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Linear Differentially Coherent Multiuser Detection for Multipath Channels*

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Abstract. By combining multipath processing, differential signal detection, and multiuser detection techniques, we develop a class of near-far resistant linear detectors for differentially coherent multipath signals. We derive and establish performance relationships among the following detectors: an optimally near-far resistant detector, a suboptimum detector which does not require knowledge of the signal coordinates, and a minimum mean square error (MMSE) detector which achieves near-optimum asymptotic efficiency. We present an adaptive multiuser detector which converges to the MMSE detector without training sequences and which requires less information than the conventional single user rake receiver.

Key words: multiuser detection, multipath channel, blind adaptive algorithm.

1. Introduction

Mobile telephony in direct sequence code-division multiple access (DS/CDMA) channels encounters a variety of communication challenges including multipath fading and multiaccess interference due to simultaneous transmissions from interfering users. Detectors which employ multipath (rake-type) combining and multiuser detection have been shown to provide effective solutions to these problems. Many of these detectors assume that reliable phase estimates are available; however, in practice, especially on the reverse (many-to-one) link, this information may not be available, and noncoherent or differentially coherent reception is required. In this paper, we will address the detection of differentially coherent multipath signals by using detectors which combine multipath processing, differential detection, and multiuser detection techniques.

By considering the signal structure of the interfering users, multiuser detection (see [1] and [2] for recent tutorials) effectively counteracts the effects of multiple access interference (MAI). These effects may be quite detrimental since variations in the received signal strengths may render conventional detection schemes useless if the desired user's signal is much weaker than the interferers' signals. In this case, it is necessary to use multiuser detection to combat this so-called near-far problem. On the other hand, even if a system employs perfect power control to ensure equal-strength signals at the receiver, multiuser detection still potentially provides significant performance gains over conventional detection techniques which neglect the presence of interfering users.

Using a judicious combination of multipath combining schemes and low complexity multiuser detection techniques, our proposed detectors will provide attractive near-far resistant performance at very little cost in terms of increased complexity or information requirements.

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In particular, we will present an adaptive near-far resistant detector which requires even less information than the conventional rake receiver.

To put this work in context, we briefly discuss multiuser detection and its extensions to differentially coherent detection, multipath processing, and adaptive detection.

- **Multiuser detection**: The optimum multiuser detector, developed in [3], consists of a bank of matched filters matched to each of the user's signals followed by decision function whose complexity is exponential in the number of users. Because of the optimum detector's high complexity, a number of suboptimum detectors were derived which are less complex but, like the optimum detector, effectively combat the near-far problem or provide significant performance gains over the conventional detector even in the presence of perfect power control. The decorrelating detector [4] is a particularly easy multiuser detector to implement since it replaces the decision function of the optimum detector with a simple linear transformation and a bank of hard limiters which are used for the bit decisions. The linear transformation, called the decorrelator, is based on the correlations of the users' signals and essentially separates the signal components of each user at the output. The decorrelating detector is optimally near-far resistant [4].
- **Multiuser detection for differentially coherent signals**: In many communication systems, the lack of an absolute phase reference will require the use of noncoherent reception schemes. Reference [5] derives an optimally near-far resistant multiuser detector for differentially coherent (DPSK modulated) signals. The detector is similar to the coherent decorrelating detector except that the bit decisions are based on the phase difference of the decorrelator output at two successive bit intervals.
- **Multiuser detection for multipath signals**: The matched filter for multipath signals is the well-known rake filter (see for example, [6]) which coherently combines the outputs of filters matched to the time translates of a user's transmitted signal. As such, the optimum multiuser detector for multipath signals [7] uses a bank of rake filters in order to provide sufficient statistics. The rake decorrelating detector [8], [9] is a bank of rake filters followed by a linear transformation based on the correlations of the users' multipath signals. If the complex channel coefficient estimates are unreliable, one can perform the coherent combining after the decorrelator instead of at the front-end rake filter [10]. While this alternative detector does not perform as well as the original rake decorrelating detector when the coefficients are known exactly, it is much more robust to coefficient mismatch and, unlike the rake decorrelating detector, retains its near-far resistant characteristics in these situations [8]. Alternatively, when the coefficients are unknown and when the signals are modulated using DPSK, we can replace each user's coherent combiner with an equal gain combiner which sums the phase differences of a user's decorrelator output. We will rely on this technique in deriving the adaptive multiuser detector in this paper.

  We will use the notion of *multi-dimensional* signals in this paper as a generalization of multipath signals with negligible intersymbol interference. A multi-dimensional signal is a linear combination of a particular user's subsignals, which, in the special case of multipath signals, are time translates of that user's transmitted signal. This generalization allows us to take a more intuitive subspace-based approach in developing our multiuser detectors.
- **Adaptive multiuser detection**: Adaptive multiuser detectors are necessary for systems in which the detector does not acquire the crosscorrelations with the interfering signals and which, therefore, must rely on adaptive methods to "learn" about and to reject the MAI.

> Reference [11] derives a coherent, linear minimum mean square error (MMSE) detector which can be adaptively implemented with a training data sequence. This MMSE detector exhibits the attractive property of being equivalent to the (optimally near-far resistant) decorrelating detector as the background noise level vanishes. However, its disadvantage is that whenever a sudden and significant change in the channel occurs (due to a severe fade or the arrival or departure of a strong interferer), it requires the retransmission of training sequences to reinitialize the detector. To overcome this burden, [12] shows how the MMSE multiuser detector can be adaptively implemented without training sequences; this *blind adaptive multiuser detector* requires only knowledge of the desired user's signature waveform and timing, i.e., the same requirement as the conventional single-user matched filter.

After establishing some preliminaries in Section 2, we will devote Sections 3 and 4 to nonadaptive linear detectors. In Section 3, we examine the optimum linear detector for multidimensional differentially coherent signals, and in Section 4, we examine a quasi-optimum linear detector which does not require knowledge of the subsignal coordinates (i.e., multipath coefficients). These detectors are most suitable for use at the base station receiver (as opposed to the mobile receiver) since they require information about all of the users' signals. The blind adaptive detector, presented in Section 5, does not require explicit knowledge of the interfering signature waveforms and therefore could be used at both the base station and the mobile receiver. In Section 6 we quantitatively compare the performances of our three detectors along with conventional rake detectors.

2. Preliminaries

We will represent the signals using a complex baseband representation and as finite N-vectors with respect to some basis. The received signal is composed of K users' L multipath signals:

$$\mathbf{r}[j] = \sum_{k=1}^{K} \left[A_k \, \mathrm{e}^{j\theta_k} b_k[j] \sum_{l=1}^{L} c^{\circ}_{k,l} \mathbf{s}^{\circ}_{k,l} \right] + \mathbf{n}[j], \tag{1}$$

where j is the symbol interval; A_k is the kth user's amplitude; $\theta_k \in [\Leftrightarrow\pi, \pi)$ is the kth user's unknown, uniformly distributed phase; $b_k[j] \in \{\Leftrightarrow 1, 1\}$ is the differentially encoded bit whose corresponding data bit is $d_k[j] = \begin{cases} 1, & b_k[j] = b_k[j-1] \\ -1, & b_k[j] = -b_k[j-1] \end{cases}$; $c^{\circ}_{k,l} \in \mathbf{C}$ is the complex channel coefficient of the kth user's lth multipath signal; $\mathbf{s}^{\circ}_{k,l}$ is the real N-vector denoting the kth user's lth multipath signal. The signals are normalized to have unit energy: $\langle \mathbf{s}^{\circ}_{k,l}, \mathbf{s}^{\circ}_{k,l} \rangle = 1$. $\mathbf{n}[j]$ is the complex zero-mean additive white Gaussian noise whose real and imaginary components are independent, identically distributed with variance σ^2.

It is convenient to use a vector subspace interpretation of these signals. In other words, we think of the L_k multipath signals for the kth user, $\mathbf{s}^{\circ}_{k,1} \ldots \mathbf{s}^{\circ}_{k,L}$ as vectors which span an L_k dimensional subspace denoted by $\mathcal{S}_k$. In addition, we can think of $c^{\circ}_{k,1} \ldots c^{\circ}_{k,L}$ as their corresponding complex amplitudes. The kth user's multipath signals form a spanning set for $\mathcal{S}_k$ but do not necessarily form an orthonormal basis for $\mathcal{S}_k$. Note, however, that given $\mathcal{S}_k$, we can easily find an orthonormal basis, $\mathbf{s}_{k,1} \ldots \mathbf{s}_{k,L}$, via the Gram–Schmidt process. Since these orthonormal vectors, which we will call *subsignals*, form a basis for $\mathcal{S}_k$, $\sum_{l=1}^{L_k} c^{\circ}_{k,l} \mathbf{s}^{\circ}_{k,l}$

lies in $\mathcal{S}_k$, and this combined signal can be written as a complex linear combination of the basis vectors:

$$\sum_{l=1}^{L_k} c_{k,l}^{\circ} \mathbf{s}_{k,l}^{\circ} = \sum_{l=1}^{L_k} c_{k,l} \mathbf{s}_{k,l}. \tag{2}$$

We call $c_{k,l}$ the complex coordinate of the kth user's lth subsignal $\mathbf{s}_{k,l}$. Using (2) to rewrite (1), we have

$$\mathbf{r}[j] = \sum_{k=1}^{K} \left[A_k \, \mathrm{e}^{j\theta_k} b_k[j] \sum_{l=1}^{L_k} c_{k,l} \mathbf{s}_{k,l} \right] + \mathbf{n}[j]. \tag{3}$$

We make the following assumptions in this paper:

(1) The received signals are symbol synchronous. This assumption is made to simplify the analysis and presentation.
(2) In the context of multipath channels, we assume that the symbol rate is slow compared to the time coherence of the channel. Hence, the channel is slowly varying, and in the larger context of multidimensional signals, this assumption implies that the subsignal coordinates are time invariant over two successive symbol intervals.
(3) Defining $\mathbf{c}_k \equiv [c_{k,1} \dots c_{k,L}]^T$, we assume that $||\mathbf{c}_k|| = 1$ for all $k = 1, \dots, K$. We can make this assumption without loss of generality since the normalizing factor for each user's set of coordinates can be incorporated into the signal amplitude A_k.
(4) The subspace vectors of all K users are linearly independent. In other words, the $N \times L$ matrix

$$\mathbf{S}_k \equiv [\mathbf{s}_{k,1} \dots \mathbf{s}_{k,L}], \quad k = 1, \dots, K$$

has rank L, and the $N \times KL$ matrix

$$\mathbf{S} \equiv [\mathbf{S}_1 \dots \mathbf{S}_K] \tag{4}$$

has rank KL.

