(\sum_{k=1}^{K} P_k d_{ak} d_{bk} - K Q_{ab}\right)\right]\left(\sideset{}{'}\prod_{a\leq b}^{u} \frac{\mathrm{d}\tilde{Q}_{ab}}{2\pi j}\right)\right\} && (212)\\
=\;& \int \mathop{\mathsf{E}}_{\{d_{ak}\}}\left\{\exp\left[\sum_{k=1}^{K} P_k \sideset{}{'}\sum_{a\leq b}^{u} \tilde{Q}_{ab} d_{ak} d_{bk}\right]\right\}\sideset{}{'}\prod_{a\leq b}^{u} \exp\left[-K\tilde{Q}_{ab}Q_{ab}\right]\left(\sideset{}{'}\prod_{a\leq b}^{u} \frac{\mathrm{d}\tilde{Q}_{ab}}{2\pi j}\right) && (213)\\
=\;& \int \prod_{k=1}^{K} \mathop{\mathsf{E}}_{\{d_a\}}\left\{\exp\left[P_k \sideset{}{'}\sum_{a\leq b}^{u} \tilde{Q}_{ab} d_a d_b\right]\right\}\sideset{}{'}\prod_{a\leq b}^{u} \exp\left[-K\tilde{Q}_{ab}Q_{ab}\right]\left(\sideset{}{'}\prod_{a\leq b}^{u} \frac{\mathrm{d}\tilde{Q}_{ab}}{2\pi j}\right) && (214)\\
=\;& \int \exp\left[K\left(\mathop{\mathsf{E}}_{P}\left\{\log \mathop{\mathsf{E}}_{\{d_a\}}\left\{\exp\left(P \sideset{}{'}\sum_{a\leq b}^{u} \tilde{Q}_{ab} d_a d_b\right)\right\}\right\} - \sideset{}{'}\sum_{a\leq b}^{u} \tilde{Q}_{ab} Q_{ab}\right)\right]\left(\sideset{}{'}\prod_{a\leq b}^{u} \frac{\mathrm{d}\tilde{Q}_{ab}}{2\pi j}\right) && (215)
\end{aligned}
$$

where the integral over each $\tilde{Q}_{ab}$ is from $-j\infty + t_{ab}$ to $j\infty + t_{ab}$ for some real number t_{ab}. It is then straightforward to write (118) by defining $X(u, P)$ as (119).

The integrand in (118) is an analytical function on the multi-dimensional complex space. Due to the exponential factor K, the integral (118) is dominated by the value of the integrand evaluated at its maximum. This can be justified as follows. The saddle point of the integrand can be found by setting its derivative with respect to each $\tilde{Q}_{ab}$ to 0. The exponential function can be expanded by Taylor series at the vicinity of the saddle point. Higher order terms can be shown to diminish with a rate faster than $\frac{1}{K}$. Therefore, the leading term dominates the integral in the $K \to \infty$ limit and the rate of $\mu_K^{(u)}$ is given as (120).

In the linear detector case, assuming replica symmetry (123), we have

$$
\begin{aligned}
&X(u, P) \\
=\;& \mathop{\mathsf{E}}_{\{d_a\}}\left\{\exp\left[P\sum_{a=1}^{u} \tilde{Q}_{0a} d_a + P \sum_{1\leq a\leq b}^{u} \tilde{Q}_{ab} d_a d_b\right]\right\} && (216)\\
=\;& \mathop{\mathsf{E}}_{\{d_a\}}\left\{\exp\left[PE\sum_{a=1}^{u} d_a + PF \sum_{1\leq a<b}^{u} d_a d_b + \frac{1}{2}PG\sum_{a=1}^{u} d_a^2\right]\right\} && (217)\\
=\;& \mathop{\mathsf{E}}_{\{d_a\}}\left\{\exp\left[PE\sum_{a=1}^{u} d_a + \frac{1}{2}PF\left(\sum_{a=1}^{u} d_a\right)^2 - \frac{1}{2}P(F-G)\sum_{a=1}^{u} d_a^2\right]\right\}. && (218)
\end{aligned}
$$

