High-SNR Power Offset in Multi-Antenna Communication

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I. INTRODUCTION

The analysis of the high-SNR multi-antenna capacity with coherent receivers has focused on the *multiplexing gain*, i.e., the multiplicative increase as function of the number of antennas. For most channels of interest, such multiplexing gain equals the minimum of the number of transmit and receive antennas. This traditional characterization, however, is unable to quantify the impact of many relevant channel features. As a function of $\mathsf{SNR}|_{\mathrm{dB}}$, the capacity is very well approximated, from moderate SNR on, as an affine function. The impact of the various channel features is captured in the *power offset* (in dB) or zero-order term in the affine expansion.

With n_{T} transmit and n_{R} receive antennas, the complex model we consider is $\mathbf{y}{=}\mathbf{H}\,\mathbf{x}+\mathbf{n}$ where $\mathbf{x}$ and $\mathbf{y}$ are the input and output vectors while $\mathbf{n}$ is white Gaussian noise. The channel is represented by the $(n_{\mathrm{R}}{\times}n_{\mathrm{T}})$ zero-mean random matrix $\mathbf{H}$ normalized such that $E[\mathrm{Tr}\{\mathbf{H}\mathbf{H}^\dagger\}]{=}n_{\mathrm{R}}n_{\mathrm{T}}$. The covariance of the input is

$$\mathbf{\Phi} = \frac{E[\mathbf{x}\mathbf{x}^\dagger]}{\frac{1}{n_{\mathrm{T}}}E[\|\mathbf{x}\|^2]}$$

When $\mathbf{H}$ is known at the receiver, the average mutual information (b/s/Hz) achieved by a zero-mean Gaussian input is

$$I(\mathsf{SNR},\mathbf{\Phi}) = E\left[\log_2\det\left(\mathbf{I}+\tfrac{\mathsf{SNR}}{n_{\mathrm{T}}}\,\mathbf{H}\mathbf{\Phi}\mathbf{H}^\dagger\right)\right] \qquad (1)$$

with

$$\mathsf{SNR} = \frac{E[\|\mathbf{x}\|^2]}{\frac{1}{n_{\mathrm{R}}}E[\|\mathbf{n}\|^2]}$$

In terms of antenna correlation, we adhere to the *separable* model whereby $\mathbf{H}{=}\mathbf{\Theta}_{\mathrm{R}}^{1/2}\mathbf{W}\mathbf{\Theta}_{\mathrm{T}}^{1/2}$ where $\mathbf{W}$ has IID zero-mean unit-variance Gaussian entries, $\mathbf{\Theta}_{\mathrm{R}}$ and $\mathbf{\Theta}_{\mathrm{T}}$ are positive-definite $(n_{\mathrm{R}}{\times}n_{\mathrm{R}})$ and $(n_{\mathrm{T}}{\times}n_{\mathrm{T}})$ correlation matrices indicating the correlation between receive antennas and between transmit antennas, and $(\cdot)_{i,j}$ denotes the (i,j)th entry.

Since achieving capacity requires signalling on the eigenvectors of $\mathbf{\Theta}_{\mathrm{T}}$ [1], we shall consider that $\mathbf{\Phi}{=}\mathbf{V}\mathbf{P}\mathbf{V}^\dagger$ where $\mathbf{V}$ contains the eigenvectors of $\mathbf{\Theta}_{\mathrm{T}}$ while $\mathbf{P}{=}\mathrm{diag}\{p_1,p_2,\ldots,p_{n_{\mathrm{T}}}\}$ with $p_j{>}0$ the power allocated to the jth such eigenvector.

An affine expansion of (1), with SNR expressed in dB, yields

$$I(\mathsf{SNR},\mathbf{\Phi}) = S_\infty\left(\frac{\mathsf{SNR}|_{\mathrm{dB}}}{3\,\mathrm{dB}} - \mathcal{L}_\infty\right) + o(1).$$

where S_∞ is the multiplexing gain in b/s/Hz/(3 dB), given by

$$S_\infty = \lim_{\mathsf{SNR}\to\infty}\frac{I(\mathsf{SNR},\mathbf{\Phi})}{\log_2\mathsf{SNR}} = \min(n_{\mathrm{T}},n_{\mathrm{R}})$$

which is insensitive to correlation, to $\max(n_{\mathrm{T}},n_{\mathrm{R}})$, and to the input power allocation (as long as $p_j{>}0\ \forall j$). These features however, are strongly reflected in $\mathcal{L}_\infty$, which represents the *power offset* in 3-dB units with respect to a reference channel whose dimensions are unfaded and orthogonal, i.e., such that $\frac{1}{n_{\mathrm{T}}}\mathbf{H}\mathbf{H}^\dagger{=}\mathbf{I}$. This measure, introduced in [2], is obtained as[1]

$$\mathcal{L}_\infty(\mathbf{\Phi}) = \lim_{\mathsf{SNR}\to\infty}\log_2\mathsf{SNR} - \frac{I(\mathsf{SNR},\mathbf{\Phi})}{S_\infty}$$