In the study of multiuser detectors, a commonly used performance measure is the *asymptotic efficiency* [4], [5] which measures the rate at which the bit error probability approaches zero as the background noise goes to zero. The asymptotic efficiency of a multiuser detector with probability of error $P_k(\sigma)$ for differentially coherent multidimensional signals is

$$\eta_k = \sup \left\{ 0 \leqslant r \leqslant 1; \lim_{\sigma \to 0} \frac{P_k(\sigma)}{\exp\left[\frac{-rA_k^2}{2\sigma^2}\right]} < \infty \right\}. \tag{5}$$

A related performance measure, the *near-far resistance*, is the infimum of the asymptotic efficiency as the interferers' energies, phases, and subsignal coordinates are allowed to vary:

$$\bar{\eta}_k = \inf_{\substack{A_j \geqslant 0 \\ \theta_j \in [-\pi, \pi) \\ \mathbf{c}_j \in \mathbf{C}^L \\ j \neq k}} [\eta_k]. \tag{6}$$

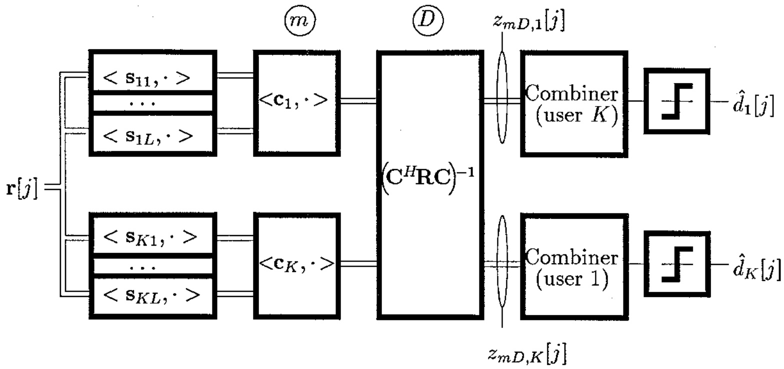


Figure 1. mD detector.

We now introduce some notation which will simplify the presentation. An alternative way of writing the received signal (3) is by using matrix notation:

$$\mathbf{r} = \mathbf{S}\mathbf{C}\mathbf{\Psi}\mathbf{A}\mathbf{b} + \mathbf{n},$$

where $\mathbf{S}$ is defined in (4), $\mathbf{C} \equiv \text{diag}(\mathbf{c}_1 \ldots \mathbf{c}_K)$ is the $KL \times K$ matrix of user coordinates, $\mathbf{A} \equiv \text{diag}(A_1 \ldots A_K)$, $\mathbf{\Psi} \equiv diag[\mathrm{e}^{j\theta_1} \ldots \mathrm{e}^{j\theta_K}]$, and $\mathbf{b} \equiv [b_1 \ldots b_K]^T$. We define $\mathbf{R} \equiv \mathbf{S}^T\mathbf{S}$ as the $KL \times KL$ cross-correlation matrix of the users' subsignal vectors.

Given a matrix $\mathbf{X}$, let $\mathbf{X}_{(a,b)}$ be the element in the ath row and bth column. If $\mathbf{X}$ is square and positive definite, we denote its inverse as $\mathbf{X}^+$. If $\mathbf{X}$ is a $KL \times KL$ matrix, we can partition it as

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_{[11]} & \mathbf{X}_{[1I]} \\ \mathbf{X}_{[I1]} & \mathbf{X}_{[II]} \end{bmatrix},$$

such that $\mathbf{X}_{[11]}$, $\mathbf{X}_{[1I]}$, $\mathbf{X}_{[I1]}$, and $\mathbf{X}_{[II]}$ are respectively $L \times L$, $L \times (K \Leftrightarrow 1)L$, $(K \Leftrightarrow 1)L \times L$, and $(K \Leftrightarrow 1)L \times (K \Leftrightarrow 1)L$ submatrices of $\mathbf{X}$. Let $\mathbf{I}_{[a]}$ denote an $a \times a$ identity matrix, let $\mathbf{0}_{[a \times b]}$ denote an $a \times b$ matrix of zeros, and let $\mathbf{0}_a$ denote an a-vector of zeros.

3. The Multipath-Combining Decorrelating (mD) Detector

The derivation of the optimally near-far resistant linear multiuser detector for differentially coherent multi-dimensional signals, given originally in [8], follows directly from the equivalent multiuser detector for single dimensional signals [5]. Assuming that the subsignal coordinates of all the users are known exactly, the multidimensional signal detection problem reduces to a single-dimensional problem since the linear combination of the kth user's subsignals can be redefined as the combined signal $\tilde{\mathbf{s}}_k \equiv \sum_{l=1}^{L} c_{k,l}\mathbf{s}_{k,l} = \mathbf{S}_k\mathbf{c}_k$. It follows from [5] that the resulting optimally near-far resistant linear detector, given in Figure 1, consists of a bank of filters matched to $\tilde{\mathbf{s}}_k$ followed by a decorrelator [4] based on the combined signals' crosscorrelations. The kth user's matched filter output, given by $\tilde{\mathbf{s}}_k^H\mathbf{r} = \mathbf{c}_k^H\mathbf{S}_k^T\mathbf{r}$, can be reinterpreted as a bank of filters matched to the user's L subsignal vectors followed by a combiner designated by $\langle \mathbf{c}_k, \cdot \rangle$. In the multipath signal context, the matched filter $\langle \tilde{\mathbf{s}}_k, \cdot \rangle$ is equivalent to a rake matched filter. Passing the received signal $\mathbf{r}[j]$ through a parallel bank of K matched filters, the resulting decision statistic is a K-vector

$$\begin{aligned} \mathbf{C}^H\mathbf{S}^T\mathbf{r}[j] &= \mathbf{C}^H\mathbf{S}^T\mathbf{S}\mathbf{C}\mathbf{\Psi}\mathbf{A}\mathbf{b}[j] + \mathbf{C}^H\mathbf{S}^T\mathbf{n}[j] && (7) \\ &= \mathbf{C}^H\mathbf{R}\mathbf{C}\mathbf{\Psi}\mathbf{A}\mathbf{b}[j] + \mathbf{C}^H\mathbf{S}^T\mathbf{n}[j], && (8) \end{aligned}$$

where the first term of (8) is due to the users' signals and the second term is due to background noise. We can separate or *decorrelate* the users' signals by multiplying the matched filter outputs (8) with the $K \times K$ decorrelator matrix $(\mathbf{C}^H\mathbf{R}\mathbf{C})^{-1}$

$$\begin{aligned} \mathbf{z}_{mD}[j] &\equiv (\mathbf{C}^H\mathbf{R}\mathbf{C})^{-1}\mathbf{C}^H\mathbf{R}\mathbf{C}\Psi A b[j] + (\mathbf{C}^H\mathbf{R}\mathbf{C})^{-1}\mathbf{C}^H\mathbf{S}^T\mathbf{n}[j] && (9) \\ &= \mathbf{\Psi}\mathbf{A}\mathbf{b}[j] + \mathbf{n}'[j], && (10) \end{aligned}$$

where $\mathbf{n}'[j]$ is a zero-mean complex Gaussian random vector with covariance

$$\begin{aligned} & E\left[\begin{bmatrix} \mathrm{Re}(\mathbf{n}'[j]) \\ \mathrm{Im}(\mathbf{n}'[j]) \end{bmatrix} \begin{bmatrix} \mathrm{Re}(\mathbf{n}'[j])^T & \mathrm{Im}(\mathbf{n}'[j])^T \end{bmatrix}\right] \\ & = \sigma^2 \begin{bmatrix} \mathrm{Re}([\mathbf{C}^H\mathbf{R}\mathbf{C}]^{-1}) & \Leftrightarrow \mathrm{Im}([\mathbf{C}^H\mathbf{R}\mathbf{C}]^{-1}) \\ \mathrm{Im}([\mathbf{C}^H\mathbf{R}\mathbf{C}]^{-1}) & \mathrm{Re}([\mathbf{C}^H\mathbf{R}\mathbf{C}]^{-1}) \end{bmatrix} \qquad (11) \end{aligned}$$

and is uncorrelated with $\mathbf{n}'[j \Leftrightarrow 1]$. We call this the *multipath-combining decorrelating (mD) detector*. Hence $\mathbf{z}_{mD}$ is a K-vector whose kth element consists of only the kth user's complex amplitude ($e^{j\theta_k}A_k b_k[j]$) and background noise. Given $z_{mD,k}[j]$, the kth component of $\mathbf{z}_{mD}[j]$, and $z_{mD,k}[j \Leftrightarrow 1]$ as the statistics for user k, decisions on its information bit d_k are obtained as we would with a pair of single user DPSK statistics. Namely, we take their phase difference and use a hard limiter to decide whether this phase change is closer to 0 or to π:

$$\widehat{d}_k = \mathrm{sgn}\left[\mathrm{Re}(z^H_{mD,k}[j] z_{mD,k}[j \Leftrightarrow 1])\right]. \qquad (12)$$

Note that in the absence of background noise, the decorrelator output (9) is $\mathbf{\Psi}\mathbf{A}\mathbf{b}[j]$, and the bit estimates (12)

$$\begin{aligned} \widehat{d}_k &= \mathrm{sgn}\left[A_k^2 b_k[j] b_k[j \Leftrightarrow 1]\right] \\ &= \left\{ \begin{array}{rcl} 1, \text{ if } b_k[j] &=& b_k[j \Leftrightarrow 1] \\ \Leftrightarrow 1, \text{ if } b_k[j] &=& \Leftrightarrow b_k[j \Leftrightarrow 1] \end{array} \right\} \end{aligned}$$

are perfect.

We observe that if the signal subspaces of all users are mutually orthogonal, the cross-correlation matrix $\mathbf{R}$ reduces to the identity matrix, and the mD detector reduces to a bank of single user rake receivers. This situation is the multidimensional analog of the single-dimensional situation in which the users' signals are mutually orthogonal.