We use a standard trick to linearize the exponent,

$$
e^{\frac{1}{2}x^2} = \mathop{\mathsf{E}}_{z}\left\{e^{xz}\right\}, \quad \forall x \qquad (219)
$$

where we use z to represent a standard Gaussian random variable throughout this paper. We have

$$\begin{aligned}
& X(u,P) \\
= \;& \mathop{\mathsf{E}}_{z}\left\{ \mathop{\mathsf{E}}_{\{d_a\}}\left\{ \exp\left[\left(PE+\sqrt{PF}\,z\right)\sum_{a=1}^{u} d_a - \frac{1}{2}P(F-G)\sum_{a=1}^{u} d_a^2 \right] \right\} \right\} && (220) \\
= \;& \mathop{\mathsf{E}}_{z}\left\{ \mathop{\mathsf{E}}_{d}\left\{ \exp\left[\left(PE+\sqrt{PF}\,z\right) d - \frac{1}{2}P(F-G)d^2 \right] \right\} \right\}^{u} && (221) \\
= \;& \mathop{\mathsf{E}}_{z}\left\{ (1+P(F-G))^{-\frac{u}{2}} \exp\left[\frac{u\left(PE+\sqrt{PF}\,z\right)^2}{2(1+P(F-G))} \right] \right\} && (222) \\
= \;& (1+P(F-G))^{-\frac{u-1}{2}}(1+(1-u)PF-PG)^{-\frac{1}{2}} \exp\left[\frac{uP^2E^2}{2(1+(1-u)PF-PG)} \right]. && (223)
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
\sum_{a\le b}^{u}{}' \tilde{Q}_{ab}Q_{ab} &= \sum_{b=1}^{u} \tilde{Q}_{0b}Q_{0b} + \sum_{1\le a<b}^{u} \tilde{Q}_{ab}Q_{ab} + \sum_{a=1}^{u} \tilde{Q}_{aa}Q_{aa} && (224) \\
&= uEm + \frac{u(u-1)}{2}Fq + \frac{u}{2}Gp. && (225)
\end{aligned}$$

Hence, (120) yields the result (125).

12 Evaluation of the Correlation

The evaluation of the correlation follows from the evaluation of the free energy in Section 4.1 and Appendix 11.

By (92), the modified partition function for the linear replicated system is

$$Z^{(u)}(\mathbf{r},\mathbf{S};h) = \mathop{\mathsf{E}}_{\{d_{ak}\}}\left\{ \exp\left[h\sum_{k=1}^{K_1}\prod_{m=1}^{i} d_{a_m k} \right] \prod_{a=1}^{u} \exp\left[-\frac{1}{2\sigma^2}\|\mathbf{r}-\mathbf{S}\mathbf{A}\mathbf{d}_a\|^2 \right] \right\}. \tag{226}$$

Similar to (99),

$$\begin{aligned}
& \left\langle\!\!\left\langle Z^{(u)}(\mathbf{r},\mathbf{S};h) \right\rangle\!\!\right\rangle \\
=\, & \mathop{\mathsf{E}}_{\{d_{ak}\}}\left\{ \exp\left[h\sum_{k=1}^{K_1}\prod_{m=1}^{i} d_{a_m k} \right] \right. \\
& \left. \left[\int (2\pi\sigma_0^2)^{-\frac{1}{2}} \mathop{\mathsf{E}}_{\mathbf{v}}\left\{ \exp\left[-\frac{1}{2\sigma_0^2}\left(r-\sqrt{\beta}\,v_0\right)^2 \right] \prod_{a=1}^{u} \exp\left[-\frac{1}{2\sigma^2}\left(r-\sqrt{\beta}\,v_a\right)^2 \right] \right\} \mathrm{d}r \right]^{N} \right\}.
\end{aligned} \tag{227}$$