II. HIGH-SNR POWER OFFSET

Let $m{=}\min(n_{\mathrm{T}},n_{\mathrm{R}})$, $n{=}\max(n_{\mathrm{T}},n_{\mathrm{R}})$ and $\mathbf{\Theta}{=}\mathbf{\Theta}_{\mathrm{T}}^{1/2}\mathbf{\Phi}\mathbf{\Theta}_{\mathrm{T}}^{1/2}$. Let $\mathbf{\Theta}_m{=}\mathbf{\Theta}_{\mathrm{R}}$ and $\mathbf{\Theta}_n{=}\mathbf{\Theta}$ if $n_{\mathrm{T}}{\geq}n_{\mathrm{R}}$, vice versa if $n_{\mathrm{T}}{\leq}n_{\mathrm{R}}$. Denoting by λ_j the jth eigenvalue of $\mathbf{\Theta}_n$, further let $\mathbf{\Psi}_i$, $i\in\{1,\ldots,m\}$, be a $(m{\times}m)$ matrix with (k,ℓ)th entry

$$(\mathbf{\Psi}_i)_{k,\ell} = \mu_1\,\lambda_{n-m+k}^{n-m-1+\ell} - \sum_{d=1,q=1}^{n-m}\mu_2\,(\mathbf{\Upsilon}^{-1})_{d,q}\,\lambda_{n-m+k}^{d-1}\,\lambda_q^{n-m-1+\ell}$$

where $\mathbf{\Omega}$ is the $(n{\times}n)$ Vandermonde matrix defined as $(\mathbf{\Omega})_{i,j}{=}\lambda_i^{j-1}$, whose $(n-m){\times}(n-m)$ principal submatrix is $\mathbf{\Upsilon}$, and $\mu_1{=}\mu_2{=}1$ if $\ell{\neq}i$ whereas, for $\ell{=}i$,

$$\mu_1 = -\gamma+\sum_{u=1}^{\ell-1}\tfrac{1}{u}+\log_e\lambda_{n-m+k} \qquad \mu_2 = -\gamma+\sum_{u=1}^{\ell-1}\tfrac{1}{u}+\log_e\lambda_q$$

where $\gamma{=}0.5772\ldots$ is the Euler-Mascheroni constant.

Proposition 1 *The high-SNR power offset, in 3-dB units, is*

$$\mathcal{L}_\infty = \log_2 n_{\mathrm{T}} - \tfrac{\det\mathbf{\Upsilon}}{\det\mathbf{\Omega}}\tfrac{\log_2 e}{m}\sum_{i=1}^{m}\det\mathbf{\Psi}_i - \tfrac{1}{m}\log_2\det\mathbf{\Theta}_m$$

which, for $n_{\mathrm{T}}{=}n_{\mathrm{R}}{=}n$, simplifies to

$$\mathcal{L}_\infty = \log_2 n + \left(\gamma - \sum_{u=2}^{n}\tfrac{1}{u}\right)\log_2 e - \tfrac{1}{n}\log_2\det(\mathbf{\Theta}_{\mathrm{R}}\mathbf{\Theta})$$

Example 1 *If $n_{\mathrm{T}}{=}2$ and $n_{\mathrm{R}}{=}1$ with λ_1 and λ_2 the eigenvalues of $\mathbf{\Theta}_{\mathrm{T}}$, then*

$$\mathcal{L}_\infty = 1+\gamma\log_2 e - \frac{p_2\lambda_2\log_2(p_2\lambda_2)-p_1\lambda_1\log_2(p_1\lambda_1)}{p_2\lambda_2-p_1\lambda_1}$$

from which the capacity-achieving powers at high SNR follow.

REFERENCES

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[3] A. Lapidoth and S. M. Moser, "Capacity bounds via duality with applications to multi-antenna systems on flat-fading channels," *IEEE Trans. on Inform. Theory*, vol. 49, Oct 2003.

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[1] Partially supported by U.S. Army Research Laboratory under contract DAAD 19-01-2-00

[1] Its zero-order counterpart for $\mathbf{H}$ unknown to the receiver (non-coherent communication) is the *fading number* introduced in [3].