Since the decision statistics $z_{mD,k}[j]$ and $z_{mD,k}[j \Leftrightarrow 1]$ are decorrelated from those of the other users, the bit error rate of the mD detector for the kth user is identical to a single user DPSK detector using the same statistics. For the jth symbol interval,

$$z_{mD,k}[j] = \mathrm{e}^{j\theta_k} A_k b_k[j] + n'_k[j],$$

$n'_k[j]$ is a zero mean complex Gaussian random variable whose real and imaginary components are i.i.d. with variance given by the kth diagonal term of $\sigma^2(\mathbf{C}^H\mathbf{R}\mathbf{C})^{-1}$, denoted by

$\sigma^2(\mathbf{C}^H\mathbf{R}\mathbf{C})^{+}_{(k,k)}$. (Note that since $\mathbf{C}^H\mathbf{R}\mathbf{C}$ is Hermitian, the diagonal elements of its inverse are real). Hence the probability of error for the kth user [5] is

$$P_{mD,k}(\sigma) = \tfrac{1}{2}\exp\left[\frac{\Leftrightarrow A_k^2}{2\sigma^2(\mathbf{C}^H\mathbf{R}\mathbf{C})^{+}_{(k,k)}}\right]. \tag{13}$$

From (5) and (13), the asymptotic efficiency of the mD detector is

$$\bar{\eta}_{mD,k} = \frac{1}{(\mathbf{C}^H\mathbf{R}\mathbf{C})^{+}_{(k,k)}}$$

and it follows from [5] that this is the optimum efficiency achievable for a given set of complex subsignal coordinates.

Note that the signal cross-correlation matrix $(\mathbf{C}^H\mathbf{R}\mathbf{C})$ depends on the users' combined signals $\tilde{\mathbf{s}}_k$ but not on the spanning set chosen to represent them; in other words, we can write $\tilde{\mathbf{s}}_k = \mathbf{S}'_k\mathbf{c}'_k$, where $\mathbf{c}'_k$ is the corresponding coordinate vector for some arbitrary set of spanning vectors for user k given by the columns of $\mathbf{S}'_k$.

So far, we have only considered the case where the subsignal coordinates of all users are known exactly. In practice, we will use (nonideal) estimates of these parameters in the detector. If we denote the estimates of $\mathbf{c}_k$ as $\widehat{\mathbf{c}}_k(k = 1\ldots K)$ and denote $\widehat{\mathbf{C}} \equiv \mathrm{diag}(\widehat{\mathbf{c}}_1\ldots\widehat{\mathbf{c}}_K)$, the decorrelator output is

$$\mathbf{z}_{mD}[j] = (\widehat{\mathbf{C}}\mathbf{R}\widehat{\mathbf{C}})^{-1}(\widehat{\mathbf{C}}\mathbf{R}\mathbf{C})\mathbf{\Psi}\mathbf{A}\mathbf{b}[j] + \mathbf{n}'[j],$$

where the noise component $\mathbf{n}'[j]$ is unchanged from (9). Suppose that user k is the desired user and that there is coordinate mismatch for at least one of the interferers, i.e., $\widehat{\mathbf{c}}_j \neq \mathbf{c}_j$ for at least one $j = 1,\ldots,K$, $j \neq k$. Then there will be "leakage" from the interfering users which contaminates user k's decorrelator output. The probability of error can then be driven arbitrarily high (approaching $\frac{1}{2}$) by increasing the interferers' energies. Hence the mD detector has a near-far resistance of zero if there is coordinate mismatch for at least one of the interfering users.

This shortcoming of the mD detector motivates the consideration of a more robust multiuser detector which retains its near-far resistant characteristics in the presence of interferer coordinate mismatch. We address this problem in the next section.

4. Decorrelating Multipath-Combining (Dm) Detector

By performing decorrelation prior to multipath combining, the decorrelating multipath combining (Dm) detector (also known as the *multipath decorrelating equal-gain combiner* in the original source [10]) not only achieves the goal of providing near-far resistant performance in the presence of interferer coordinate mismatch but surpasses the goal since it does not require any coordinate estimates at all. This detector only requires knowledge of all the users' signal subspaces.

4.1. Derivation of the Dm Detector

We now derive the structure of this detector, as given in Figure 2. The complex received signal $\mathbf{r}[j]$ is passed through a bank of KL filters matched to each individual subsignal vector.

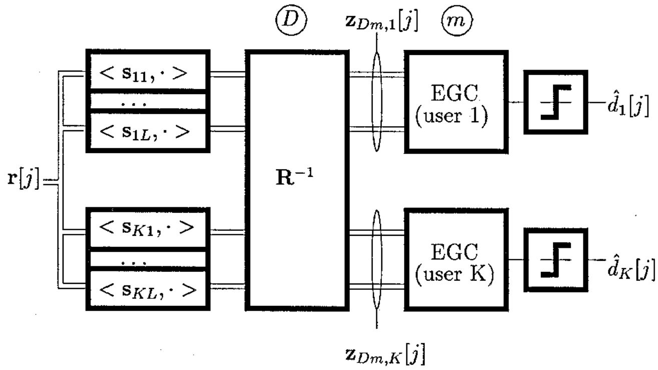


Figure 2. Dm detector.

Instead of combining each user's matched filter outputs as we did with the mD detector, the KL outputs are decorrelated using a linear transformation designated by the inverse of the subsignal cross correlation matrix ($\mathbf{R}^{-1}$). Since the decorrelator input can be written as

$$\begin{aligned}\mathbf{y}[j] &\equiv \mathbf{S}^T\mathbf{r}[j] \\ &= \mathbf{S}^T\mathbf{SC\Psi Ab}[j] + \mathbf{n}'[j] \\ &= \mathbf{RC\Psi Ab}[j] + \mathbf{n}'[j],\end{aligned}$$

where $\mathbf{n}'[j]$ is a complex Gaussian random vector with i.i.d. $\eta(\mathbf{0}_{KL}, \sigma^2\mathbf{R})$ real and imaginary components, the decorrelator output is

$$\mathbf{R}^{-1}\mathbf{y}[j] = \mathbf{C\Psi Ab}[j] + \mathbf{R}^{-1}\mathbf{n}'[j].$$

Define the decision statistic $\mathbf{z}_{Dm,k}$ for the kth user as the kth L-vector of the decorrelator output. For the first user

$$\mathbf{z}_{Dm,1}[j] = \begin{bmatrix}\mathbf{R}^+_{[11]} \\ \mathbf{R}^+_{[I1]}\end{bmatrix}^T \mathbf{y}[j] \tag{14}$$

$$= \mathbf{c}_1\psi_1 A_1 b_1[j] + \mathbf{n}_1[j] \tag{15}$$

where $\mathbf{n}_1[j]$ is a complex Gaussian random vector whose real and imaginary components are i.i.d. $\eta(\mathbf{0}_L, \sigma^2\mathbf{R}^+_{[11]})$ and is uncorrelated with $\mathbf{n}_1[j \Leftrightarrow 1]$. If we assume that $\mathbf{c}_1$, ψ_1, and A_1 are unknown, the bit decision for $\widehat{d}_1[j]$ is equivalent to the following binary hypothesis test:

$$H_0 : \begin{bmatrix}\mathrm{Re}(\mathbf{z}_{Dm,1}[j]) \\ \mathrm{Im}(\mathbf{z}_{Dm,1}[j]) \\ \mathrm{Re}(\mathbf{z}_{Dm,1}[j \Leftrightarrow 1]) \\ \mathrm{Im}(\mathbf{z}_{Dm,1}[j \Leftrightarrow 1])\end{bmatrix} = \begin{bmatrix}\mathbf{x} \\ \mathbf{x}\end{bmatrix} + \mathbf{n}, \tag{16}$$

$$H_1 : \begin{bmatrix}\mathrm{Re}(\mathbf{z}_{Dm,1}[j]) \\ \mathrm{Im}(\mathbf{z}_{Dm,1}[j]) \\ \mathrm{Re}(\mathbf{z}_{Dm,1}[j \Leftrightarrow 1]) \\ \mathrm{Im}(\mathbf{z}_{Dm,1}[j \Leftrightarrow 1])\end{bmatrix} = \begin{bmatrix}\mathbf{x} \\ \Leftrightarrow\mathbf{x}\end{bmatrix} + \mathbf{n}, \tag{17}$$

where **x** is an unknown, real $2L$-vector with energy $A_1^2||c_1||^2$ and **n** is a real Gaussian random vector with mean $\mathbf{0}_{[4L\times 4L]}$ and covariance

$$\sigma^2 \begin{bmatrix} \mathbf{R}^+_{[11]} & & & \\ & \mathbf{R}^+_{[11]} & & \\ & & \mathbf{R}^+_{[11]} & \\ & & & \mathbf{R}^+_{[11]} \end{bmatrix}.$$

It can be shown that the maximum likelihood (ML) decision function is

$$\begin{aligned} \widehat{d}_1[j] \;=\; \operatorname{sgn}\Big[&\mathrm{Re}(\mathbf{z}^T_{Dm,1}[j])(\mathbf{R}^+_{[11]})^{-2}\,\mathrm{Re}(\mathbf{z}_{Dm,1}[j \Leftrightarrow 1]) \\ &+ \mathrm{Im}(\mathbf{z}^T_{Dm,1}[j])(\mathbf{R}^+_{[11]})^{-2}\,\mathrm{Im}(\mathbf{z}_{Dm,1}[j \Leftrightarrow 1])\Big]. \end{aligned} \tag{18}$$

While this decision function only requires information about the users' signal subspaces (and not their coordinates), a similar but less complex decision function is based on an equal gain combiner (EGC) [10]:

$$\widehat{d}_1[j] \;=\; \operatorname{sgn}[\mathrm{Re}(\mathbf{z}^T_{Dm,1}[j])\,\mathrm{Re}(\mathbf{z}_{Dm,1}[j \Leftrightarrow 1]) + \mathrm{Im}(\mathbf{z}^T_{Dm,1}[j])\,\mathrm{Im}(\mathbf{z}_{Dm,1}[j \Leftrightarrow 1])], \tag{19}$$

$$\;=\; \operatorname{sgn}[\mathrm{Re}(\mathbf{z}^H_{Dm,1}[j]\mathbf{z}_{Dm,1}[j \Leftrightarrow 1])]. \tag{20}$$

Note that (20) is the maximum likelihood decision function (18) if the elements of **n** are i.i.d., i.e. if $\mathbf{R}^+_{[11]} = \mathbf{I}_{[L]}$. Since $\mathbf{R}^+_{[11]}$ is typically strongly diagonal (except in rare cases of unfavorable crosscorrelations), the EGC is a sensible combining strategy which does not require $\mathbf{c}_k$ nor user subspace information. We will use it for the Dm detector as well as for the subsequent adaptive detector.