Therefore,

$$\left\langle\!\!\left\langle Z^{(u)}(\mathbf{r},\mathbf{S};h) \right\rangle\!\!\right\rangle = \int \exp\left[\beta^{-1}K\cdot G_K^{(u)}(\mathbf{Q}) \right] \cdot \mu_K^{(u)}(\mathbf{Q};h) \left(\prod_{a\le b}^{u}{}' \mathrm{d}Q_{ab} \right) \tag{228}$$

where $G_K^{(u)}$ is the same as defined by (111), and $\mu_K^{(u)}(\mathbf{Q};h)$ is a probability measure expressed as

$$\mu_K^{(u)}(\mathbf{Q};h) = \mathop{\mathsf{E}}_{\{d_{ak}\}} \left\{ \exp\left[h \sum_{k=1}^{K_1} \prod_{m=1}^{i} d_{a_m k} \right] {\prod_{a\le b}^{u}}' \delta\left(\sum_{k=1}^{K} P_k d_{ak} d_{bk} - K Q_{ab} \right) \right\}. \tag{229}$$

For the purpose of calculating the correlation, only $\mu_K^{(u)}(\mathbf{Q};h)$ is relevant. We have

$$\begin{aligned} & \mu_K^{(u)}(\mathbf{Q};h) \\ = & \mathop{\mathsf{E}}_{\{d_{ak}\}} \left\{ \exp\left[h \sum_{k=1}^{K_1} \prod_{m=1}^{i} d_{a_m k} \right] \int {\prod_{a\le b}^{u}}' \exp\left[\tilde{Q}_{ab} \left(\sum_{k=1}^{K} P_k d_{ak} d_{bk} - K Q_{ab} \right) \right] \left({\prod_{a\le b}^{u}}' \frac{\mathrm{d}\tilde{Q}_{ab}}{2\pi j} \right) \right\} & (230) \\ = & \int \mathop{\mathsf{E}}_{\{d_{ak}\}} \left\{ \exp\left[h \sum_{k=1}^{K_1} \prod_{m=1}^{i} d_{a_m k} \right] \exp\left[\sum_{k=1}^{K} P_k {\sum_{a\le b}^{u}}' \tilde{Q}_{ab} d_{ak} d_{bk} - K \tilde{Q}_{ab} Q_{ab} \right] \right\} \left({\prod_{a\le b}^{u}}' \frac{\mathrm{d}\tilde{Q}_{ab}}{2\pi j} \right) & (231) \\ = & \int \exp\left[K \left(\alpha_1 (\log X_1(u;h) - \log X_1(u;0)) + \mathop{\mathsf{E}}_{P} \{\log X(u,P)\} - {\sum_{a\le b}^{u}}' \tilde{Q}_{ab} Q_{ab} \right) \right] \left({\prod_{a\le b}^{u}}' \frac{\mathrm{d}\tilde{Q}_{ab}}{2\pi j} \right) & (232) \end{aligned}$$

where $X(u,P)$ is defined the same as in (119) and

$$X_1(u;h) = \mathop{\mathsf{E}}_{\{d_a\}} \left\{ \exp\left[h \prod_{m=1}^{i} d_{a_m} \right] \exp\left[P_1 {\sum_{a\le b}^{u}}' \tilde{Q}_{ab} d_a d_b \right] \right\}. \tag{233}$$

It is important to note that

$$K_1^{-1} \frac{\partial \log \langle\!\langle Z^u(\mathbf{r},\mathbf{S};h) \rangle\!\rangle}{\partial h}\bigg|_{h=0} = \frac{\partial}{\partial h} \log X_1(u;h)\bigg|_{h=0}. \tag{234}$$

Hence it suffices to consider the derivative of $X_1(u;h)$ with respect to h for the purpose of finding the correlation.