4.2. PERFORMANCE OF THE Dm DETECTOR

Reference [13] gives the expected probability of error for the Dm detector averaged over the complex Gaussian subsignal coordinates. We now give the exact probability of error over pairs of successive bit intervals for a given set of subsignal coordinates. The real and imaginary components of user 1's EGC input are

$$\begin{bmatrix} \mathbf{z}^R_{Dm,1}[j] \\ \mathbf{z}^I_{Dm,1}[j] \end{bmatrix} \sim \eta\left(\begin{bmatrix} \mu^R \\ \mu^I \end{bmatrix}, \sigma^2 \begin{bmatrix} \mathbf{\Sigma}^R & \\ & \mathbf{\Sigma}^I \end{bmatrix}\right),$$

where

$$\begin{aligned} \mu^R &= \mathbf{g}_1^R A_1 b_1[j], & \mathbf{\Sigma}^R &= \mathbf{R}^+_{[11]}, \\ \mu^I &= \mathbf{g}_1^I A_1 b_1[j], & \mathbf{\Sigma}^I &= \mathbf{R}^+_{[11]}, \end{aligned}$$

where $\mathbf{g}_1 \equiv \mathbf{c}_1\,\mathrm{e}^{j\theta_1}$ and for notational convenience, $\mathbf{g}_1^R \equiv \mathrm{Re}(\mathbf{g}_1)$ and $\mathbf{g}_1^I \equiv \mathrm{Im}(\mathbf{g}_1)$. Since the covariance matrices have full rank, we can diagonalize $\mathbf{R}^+_{[11]} = \mathbf{V}\Lambda\mathbf{V}^T$ where $\mathbf{V}$ is the $L \times L$ matrix of eigenvectors and $\Lambda \equiv \mathrm{diag}(\lambda_1 \ldots \lambda_L)$ is the matrix of L (distinct) eigenvalues. We assume without loss of generality that the eigenvalues are ordered such that $\lambda_1 < \cdots < \lambda_L$.

The probability of error for user k for a given channel realization $\mathbf{c}_k$ is the weighted sum of exponentials (see Appendix)

$$\begin{aligned} P_{Dm,1}(\sigma) \;=\; & \sum_{l=1}^{L} G_l(\lambda_l)^L \left[\prod_{i=1}^{L} \frac{1}{\lambda_l + \lambda_i} \right. \\ & \exp\left(\frac{\Leftrightarrow A_1^2}{2\sigma^2} \left\{ (\mathbf{g}_1^R)^T \mathbf{V} \,\mathrm{diag}\left(\frac{2}{\lambda_1+\lambda_l} \cdots \frac{2}{\lambda_L+\lambda_l} \right) \mathbf{V}^T \mathbf{g}_1^R \right.\right. \\ & \left.\left.\left. + (\mathbf{g}_1^I)^T \mathbf{V} \,\mathrm{diag}\left(\frac{2}{\lambda_1+\lambda_l} \cdots \frac{2}{\lambda_L+\lambda_l} \right) \mathbf{V}^T \mathbf{g}_1^I \right\} \right) \right], \end{aligned} \tag{21}$$

where

$$G_l \equiv \frac{(\lambda_l)^{L-1}}{\prod_{j\neq l}(\lambda_l \Leftrightarrow \lambda_j)}.$$

According to (5), the asymptotic efficiency of the Dm detector is the largest (least-negative) exponential argument, because, as the background noise approaches zero, the corresponding term in (21) dominates the error expression. Therefore

$$\begin{aligned} \eta_{Dm,1} \;=\; & (\mathbf{g}_1^R)^T \mathbf{V} \,\mathrm{diag}\left(\frac{2}{\lambda_1+\lambda_L}, \ldots, \frac{2}{\lambda_L+\lambda_L} \right) \mathbf{V}\mathbf{g}_1^R \\ & + (\mathbf{g}_1^I)^T \mathbf{V} \,\mathrm{diag}\left(\frac{2}{\lambda_1+\lambda_L}, \ldots, \frac{2}{\lambda_L+\lambda_L} \right) \mathbf{V}\mathbf{g}_1^I. \end{aligned} \tag{22}$$

And because the mD detector achieves optimum asymptotic efficiency, it follows that $\eta_{Dm,1} \leqslant \eta_{mD,1}$.

Up to this point in the paper, we have considered two linear receivers for differentially coherent multidimensional signals which require information not only for the desired user, but also for the interferers (subsignal vector and coordinate information for the mD detector, subsignal vector information for the Dm detector). In practice, this information about the interferers is often not available. We are thus motivated to derive a multiuser detector which requires only information about the desired user and which self-tunes to counteract the effects of multiaccess interference.

5. Blind Adaptive Detector for Multidimensional Signals

Our goal in this section is to derive a detector for differentially coherent multidimensional signals which requires only knowledge of $\mathcal{S}_1$ (i.e., we do not require $\mathbf{c}_1$, data training sequences, nor information about the interferers) but which adaptively achieves performance superior to the Dm detector. We will achieve this goal in the following steps: first, we derive a modified decorrelating multipath-combining (Dm^*) detector which is superior to the Dm detector in terms of asymptotic efficiency. However, the Dm^* detector is not readily implementable due to the amount of information it requires about the interfering users. Next, using a minimum mean-square error (MMSE) criterion, we derive a multiuser detector which is equivalent to the Dm^* detector as the background noise approaches zero ($\sigma \to 0$). Finally, we obtain a blind adaptive implementation of this MMSE detector which sidesteps the need to know any information about the interfering users.

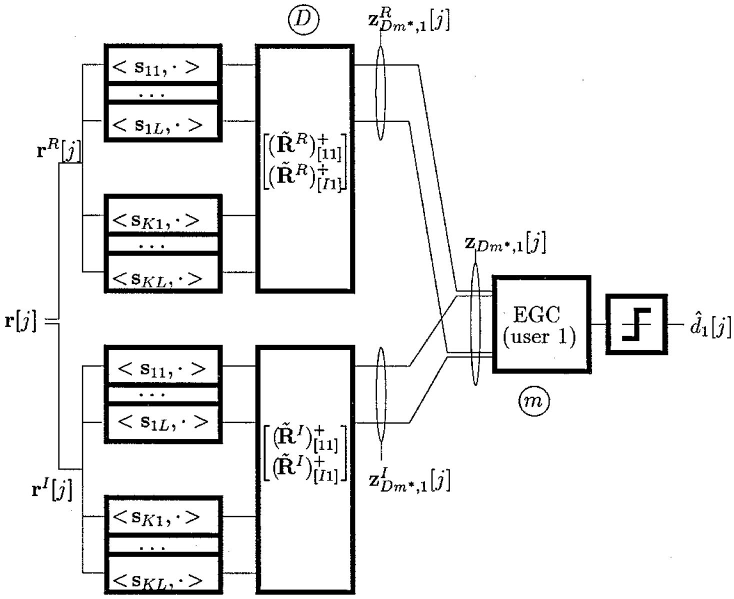


Figure 3. Dm^* detector.

5.1. MODIFIED DECORRELATING MULTIPATH-COMBINING (Dm^*) DETECTOR

As we will see in Section 5.3, in order for the MMSE detector to be implemented in a blind adaptive manner, we require that it decomposes the received signal $\mathbf{r}[j]$ into its real and imaginary components ($\mathbf{r}^R[j]$ and $\mathbf{r}^I[j]$, respectively) and processes them separately. Hence, while we can derive a Dm^* detector which processes $\mathbf{r}[j]$ in the complex domain, we will focus on a *partitioned* Dm^* detector, shown in Figure 3, which processes $\mathbf{r}^R[j]$ and $\mathbf{r}^I[j]$ separately.

As before, we assume that $\mathbf{s}_{11} \ldots \mathbf{s}_{1L}$ form an orthonormal basis for $\mathcal{S}_1$, the signal subspace for user 1. Let the combined signal for the interferers,

$$\tilde{\mathbf{s}}_k = \mathrm{e}^{i\theta_k} \sum_{l=1}^{L} c_{k,l} \mathbf{s}_{k,l} = \mathbf{S}_k \mathbf{g}_k \tag{23}$$

be redefined to include the phase term $\mathrm{e}^{i\theta_k}$, and let $\tilde{\mathbf{s}}_k^R \equiv \mathrm{Re}[\tilde{\mathbf{s}}_k]$ for the interfering users $k = 2, \ldots, K$. Consider an $L + K \Leftrightarrow 1$ pseudo-user system where the spreading codes of the pseudo-users are $\mathbf{s}_{11} \ldots \mathbf{s}_{1L}, \tilde{\mathbf{s}}_2^R \ldots \tilde{\mathbf{s}}_K^R$. Given the real part of the received signal, $\mathbf{r}^R[j]$, the decorrelating detector [4] for the $L + K \Leftrightarrow 1$ pseudo-users consists of a bank of filters matched to their spreading codes, given by

$$\mathbf{S} \begin{bmatrix} \mathbf{I}_{[L]} & \\ & \mathbf{G}_I^R \end{bmatrix} = [\mathbf{s}_{1,1} \ldots \mathbf{s}_{1,L} \quad \tilde{\mathbf{s}}_2^R \ldots \tilde{\mathbf{s}}_K^R],$$

where $\mathbf{G}_I^R \equiv \mathrm{diag}(\mathbf{g}_2^R \ldots \mathbf{g}_K^R)$, followed by the decorrelator $(\tilde{\mathbf{R}}^R)^{-1}$, where

$$\tilde{\mathbf{R}}^R \equiv \begin{bmatrix} \mathbf{I}_{[L]} & \\ & \mathbf{G}_I^R \end{bmatrix}^T \mathbf{S}^T \mathbf{S} \begin{bmatrix} \mathbf{I}_{[L]} & \\ & \mathbf{G}_I^R \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{[L]} & \\ & \mathbf{G}_I^R \end{bmatrix}^T \mathbf{R} \begin{bmatrix} \mathbf{I}_{[L]} & \\ & \mathbf{G}_I^R \end{bmatrix}$$

is the cross correlation of the pseudo-user codes. Hence the decorrelator output is

$$\left(\mathbf{S}\begin{bmatrix}\mathbf{I}_{[L]} & \\ & \mathbf{G}_I^R\end{bmatrix}(\widetilde{\mathbf{R}}^R)^{-1}\right)^T \mathbf{r}^R[j],$$

and the outputs which correspond to the first L pseudo-users (and the first "real" user's subsignals) are given by the L-vector

$$\mathbf{z}^R_{Dm^*,1}[j] = \left(\mathbf{S}\begin{bmatrix}\mathbf{I}_{[L]} & \\ & \mathbf{G}_I^R\end{bmatrix}\begin{bmatrix}(\widetilde{\mathbf{R}}^R)^+_{[11]} \\ (\widetilde{\mathbf{R}}^R)^+_{[I1]}\end{bmatrix}\right)^T \mathbf{r}^R[j]. \tag{24}$$

This statistic is decorrelated not only from the other $K \Leftrightarrow 1$ users, but each of its components is decorrelated from the other $L \Leftrightarrow 1$ components. The expected value of (24)

$$E[\mathbf{z}^R_{Dm^*,1}[j]] = A_1 b_1[j]\mathbf{c}_1^R$$

gives the real coordinates of user 1's subsignal vectors. Defining the corresponding imaginary variables

$$\widetilde{\mathbf{R}}^I \equiv \begin{bmatrix}\mathbf{I}_{[L]} & \\ & \mathbf{G}_I^I\end{bmatrix}^T \mathbf{R}\begin{bmatrix}\mathbf{I}_{[L]} & \\ & \mathbf{G}_I^I\end{bmatrix}$$

and

$$\mathbf{z}^I_{Dm^*,1}[j] = \left(\mathbf{S}\begin{bmatrix}\mathbf{I}_{[L]} & \\ & \mathbf{G}_I^I\end{bmatrix}\begin{bmatrix}(\widetilde{\mathbf{R}}^I)^+_{[11]} \\ (\widetilde{\mathbf{R}}^I)^+_{[I1]}\end{bmatrix}\right)^T \mathbf{r}^I[j],$$

we let $\mathbf{z}_{Dm^*,1}[j] \equiv \mathbf{z}^R_{Dm^*,1}[j] + i\mathbf{z}^I_{Dm^*,1}[j]$ be our complex L-vector statistic. We assume that the Dm^* detector has no knowledge of user 1's subsignal coordinates. Hence, as with the Dm detector (20), we use the decision function

$$\widehat{d}_1[j] = \text{sgn}[\text{Re}(\mathbf{z}^H_{Dm^*,1}[j]\mathbf{z}_{Dm^*,1}[j \Leftrightarrow 1])]$$

to obtain the bit estimates.

In summary, the two differences between the Dm and Dm^* detectors are as follows:

(1) The Dm detector processes the received signal $\mathbf{r}[j]$ in the complex domain whereas the Dm^* detector processes the real and imaginary components of $\mathbf{r}[j]$ separately
(2) The Dm detector users a bank of KL filters matched to each individual subsignal vector whereas the Dm^* detector, for both the real and imaginary components of $\mathbf{r}[j]$, users a bank of $L + K \Leftrightarrow 1$ filters matched to the desired user's subsignal vectors and the interfering users' real/imaginary combined signals.

Notice that in addition to the signal subspace information $\mathcal{S}_1 \ldots \mathcal{S}_K$ (which was required by the Dm detector), the Dm^* detector requires additional information about the interferers' subsignal coordinates: $\mathbf{c}_2 \ldots \mathbf{c}_K$. Therefore, our intuition tells us that the Dm^* detector's performance should be superior to the Dm's. Just like the mD and Dm detectors, the Dm^* detector's probability of error depends only on the complex Gaussian statistics of the EGC input:

$$\begin{bmatrix}\mathbf{z}^R_{Dm^*,1}[j] \\ \mathbf{z}^I_{Dm^*,1}[j]\end{bmatrix} \sim \eta\left(\begin{bmatrix}\mu^R \\ \mu^I\end{bmatrix}, \sigma^2\begin{bmatrix}\mathbf{\Sigma}^R & \\ & \mathbf{\Sigma}^I\end{bmatrix}\right),$$

where

$$\begin{aligned} \mu^R &= \mathbf{g}_1^R A_1 b_1[j], & \mathbf{\Sigma}^R &= (\widetilde{\mathbf{R}}^R)^+_{[11]}, \\ \mu^I &= \mathbf{g}_1^I A_1 b_1[j], & \mathbf{\Sigma}^I &= (\widetilde{\mathbf{R}}^I)^+_{[11]}. \end{aligned}.$$

By diagonalizing the covariance matrices $(\widetilde{\mathbf{R}}^R)^+_{[11]} = \mathbf{V}^R \Gamma^R (\mathbf{V}^R)^T$ and $(\widetilde{\mathbf{R}}^I)^+_{[11]} = \mathbf{V}^I \Gamma^I (\mathbf{V}^I)^T$ (Γ is the diagonal matrix of distinct eigenvalues $\mathrm{diag}(\gamma_1 \ldots \gamma_L)$) and $\gamma_l^M \equiv \max(\gamma_l^R, \gamma_l^I)$, the probability of error for user 1 as derived in the Appendix is upper bounded by the weighted sum of exponentials

$$\begin{aligned} P_{Dm^*,1}(\sigma) \leqslant \sum_{l=1}^{L} G_l (\gamma_l^M)^L & \left[\prod_{i=1}^{L} \sqrt{\frac{1}{(\gamma_l^M + \gamma_i^R)(\gamma_l^M + \gamma_i^I)}} \right. \\ & \exp \left(\frac{\Leftrightarrow A_1^2}{2\sigma^2} \left\{ (\mathbf{g}_1^R)^T \mathbf{V}^R \,\mathrm{diag} \left(\frac{2}{\gamma_1^R + \gamma_l^M} \cdots \frac{2}{\gamma_L^R + \gamma_l^M} \right) (\mathbf{V}^R)^T \mathbf{g}_1^R \right. \right. \\ & \left. \left. \left. + (\mathbf{g}_1^I)^T \mathbf{V}^I \,\mathrm{diag} \left(\frac{2}{\gamma_1^I + \gamma_l^M} \cdots \frac{2}{\gamma_L^I + \gamma_l^M} \right) (\mathbf{V}^I)^T \mathbf{g}_1^I \right\} \right) \right], \end{aligned} \qquad (25)$$

where

$$G_l \equiv \frac{(\gamma_l^M)^{L-1}}{\prod_{j \neq l} (\gamma_l^M \Leftrightarrow \gamma_j^M)}.$$

The corresponding lower bound for the asymptotic efficiency is

$$\begin{aligned} \eta_{Dm^*,1} \geqslant\ & (\mathbf{g}_1^R)^T \mathbf{V}^R \,\mathrm{diag} \left(\frac{2}{\gamma_1^R + \gamma_L^M} \cdots \frac{2}{\gamma_L^R + \gamma_L^M} \right) (\mathbf{V}^R)^T \mathbf{g}_1^R \\ & + (\mathbf{g}_1^I)^T \mathbf{V}^I \,\mathrm{diag} \left(\frac{2}{\gamma_1^I + \gamma_L^M} \cdots \frac{2}{\gamma_L^I + \gamma_L^M} \right) (\mathbf{V}^I)^T \mathbf{g}_1^I \end{aligned} \qquad (26)$$

which, as we see in the following proposition, is lower bounded by the asymptotic efficiency of the Dm detector.

PROPOSITION 1. $\eta_{Dm^*,1} \geqslant \eta_{Dm,1}$

Proof. Note that $(\widetilde{\mathbf{R}}^R)^+_{[11]} \preceq \mathbf{R}^+_{[11]}$ (i.e., $\mathbf{R}^+_{[11]} \Leftrightarrow (\widetilde{\mathbf{R}}^R)^+_{[11]}$ is positive semidefinite) is equivalent to $\left((\widetilde{\mathbf{R}}^R)^+_{[11]}\right)^{-1} \succeq \left(\mathbf{R}^+_{[11]}\right)^{-1}$ and implies $\gamma_L^R \leqslant \lambda_L$. Similarly, $(\widetilde{\mathbf{R}}^I)^+_{[11]} \preceq \mathbf{R}^+_{[11]}$ is equivalent to $\left((\widetilde{\mathbf{R}}^I)^+_{[11]}\right)^{-1} \succeq \left(\mathbf{R}^+_{[11]}\right)^{-1}$ and implies $\gamma_L^I \leqslant \lambda_L$. Hence from (22) and (26), it is sufficient to show that $(\widetilde{\mathbf{R}}^R)^+_{[11]} \preceq \mathbf{R}^+_{[11]}$ and $(\widetilde{\mathbf{R}}^I)^+_{[11]} \preceq \mathbf{R}^+_{[11]}$. From [14]

$$(\widetilde{\mathbf{R}}^R)^+_{[11]} = \left[\widetilde{\mathbf{R}}^R_{[11]} \Leftrightarrow \widetilde{\mathbf{R}}^R_{[1I]} (\widetilde{\mathbf{R}}^R_{[II]})^{-1} \widetilde{\mathbf{R}}^R_{[I1]} \right]^{-1} \qquad (27)$$

$$= \left[\mathbf{R}_{[11]} \Leftrightarrow \mathbf{R}_{[1I]} \mathbf{G}_I^R \left((\mathbf{G}_I^R)^T \mathbf{R}_{[II]} \mathbf{G}_I^R \right)^{-1} (\mathbf{G}_I^R)^T \mathbf{R}_{[I1]} \right]^{-1}, \qquad (28)$$

$$\mathbf{R}^+_{[11]} = \left[\mathbf{R}_{[11]} \Leftrightarrow \mathbf{R}_{[1I]} (\mathbf{R}_{[II]})^{-1} \mathbf{R}_{[I1]} \right]^{-1}. \qquad (29)$$

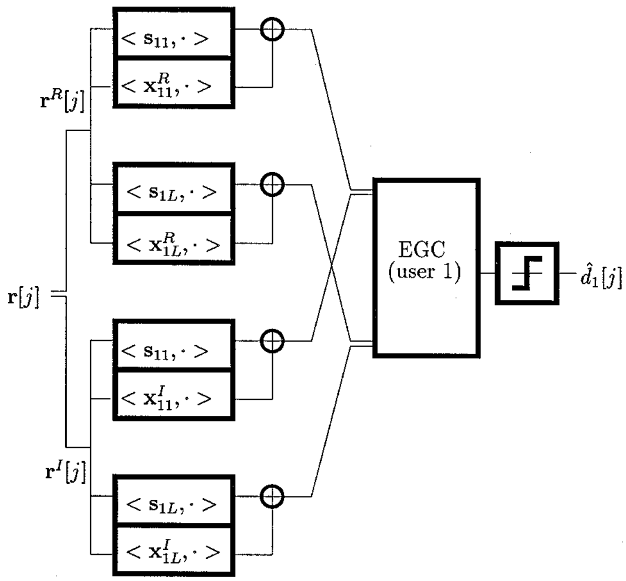


Figure 4. Orthogonally anchored MMSE detector.