Assume replica symmetry as before but note that $G = F - E$ and $p = 1 - m + q$ in the limit of $h \to 0$. Then

$$X_1(u;h) = \mathop{\mathsf{E}}_{z} \left\{ \mathop{\mathsf{E}}_{\{d_a\}} \left\{ \exp\left[h \prod_{m=1}^{i} d_{a_m} \right] \exp\left(P_1 E + \sqrt{P_1 F}\, z \right) \sum_{a=1}^{u} d_a - \frac{1}{2} P_1 E \sum_{a=1}^{u} d_a^2 \right\} \right\} \tag{235}$$

Therefore,

$$\begin{aligned} & \frac{\partial}{\partial h} \log X_1(u;h)\bigg|_{h=0} \\ = & \frac{\mathop{\mathsf{E}}_{z} \left\{ \mathop{\mathsf{E}}_{\{d_a\}} \left\{ \left[\prod_{m=1}^{i} d_{a_m} \right] \exp\left[\left(P_1 E + \sqrt{P_1 F}\, z \right) \sum_{a=1}^{u} d_a - \frac{1}{2} \bar{P}_1 E \sum_{a=1}^{u} d_a^2 \right] \right\} \right\}}{\mathop{\mathsf{E}}_{z} \left\{ \mathop{\mathsf{E}}_{\{d_a\}} \left\{ \exp\left[\left(P_1 E + \sqrt{P_1 F}\, z \right) \sum_{a=1}^{u} d_a - \frac{1}{2} \bar{P}_1 E \sum_{a=1}^{u} d_a^2 \right] \right\} \right\}} & (236) \\ = & \frac{\mathop{\mathsf{E}}_{z} \left\{ [f_0(z)]^{(u-i)} [f_1(z)]^i \right\}}{\mathop{\mathsf{E}}_{z} \left\{ [f_0(z)]^u \right\}} & (237) \end{aligned}$$

where

$$f_0(z) \quad = \quad \underset{d}{\mathsf{E}}\left\{\exp\left[\left(P_1E+\sqrt{P_1F}\,z\right)d-\frac{1}{2}P_1Ed^2\right]\right\} \tag{238}$$

$$= \quad (1+P_1E)^{-\frac{1}{2}}\exp\left[\frac{\left(P_1E+\sqrt{P_1F}\,z\right)^2}{2(1+P_1E)}\right] \tag{239}$$

and

$$f_1(z) \quad = \quad \underset{d}{\mathsf{E}}\left\{d\cdot\exp\left[\left(P_1E+\sqrt{P_1F}\,z\right)d-\frac{1}{2}P_1Ed^2\right]\right\} \tag{240}$$

$$= \quad \left(P_1E+\sqrt{P_1F}\,z\right)(1+P_1E)^{-\frac{3}{2}}\exp\left[\frac{\left(P_1E+\sqrt{P_1F}\,z\right)^2}{2(1+P_1E)}\right]. \tag{241}$$

Therefore,

$$\frac{\partial}{\partial h}\log X_1(u;h)\bigg|_{h=0} = \frac{\underset{z}{\mathsf{E}}\left\{\left(P_1E+\sqrt{P_1F}\,z\right)^i(1+P_1E)^{-i}\exp\left[\frac{u\left(P_1E+\sqrt{P_1F}\,z\right)^2}{2(1+P_1E)}\right]\right\}}{\underset{z}{\mathsf{E}}\left\{(1+P_1E)^{-\frac{u}{2}}\exp\left[\frac{u\left(P_1E+\sqrt{P_1F}\,z\right)^2}{2(1+P_1E)}\right]\right\}}. \tag{242}$$

Taking the limit $u\to 0$, we have

$$\lim_{u\to 0}\frac{\partial}{\partial h}\log X_1(u;h)\bigg|_{h=0} = \underset{z}{\mathsf{E}}\left\{\left(\frac{P_1E+\sqrt{P_1F}\,z}{1+P_1E}\right)^i\right\}. \tag{243}$$