Hence

$$(\widetilde{\mathbf{R}}^R)^+_{[11]} \leqslant \mathbf{R}^+_{[11]} \;\Leftrightarrow\; \mathbf{G}_I^R \left((\mathbf{G}_I^R)^T \mathbf{R}_{[II]} \mathbf{G}_I^R \right)^{-1} (\mathbf{G}_I^R)^T \leqslant \mathbf{R}^{-1}_{[II]} \tag{30}$$

$$\Leftrightarrow\; \rho(\mathbf{U}) \leqslant 1, \tag{31}$$

where $\mathbf{U} \equiv \mathbf{G}_I^R \left((\mathbf{G}_I^R)^T \mathbf{R}_{[II]} \mathbf{G}_I^R \right)^{-1} (\mathbf{G}_I^R)^T \mathbf{R}_{[II]}$ and $\rho(\mathbf{U})$ denotes its spectral radius. Note however, that $\mathbf{UU} = \mathbf{U}$ implies that the eigenvalues of $\mathbf{U}$ are either 0 or 1. Hence $(\widetilde{\mathbf{R}}^R)^+_{[11]} \preceq \mathbf{R}^+_{[11]}$. The same arguments apply for showing $(\widetilde{\mathbf{R}}^I)^+_{[11]} \preceq \mathbf{R}^+_{[11]}$. $\square$

Some intuition on this proposition can be gained in considering the size of the decorrelation matrices of the two detectors ($KL \times KL$ for Dm and $K + L \Leftrightarrow 1 \times K + L \Leftrightarrow 1$ for Dm^*). Since the decorrelation matrix is simply a projection matrix which projects each matched filter onto the subspace orthogonal to the other matched filters, more decorrelator inputs lead to more restrictive projections. Hence the detector with the larger decorrelator (Dm) yields a signal with smaller signal-to-noise ratio at the decorrelator output which results in inferior performance. In addition, recall that the Dm^* detector requires more information than the original Dm detector.

5.2. Orthogonally Anchored MMSE Detector

In this subsection, we use the minimum mean square error (MMSE) criterion to derive a multiuser detector which is equivalent to the Dm^* detector as $\sigma \to 0$. As seen in Figure 4, this *orthogonally anchored minimum mean square error (OAMMSE)* detector replaces the $L + K \Leftrightarrow 1$ matched filters and decorrelator with a set of L linear transformations. The lth real (or imaginary) linear transformation consists of a filter matched to a subsignal vector $\mathbf{s}_{1l}$ and a filter $\mathbf{x}_{1l}^R$ (or $\mathbf{x}_{1l}^I$) which satisfies the following MMSE criterion:

$$\mathbf{x}_{1l}^R = \arg\min_{\mathbf{x}\in\mathcal{S}_1^\perp} E[\{(\mathbf{s}_{1l}+\mathbf{x})^T\mathbf{r}^R[j] \Leftrightarrow A_1 b_1[j] g_{1l}^R\}^2], \tag{32}$$

where $\mathcal{S}_1^\perp$ is the subspace orthogonal to $\mathcal{S}_1$. Therefore $\mathbf{x}_{1l}^R$, which we call the orthogonal component of the lth linear transformation, is a real N-vector which is anchored in the null space of $\mathcal{S}_1$ and minimizes the mean square error between the lth real branch output and the lth real coordinate of the desired user 1.

In order to obtain an explicit expression for $\mathbf{x}_{1l}^R$, we first notice that

$$\mathbf{S}^T\mathbf{S}\begin{bmatrix}\mathbf{R}_{[1I]}^+\\ \mathbf{R}_{[II]}^+\end{bmatrix} = \begin{bmatrix}\mathbf{0}_{[L\times L(K-1)]}\\ \mathbf{I}_{[L(K-1)]}\end{bmatrix}.$$

Hence the columns of

$$\mathbf{S}\begin{bmatrix}\mathbf{R}_{[1I]}^+\\ \mathbf{R}_{[II]}^+\end{bmatrix}$$

each lie in the subspace $\mathcal{S}_1^\perp$. Therefore $\mathbf{x}_{1l}^R$ can be given in terms of a linear combination of the columns of this matrix. Letting $\mathbf{m}_{1l}^R$ be the $KL \Leftrightarrow L$ vector which designates this linear combination, finding $\mathbf{x}_{1l}^R$ from (32) is equivalent to finding $\mathbf{m}_{1l}^R$ (a real $K \Leftrightarrow 1$ vector) which minimizes

$$E\left[\left\{\left(\mathbf{s}_{1l}+\mathbf{S}\begin{bmatrix}\mathbf{R}_{[1I]}^+\\ \mathbf{R}_{[II]}^+\end{bmatrix}\mathbf{m}_{1l}^R\right))^T\mathbf{r}[j] \Leftrightarrow A_1 b_1[j] g_{1l}^R\right\}^2\right]. \tag{33}$$

Moreover, letting $\mathbf{M}_1^R \equiv [\mathbf{m}_{11}^R \ldots \mathbf{m}_{1L}^R]$,

$$\mathbf{P} \equiv \begin{bmatrix}\mathbf{I}_{[L]} & \mathbf{R}_{[1I]}^+\\ \mathbf{0}_{[L(K-1)\times L]} & \mathbf{R}_{[II]}^+\end{bmatrix},$$

$$\mathbf{Q} \equiv E[(\mathbf{SP})^T\mathbf{r}^R[j](\mathbf{r}^R[j])^T\mathbf{SP}] = \mathbf{P}^T\mathbf{R}\mathbf{G}^R\mathbf{A}^2(\mathbf{G}^R)^T\mathbf{R}\mathbf{P} + \sigma^2\mathbf{P}^T\mathbf{R}\mathbf{P},$$

and $\mathbf{G}^R \equiv \mathrm{diag}(\mathbf{g}_1^R \ldots \mathbf{g}_K^R)$, we have, from (33) that

$$\mathbf{M}_1^R = \arg\min_{\mathbf{M}\in\mathbf{R}^{(K-1)\times L}} E\left[\left\{\left(\mathbf{SP}\begin{bmatrix}\mathbf{I}_{[L]}\\ \mathbf{M}\end{bmatrix}\right)^T\mathbf{r}^R[j] \Leftrightarrow A_1 b_1[j]\mathbf{g}_1^R\right\}^2\right] \tag{34}$$

$$= \Leftrightarrow \mathbf{Q}_{[II]}^{-1}\mathbf{Q}_{[I1]} \tag{35}$$

$$= \Leftrightarrow (\mathbf{R}_{[II]}^+)^{-1}\mathbf{G}_I^R\mathbf{A}\left(\mathbf{A}(\mathbf{G}_I^R)^T(\mathbf{R}_{[II]}^+)^{-1}\mathbf{G}_I^R\mathbf{A}+\sigma^2\mathbf{I}_{[L]}\right)^{-1}\mathbf{A}(\mathbf{G}_I^R)^T\mathbf{R}_{[I1]}\mathbf{R}_{[11]}^{-1}, \tag{36}$$

where $\mathbf{G}_I^R \equiv \mathrm{diag}(\mathbf{g}_2^R \ldots \mathbf{g}_K^R)$. Hence the orthogonal component $\mathbf{x}_{1l}^R$ is the lth column of

$$\mathbf{S}\begin{bmatrix}\mathbf{R}_{[1I]}^+\\ \mathbf{R}_{[II]}^+\end{bmatrix}\mathbf{M}_1^R.$$

The imaginary orthogonal linear transformations can be obtained similarly.

We now show the equivalence between the $OAMMSE$ and Dm^* detectors as the noise level approaches 0. We will show this result for the real part of the detector which acts on $\mathbf{r}^R[j]$; a similar derivation holds for the imaginary part. From (24) and (34), we can write the real EGC inputs for the $OAMMSE$ and Dm^* detectors respectively as

$$\left(\mathbf{SP}\begin{bmatrix}\mathbf{I}_{[L]}\\ \mathbf{M}_1^R\end{bmatrix}\right)\mathbf{r}^R[j] \quad \text{and} \quad \left(\mathbf{S}\begin{bmatrix}\mathbf{I}_{[L]} & \\ & \mathbf{G}_I^R\end{bmatrix}\begin{bmatrix}(\widetilde{\mathbf{R}}^R)^+_{[11]}\\ (\widetilde{\mathbf{R}}^R)^+_{[I1]}\end{bmatrix}\right)\mathbf{r}^R[j].$$

Hence we must prove the following proposition.

PROPOSITION 2.

$$\lim_{\sigma\to 0}\mathbf{P}\begin{bmatrix}\mathbf{I}_{[L]}\\ \mathbf{M}_1^R\end{bmatrix} = \begin{bmatrix}\mathbf{I}_{[L]} & \\ & \mathbf{G}_I^R\end{bmatrix}\begin{bmatrix}(\widetilde{\mathbf{R}}^R)^+_{[11]}\\ (\widetilde{\mathbf{R}}^R)^+_{[I1]}\end{bmatrix}.$$

Proof. From (36),

$$\lim_{\sigma\to 0}\mathbf{M}_1^R = \Leftrightarrow(\mathbf{R}^+_{[II]})^{-1}\mathbf{G}_I^R\mathbf{A}\left(\mathbf{A}(\mathbf{G}_I^R)^T(\mathbf{R}^+_{[II]})^{-1}\mathbf{G}_I^R\mathbf{A}\right)^{-1}\mathbf{A}(\mathbf{G}_I^R)^T\mathbf{R}_{[I1]}\mathbf{R}_{[11]}^{-1} \tag{37}$$

$$= \Leftrightarrow(\mathbf{R}^+_{[II]})^{-1}\mathbf{G}_I^R\left((\mathbf{G}_I^R)^T(\mathbf{R}^+_{[II]})^{-1}\mathbf{G}_I^R\right)^{-1}(\mathbf{G}_I^R)^T\mathbf{R}_{[I1]}\mathbf{R}_{[11]}^{-1}. \tag{38}$$

Hence

$$\lim_{\sigma\to 0}\mathbf{P}\begin{bmatrix}\mathbf{I}_{[L]}\\ \mathbf{M}_1^R\end{bmatrix} = \mathbf{P}\begin{bmatrix}\mathbf{I}_{[L]}\\ \Leftrightarrow(\mathbf{R}^+_{[II]})^{-1}\mathbf{G}_I^R\left((\mathbf{G}_I^R)^T(\mathbf{R}^+_{[II]})^{-1}\mathbf{G}_I^R\right)^{-1}(\mathbf{G}_I^R)^T\mathbf{R}_{[I1]}\mathbf{R}_{[11]}^{-1}\end{bmatrix} \tag{39}$$

$$= \begin{bmatrix}\mathbf{I}_{[L]} + \mathbf{R}_{[1I]}\mathbf{G}_I^R\left((\mathbf{G}_I^R)^T(\mathbf{R}^+_{[II]})^{-1}\mathbf{G}_I^R\right)^{-1}(\mathbf{G}_I^R)^T\mathbf{R}_{[I1]}\mathbf{R}_{[11]}^{-1}\\ \Leftrightarrow\mathbf{G}_I^R\left((\mathbf{G}_I^R)^T(\mathbf{R}^+_{[II]})^{-1}\mathbf{G}_I^R\right)^{-1}(\mathbf{G}_I^R)^T\mathbf{R}_{[I1]}\mathbf{R}_{[11]}^{-1}\end{bmatrix} \tag{40}$$

$$= \begin{bmatrix}\left(\mathbf{R}_{[11]}\Leftrightarrow\mathbf{R}_{[1I]}\mathbf{G}_I^R\left((\mathbf{G}_I^R)^T(\mathbf{R}^+_{[II]})^{-1}\mathbf{G}_I^R\right)^{-1}(\mathbf{G}_I^R)^T\mathbf{R}_{[I1]}\right)^{-1}\\ \Leftrightarrow\mathbf{G}_I^R(\widetilde{\mathbf{R}}^R)^+_{[II]}\widetilde{\mathbf{R}}^R_{[I1]}\mathbf{R}_{[11]}^{-1}\end{bmatrix} \tag{41}$$

$$= \begin{bmatrix}(\widetilde{\mathbf{R}}^R)^+_{[11]}\\ \mathbf{G}_I^R(\widetilde{\mathbf{R}}^R)^+_{[I1]}\end{bmatrix} \tag{42}$$

$$= \begin{bmatrix}\mathbf{I}_{[L]} & \\ & \mathbf{G}_I^R\end{bmatrix}\begin{bmatrix}(\widetilde{\mathbf{R}}^R)^+_{[11]}\\ (\widetilde{\mathbf{R}}^R)^+_{[I1]}\end{bmatrix}. \qquad \square \tag{43}$$

As a consequence of this proposition, we would expect the $OAMMSE$ detector to exhibit similar performance characteristics as the Dm^* detector in the high SNR region. Specifically, it follows from Propositions 1 and 2 that

$$\eta_{OAMMSE,k} = \eta_{Dm^*,k} \geqslant \eta_{Dm,k}.$$

For lower SNRs, similar results comparing the bit error rates of the three detectors have not been derived. A numerical analysis (Section 6) indicates that the Dm^* bit error rate is better than that of the Dm's, as expected. However, the performance relationship between the $OAMMSE$ and Dm^* detectors has not been established. (Even in the coherent single dimensional signal case, it has not yet been shown that the MMSE detector has better bit error rate than the decorrelating detector. See [15] for a discussion and partial proof of this conjecture.)

5.3. Blind Adaptive Implementation

While the $OAMMSE$ detector itself does not lend any apparent advantages over the Dm^* detector (both require knowledge of $\mathcal{S}_1 \ldots \mathcal{S}_K$ and $\mathbf{g}_2 \ldots \mathbf{g}_K$, and the systems become equivalent as $\sigma \to 0$), we will show in this subsection how the $OAMMSE$ detector can be obtained adaptively with only knowledge of $\mathcal{S}_1$ and without the use of training sequences.

Let

$$MSE_{1l}(\mathbf{x}) \equiv E[\{(\mathbf{s}_{1l} + \mathbf{x})^T \mathbf{r}^R[j] \Leftrightarrow A_1 b_1 g_{1l}^R[j]\}^2]$$

be the mean square error function defined over all vectors in $\mathcal{S}_1^\perp$. The adaptive component of the MMSE linear transformation (32) minimizes this expression. Let

$$MOE_{1l}(\mathbf{x}) \equiv E[\{(\mathbf{s}_{1l} + \mathbf{x})^T \mathbf{r}^R[j]\}^2]$$

denote the mean output energy for the lth real branch.

PROPOSITION 3. *For* $l = 1, \ldots, L$, $\mathbf{x} \in \mathcal{S}_1^\perp \Rightarrow MSE_{1l}(\mathbf{x}) = MOE_{1l}(\mathbf{x}) \Leftrightarrow (A_1 g_{1l}^R)^2$

Proof.

$$\begin{aligned} MSE_{1l}(\mathbf{x}) &\equiv E[\{(\mathbf{s}_{1l} + \mathbf{x})^T \mathbf{r}_{[}j] \Leftrightarrow A_1 b_1 g_{1l}^R[j]\}^2] && (44)\\ &= MOE_{1l}(\mathbf{x}) \Leftrightarrow 2E[\{(\mathbf{s}_{1l} + \mathbf{x})^T \mathbf{r}_{[}j]\} A_1 b_1[j] g_{1l}^R] + (A_1 g_{1l}^R)^2 && (45)\\ &= MOE_{1l}(\mathbf{x}) \Leftrightarrow 2[(A_1 g_{1l}^R)^2] + (A_1 g_{1l}^R)^2 && (46)\\ &= MOE_{1l}(\mathbf{x}) \Leftrightarrow (A_1 g_{1l}^R)^2. && (47)\end{aligned}$$

The consequence of this proposition is that the argument lying in $\mathcal{S}_1^\perp$ which minimizes $MSE_{1l}(\mathbf{x})$ also minimizes $MOE_{1l}(\mathbf{x})$. Hence we can use gradient descent methods based on the gradient of the MOE to converge on the desired adaptive MMSE linear transformation. The adaptive LMS algorithm for the lth real branch is given by [12]

$$\mathbf{x}_{1l}^R[j+1] = \mathbf{x}_{1l}^R[j] \Leftrightarrow \mu z_{1l}^R[j] \mathbf{p}^R[j], \quad l = 1, \ldots, L, \tag{48}$$

where

$$\begin{aligned} z_{1l}^R[j] &,= [(\mathbf{s}_{1l} + \mathbf{x}_{1l}^R[j])^T \mathbf{r}^R[j] \\ \mathbf{p}^R[j] &\equiv (\mathbf{I}_{[L]} \Leftrightarrow \mathbf{S}_1 \mathbf{S}_1^T) \mathbf{r}^R[j] \\ &= \left[\mathbf{r}^R[j] \Leftrightarrow \sum_{l=1}^{L} \mathbf{s}_{1l} (\mathbf{s}_{1l}^T \mathbf{r}^R[j])\right]\end{aligned}$$

is the projection of $\mathbf{r}^R[j]$ into $\mathcal{S}_1^\perp$, and μ is an appropriate step size. We obtain the recursive update algorithm for $\mathbf{x}_{1l}^I$ by replacing real components with their respective imaginary counterparts.

The single dimensional blind adaptive detector [12] is a special case $(L = 1, \mathbf{r}^I[j] = 0)$ of our multidimensional detector. Conversely, we can view each of the $2L$ linear transformations as a single dimensional blind adaptive detector whose adaptive component $\mathbf{x}_{1l}^R$ (or $\mathbf{x}_{1l}^I$) is constrained to be orthogonal to $\mathbf{s}_{1l}$ and to each of the other basis vectors $\mathbf{s}_{1j}$, $j \neq l$. Since the update algorithms (48) for the branches operate independently of each other, we can further view our multidimensional blind detector as a bank of $2L$ parallel single dimensional blind detectors whose adaptive components are anchored in $\mathcal{S}_1^\perp$: $\mathbf{x}_{1l}^R \in \mathcal{S}_1^\perp$ (and $\mathbf{x}_{1l}^I \in \mathcal{S}_1^\perp$) for all l. As a result of this observation, two attractive characteristics of the single dimensional blind detector carry over to the multidimensional detector.

- **Global convergence**: Because $MSE_{1l}(\mathbf{x})$ is a strictly convex function over the set of detectors lying in $\mathcal{S}_1^\perp$ [12], the linear transformation which minimizes it is a unique global minimum, and because of Proposition 3, we are guaranteed to converge to this detector from any initialization using (48) and an appropriately vanishing step size μ.
- **Extension to the asynchronous channel**: While the decision function of the multidimensional detector depends on statistics from two symbol intervals, the linear part of the system (everything before the EGC) functions on a symbol-by-symbol basis. As in [12], we can easily modify our multidimensional detector for asynchronous signals by extending the observation time (or *processing window*) for the received signal $\mathbf{r}[j]$ to several symbol intervals and by redefining each of the linear transformations over the new processing window.

A shortcoming of the single dimensional $(L = 1)$ blind multiuser detector is its tendency to cancel out the desired user's signal if the signal-to-noise ratio is high and if there is a mismatch between the actual and estimated signature waveform (or, in the multipath case, between the actual and estimated combined multipath waveform) $\mathbf{s}_{11}$ for the desired user [12]. The solution, at the expense of decreased efficiency, is to place energy constraints on the orthogonal component $\mathbf{x}_{11}$. For the multidimensional blind detector, this mismatched nominal problem does not occur as long as the actual waveform lies within the assumed subspace $\mathcal{S}_1$ and this subspace does not intersect those corresponding to other users; therefore, the multidimensional detector is much more robust than the single dimensional detector in channels (like the multipath channel) in which there are uncertainties in the desired user's signal.

Of course, the situation may arise in which the actual signal subspace $\mathcal{S}_1$ is not the same as the estimated signal subspace; however, the relative occurrence of this situation is unlikely compared to the occurrence of the mismatched nominal for single dimensional blind detector. As an example, suppose that the desired user's subspace is spanned by $\mathbf{s}_{11}$, $\mathbf{s}_{12}$, and $\mathbf{s}_{13}$, but the actual signal subspace contains an additional dimension spanned by $\mathbf{s}_{14}$. Without loss of generality, we can let $\mathbf{s}_{14}$ be orthogonal to $\mathbf{s}_{11}$, $\mathbf{s}_{12}$, and $\mathbf{s}_{13}$. If the desired signal contains a nonzero component of $\mathbf{s}_{14}$, the decomposition of the multidimensional blind detector yields a mismatched nominal for each of the $2L$ single dimensional blind detectors. The obvious solution is to constrain the energy for each of the $2L$ linear transformations.