Therefore,

$$\lim_{u\to 0}K_1^{-1}\frac{\partial\log\langle\!\langle Z^u(\mathbf{r},\mathbf{S};h)\rangle\!\rangle}{\partial h}\bigg|_{h=0} = \underset{z}{\mathsf{E}}\left\{\left(\frac{P_1E+\sqrt{P_1F}\,z}{1+P_1E}\right)^i\right\}. \tag{244}$$

13 Free Energy and Correlation for Optimal Detectors

The evaluation of the free energy for the optimal detectors follows mostly that for the linear detectors, where the only difference is the postulated prior distributions. The free energy is still obtained using (117). In the evaluation of $G^{(u)}$, the covariance structure of $\mathbf{v}$ under replica symmetry is

$$Q_{0a} \quad = \quad \mathsf{E}\left\{v_0v_a\right\} = m \tag{245a}$$

$$Q_{ab} \quad = \quad \mathsf{E}\left\{v_av_b\right\} = q \tag{245b}$$

for $1\le a<b$ and with $Q_{aa}\equiv 1$ for all $a=0,\dots,u$. As a result,

$$G^{(u)} = -\frac{u-1}{2}\log\left(1+\frac{\beta}{\sigma^2}(1-q)\right)-\frac{1}{2}\log\left[1+\frac{\beta}{\sigma^2}(1-q)+\frac{u}{\sigma^2}\left(\sigma_0^2+\beta(1-2m+q)\right)\right]. \tag{246}$$

In the evaluation of I_u, the point that the derivation diverges from that for the linear detector case is the following quantity,

$$X(u,P) = \sum_{\{d_a\}}2^{-u}\exp\left[P{\sum_{a\le b}^{u}}'\tilde{Q}_{ab}d_ad_b\right] \tag{247}$$

which is an expectation over binary instead of Gaussian random variables. Assuming the same replica symmetry as in section 4, and noting that $p = 1$, we have

$$\begin{aligned} X(u,P) &= \sum_{\{d_a\}} 2^{-u} \exp\left[PE\sum_{a=1}^{u} d_a + \frac{1}{2}PF\left(\sum_{a=1}^{u} d_a\right)^2 - \frac{1}{2}PF\sum_{a=1}^{u} d_a^2 \right] && (248)\\ &= \exp\left[-\frac{u}{2}PF\right] \underset{z}{\mathsf{E}} \left\{ \sum_{\{d_a\}} 2^{-u} \exp\left[\left(P_1E + \sqrt{P_1F}\,z\right) \sum_{a=1}^{u} d_a \right] \right\} && (249)\\ &= \exp\left[-\frac{u}{2}PF\right] \underset{z}{\mathsf{E}} \left\{ \cosh^u \left(PE + \sqrt{PF}\,z\right) \right\}. && (250) \end{aligned}$$

Note also that

$$\sum_{a<b}^{u} \tilde{Q}_{ab} Q_{ab} = uEm + \frac{u(u-1)}{2} Fq. \tag{251}$$

It is then straightforward to obtain the free energy as (156) and the saddle-point equations (157).

The evaluation of the correlation for the optimal detectors is also very similar to that for the linear detectors. The quantities that differ are

$$\begin{aligned} f_0(z) &= \underset{d}{\mathsf{E}} \left\{ \exp\left[\left(P_1E + \sqrt{P_1F}\,z\right) d - \frac{1}{2}P_1Ed^2 \right] \right\} && (252)\\ &= e^{-\frac{1}{2}P_1E} \cosh\left(P_1E + \sqrt{P_1F}\,z\right) && (253) \end{aligned}$$

and

$$\begin{aligned} f_1(z) &= \underset{d}{\mathsf{E}} \left\{ d \cdot \exp\left[\left(P_1E + \sqrt{P_1F}\,z\right) d - \frac{1}{2}P_1Ed^2 \right] \right\} && (254)\\ &= e^{-\frac{1}{2}P_1E} \sinh\left(P_1E + \sqrt{P_1F}\,z\right). && (255) \end{aligned}$$

By (237),

$$\begin{aligned} \lim_{u\to 0} \frac{\partial}{\partial h} \log X_1(u;h) \bigg|_{h=0} &= \lim_{u\to 0} \frac{\underset{z}{\mathsf{E}} \left\{ \cosh^{u-i}\left(P_1E + \sqrt{P_1F}\,z\right) \sinh^i \left(P_1E + \sqrt{P_1F}\,z\right) \right\}}{\underset{z}{\mathsf{E}} \left\{ \cosh^u \left(P_1E + \sqrt{P_1F}\,z\right) \right\}} && (256)\\ &= \underset{z}{\mathsf{E}} \left\{ \tanh^i \left(P_1E + \sqrt{P_1F}\,z\right) \right\}. && (257) \end{aligned}$$

By (93), (234) and (257), we have (158).

References

[1] S. Verdú, *Multiuser Detection*. Cambridge University Press, 1998.

[2] D. Guo, L. K. Rasmussen, and T. J. Lim, “Linear parallel interference cancellation in long-code CDMA multiuser detection,” *IEEE J. Selected Areas Commun.*, vol. 17, pp. 2074–2081, Dec. 1999.

[3] A. J. Grant and P. D. Alexander, “Random sequence multisets for synchronous code-division multiple-access channels,” *IEEE Trans. Inform. Theory*, vol. 44, pp. 2832–2836, Nov. 1998.

[4] D. N. C. Tse and S. V. Hanly, "Linear multiuser receivers: effective interference, effective bandwidth and user capacity," *IEEE Trans. Inform. Theory*, vol. 45, pp. 622–640, March 1999.

[5] S. Verdú and S. Shamai, "Spectral efficiency of CDMA with random spreading," *IEEE Trans. Inform. Theory*, vol. 45, pp. 622–640, March 1999.

[6] D. N. C. Tse and S. Verdu, "Optimum asymptotic multiuser efficiency for randomly spread CDMA," *IEEE Trans. Inform. Theory*, vol. 46, pp. 2718–2722, Nov. 2000.

[7] R. R. Müller, "Multiuser receivers for randomly spread signals: fundamental limits with and without decision-feedback," *IEEE Trans. Inform. Theory*, vol. 47, pp. 268–283, Jan. 2001.

[8] P. Viswanath, D. N. C. Tse, and V. Anantharam, "Asymptotically optimal water-filling in vector multiple-access channels," *IEEE Trans. Inform. Theory*, vol. 47, pp. 241–267, Jan. 2001.

[9] J. Zhang, E. K. P. Chong, and D. N. C. Tse, "Output MAI distribution of linear MMSE multiuser receivers in DS-CDMA systems," *IEEE Trans. Inform. Theory*, vol. 47, pp. 1128–1144, March 2001.

[10] S. Shamai and S. Verdú, "The impact of frequency-flat fading on the spectral efficiency of CDMA," *IEEE Trans. Inform. Theory*, vol. 47, pp. 1302–1327, May 2001.

[11] D. Guo, S. Verdú, and L. K. Rasmussen, "Asymptotic normality of linear CDMA multiuser detection outputs," in *Proceedings 2001 IEEE International Symposium on Information Theory*, p. 307, Washington, D.C., June 2001.

[12] L. Li, A. Tulino, and S. Verdú, "Asymptotic eigenvalue moments for linear multiuser detection," *Communications in Information and Systems*, vol. 1, pp. 273–304, Sept. 2001.

[13] D. Guo, S. Verdú, and L. K. Rasmussen, "Asymptotic normality of linear multiuser receiver outputs," *accepted for IEEE Trans. on Inform. Theory*, 2001.

[14] U. Grenander and J. W. Silverstein, "Spectral analysis of networks with random topologies," *SIAM J. Appl. Math.*, vol. 32, pp. 499–519, March 1977.

[15] D. Jonssonaaa, "Some limit theorems for the eigenvalues of a sample covariance matrix," *Journal of Multivariate Analysis*, vol. 12, pp. 1–38, Dec. 1982.

[16] J. W. Silverstein and Z. D. Bai, "On the empirical distribution of eigenvalues of a class of large dimensional random matrices," *Journal of Multivariate Analysis*, vol. 54, pp. 175–192, Aug. 1995.

[17] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York : Wiley, 1991.

[18] N. Sourlas, "Spin-glass models as error-correcting codes," *Nature*, vol. 339, pp. 693–695, June 1989.

[19] N. Sourlas, "Spin-glasses, error-correcting codes and finite-temperature decoding," *Europhysics Letters*, vol. 25, Jan. 1994.

[20] A. Montanari and N. Sourlas, "The statistical mechanics of turbo codes," *Eur. Phys. J. B*, vol. 18, no. 1, pp. 107–119, 2000.

[21] Y. Kabashima and D. Saad, "Statistical mechanics of error-correcting codes," *Europhysics Letters*, vol. 45, no. 1, pp. 97–103, 1999.

[22] H. Nishimori and K. Y. M. Wong, "Statistical mechanics of image restoration and error-correcting codes," *Pysical Review E*, vol. 60, pp. 132–144, July 1999.

[23] K. Nakamura, Y. Kabashima, and D. Saad, "Statistical mechanics of low-density parity check error-correcting codes over Galois fields," *Europhysics Letters*, vol. 56, pp. 610–616, Nov. 2001.

[24] H. Nishimori, *Statistical Physics of Spin Glasses and Information Processing: An Introduction*. Number 111 in International Series of Monographs on Physics, Oxford University Press, 2001.

[25] G. I. Kechriotis and E. S. Manolakos, “Hopfield neural network implementation of the optimal CDMA multiuser detector,” *IEEE Trans. Neural Networks*, Jan. 1996.

[26] T. Tanaka, “Analysis of bit error probability of direct-sequence CDMA multiuser demodulators,” in *Advances in Neural Information Processing Systems* (T. K. Leen et al., ed.), vol. 13, pp. 315–321, The MIT Press, 2001.

[27] T. Tanaka, “Average-case analysis of multiuser detectors,” in *Proceedings 2001 IEEE International Symposium on Information Theory*, p. 287, Washington, D.C., June 2001.

[28] T. Tanaka, “A statistical mechanics approach to large-system analysis of CDMA multiuser detectors,” *accepted for IEEE Trans. Inform. Theory*, 2002.

[29] J.-P. Bouchaud and M. Potters, *Theory of Financial Risks: From Statistical Physics to Risk Management*. Cambridge University Press, 2000.

[30] D. Guo and S. Verdú, “Minimum probability of error of many-user CDMA without power control,” in *Proceedings 2002 IEEE International Symposium on Information Theory*, p. 188, Lausanne, Switzerland, June 2002.

[31] D. Guo and S. Verdú, “Multiuser detection and statistical mechanics,” *to appear in Ian Blake Festschrift*, 2002.

[32] V. Dotsenko, *The Theory of Spin Glasses and Neural Networks*. World Scientific, 1994.

[33] P. McCullagh, *Tensor methods in statistics*. Chapman and Hall, 1987.

[34] R. S. Ellis, *Entropy, large deviations, and statistical mechanics*, vol. 271 of *A series of comprehensive studies in mathematics*. Springer-Verlag, 1985.

[35] Y. C. Eldar and A. M. Chan, “The independence of wishart matrix eigenvalues and eigenvectors, and its application to the asymptotic performance of the decorrelator,” *submitted to IEEE Trans. Inform. Theory*, 2001.

[36] H. L. Royden, *Real Analysis*. Macmillan, 1988.

[37] R. R. Müller and W. H. Gerstacker, “On the capacity loss due to separation of detection and decoding,” *IEEE Trans. Inform. Theory*, 2002. submitted.