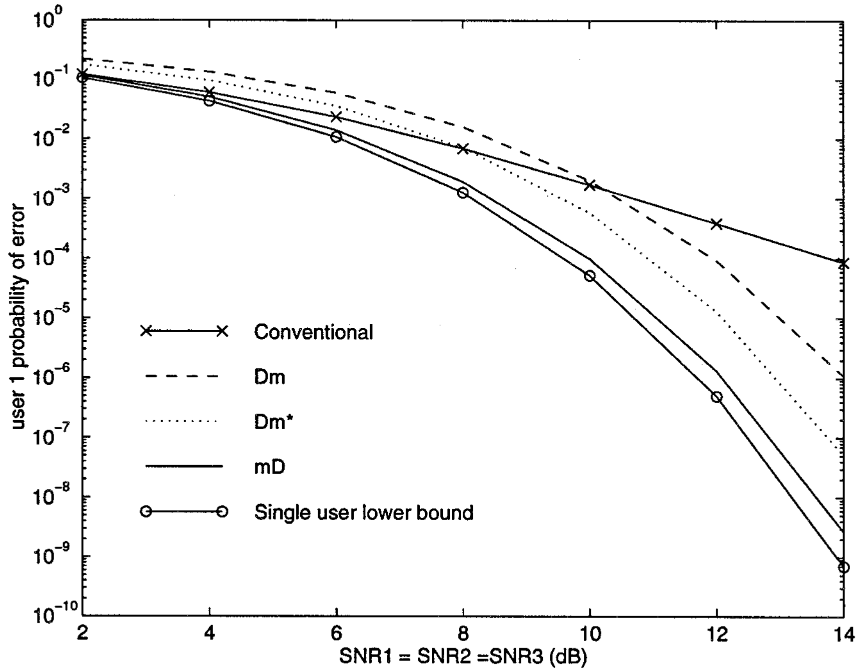


Figure 5. Detector performances with perfect parameter estimates.

6. Performance Analysis

While most of the comparative analysis has been made for the detectors' efficiencies in the high SNR range, we will now see that the relationships apply for the error probability measure at lower SNRs. We will examine the ideal error rates that each detector can attain when all of the relevant parameters (subsignal vectors and coordinates) are known exactly. We consider the probability of error for user 1 in a $K = 3$ user synchronous system where each user's signal has dimensionality $L = 3$. We assume that there is perfect power control, *i.e.,* $A_1 = A_2 = A_3$. For a given signal-to-noise (SNR) ratio (where SNR is given by A_k^2/σ^2 in decibels), we will compute the average performance over 1000 normalized random subsignal vectors $(||\mathbf{s}_{k,l}|| = 1)$ and normalized random coordinate vectors $||\mathbf{c}_k|| = 1$, $k = 1, 2, 3$. In other words, on each iteration, we compute the error rates for all detectors under consideration using,

$$\mathbf{c}_k = \frac{\mathbf{c}'_k}{||\mathbf{c}'_k||},$$

where $\mathbf{c}'_k$ $(k = 1, 2, 3)$ is a zero-mean complex Gaussian random vector of size $L = 3$, and

$$\mathbf{s}_{k,l} = \frac{\mathbf{s}'_{k,l}}{||\mathbf{s}'_{k,l}||},$$

where $\mathbf{s}_{k,l}$ $(k = 1, 2, 3;\ l = 1, 2, 3)$ is a random vector whose dimension N is equal to the number of chips per symbol.

The error probability curves are given in Figure 5 are averaged over the choice of $N = 31$ signature waveforms. The following detectors are analyzed:

- **Conventional detector** [5]: Requires $\mathcal{S}_1$, $\mathbf{c}_1$. This is the conventional rake detector operating in a multiuser environment where the other 2 interfering users are active. Note that even under perfect power control, the exponential drop in the error rate is much slower than that of the multiuser detectors. Note also that it outperforms the Dm and Dm^* detectors at low SNR because it assumes knowledge of $\mathbf{c}_1$.
- mD **detector**(13): Requires $\mathcal{S}_1 \ldots \mathcal{S}_K$, $\mathbf{c}_1 \ldots \mathbf{c}_K$. Note that there is only about a $\frac{1}{3}$ dB degradation from the single user lower bound. This detector effectively eliminates the multiaccess interference but requires knowledge of all of the users' subspaces and coordinates.
- Dm **detector**(21): Requires $\mathcal{S}_1 \ldots \mathcal{S}_K$ but no longer requires knowledge of the users' coordinates. It sustains a loss of about 2 dB loss in performance with respect to the mD detector.
- Dm^* **detector** (25): Requires $\mathcal{S}_1 \ldots \mathcal{S}_K$, $\mathbf{g}_2 \ldots \mathbf{g}_K$. The additional information about the interferers' coordinates provides a 1 dB performance gain over the Dm detector. The blind adaptive $OAMMSE$ detector, which is equivalent to the Dm^* detector as $\sigma \to 0$, provides very similar performance results for high SNRs while requiring only knowledge of $\mathcal{S}_1$. It is interesting to notice the performance gains at high SNR over the conventional detector which requires $\mathbf{c}_1$ in addition to $\mathcal{S}_1$. The blind $OAMMSE$ detector explicitly requires knowledge of only $\mathcal{S}_1$ but adaptively acquires knowledge of $\mathcal{S}_2 \ldots \mathcal{S}_K$ and $\mathbf{g}_2 \ldots \mathbf{g}_K$ (or more precisely, $\tilde{\mathbf{s}}_2 \ldots \tilde{\mathbf{s}}_K$ as defined in (23)).
- **Single user lower bound**[6]: Requires $\mathcal{S}_1$, $\mathbf{c}_1$. This is the conventional rake detector operating in an environment where no interfering users are present. Its performance gives a simple lower bound for any multiuser detection schemes where interfering users are present. Its probability of error is well-known and is given by

$$P_{\mathrm{SU},1}(\sigma) = \tfrac{1}{2} \exp\left[\frac{\Leftrightarrow A_1^2 \mathbf{c}_1^H \mathbf{R}_{[11]} \mathbf{c}_1}{2\sigma^2}\right].$$

Starting with the optimally near-far resistant mD detector, we see that even for low SNRs, its bit error rate almost achieves the lower bound. The difference between the mD detector and the lower bound is less than 0.5 dB. There is about a 2 dB performance loss for the Dm detector when knowledge of the subsignal coordinates is not assumed. For the Dm^* detector, knowledge of the interferer's coordinates improves the performance about 1 dB. For high SNRs, the Dm^* detector's performance should be identical to the converged blind adaptive $OAMMSE$ detector's. Hence while the blind $OAMMSE$ detector only requires information about the desired user's subspace, its performance is superior to the Dm detector which requires this information for all of the users. Finally, we note that even under perfect power control, the multiuser detectors outperform the conventional rake detector in the high SNR range.

7. Appendix

We compute the probability of error bounds on the EGC and hard-limiter decision function for a Gaussian EGC input with a full rank covariance matrix. These error expressions are used for the Dm and Dm^* detectors. Let $\mathbf{x}$ be the Gaussian EGC input with statistics

$$\begin{bmatrix} \mathbf{x}^R \\ \mathbf{x}^I \end{bmatrix} \sim \eta\left(\begin{bmatrix} \mu^R \\ \mu^I \end{bmatrix}, \sigma^2 \begin{bmatrix} \Sigma^R & \mathbf{0}_L \\ \mathbf{0}_L & \Sigma^I \end{bmatrix}\right),$$

where Σ^R and Σ^I have full rank. We can diagonalize the covariance matrices as $\Sigma^R = \mathbf{V}^R\Lambda^R(\mathbf{V}^R)^T$ and $\Sigma^I = \mathbf{V}^I\Lambda^I(\mathbf{V}^I)^T$ where $\mathbf{V}^R$ ($\mathbf{V}^I$) is the $L \times L$ matrix of eigenvectors and $\Lambda^R \equiv \mathrm{diag}(\lambda_1^R \ldots \lambda_L^R)$ ($\Lambda^I \equiv \mathrm{diag}(\lambda_1^I \ldots \lambda_L^I)$) is the diagonal matrix of distinct eigenvalues of Σ^R (Σ^I). Using the equal-gain combiner, the decision function is

$$\widehat{\mathbf{d}}[j] = \mathrm{sgn}\left[\mathrm{Re}\left\{\mathbf{x}^H[j]\mathbf{x}[j \Leftrightarrow 1]\right\}\right].$$

Letting

$$\begin{aligned} \widetilde{\mathbf{x}}_{-1}^R &\equiv \mathbf{x}^R[j] \Leftrightarrow \mathbf{x}^R[j \Leftrightarrow 1], & \widetilde{\mathbf{x}}_{-1}^I &\equiv \mathbf{x}^I[j] \Leftrightarrow \mathbf{x}^I[j \Leftrightarrow 1], \\ \widetilde{\mathbf{x}}_1^R &\equiv \mathbf{x}^R[j] + \mathbf{x}^R[j \Leftrightarrow 1], & \widetilde{\mathbf{x}}_1^I &\equiv \mathbf{x}^I[j] + \mathbf{x}^I[j \Leftrightarrow 1], \end{aligned} \tag{49}$$

when a "$\Leftrightarrow$1" is transmitted ($b[j] = \Leftrightarrow b[j \Leftrightarrow 1]$),

$$\widetilde{\mathbf{x}}_{-1} \equiv \begin{bmatrix} \widetilde{\mathbf{x}}_{-1}^R \\ \widetilde{\mathbf{x}}_{-1}^I \end{bmatrix} \sim \eta\left(\begin{bmatrix} 2\mu^R \\ 2\mu^I \end{bmatrix}, \sigma^2\begin{bmatrix} 2\Sigma^R & \mathbf{0}_L \\ \mathbf{0}_L & 2\Sigma^I \end{bmatrix}\right), \tag{50}$$

$$\widetilde{\mathbf{x}}_1 \equiv \begin{bmatrix} \widetilde{\mathbf{x}}_1^R \\ \widetilde{\mathbf{x}}_1^I \end{bmatrix} \sim \eta\left(\begin{bmatrix} \mathbf{0}_L \\ \mathbf{0}_L \end{bmatrix}, \sigma^2\begin{bmatrix} 2\Sigma^R & \mathbf{0}_L \\ \mathbf{0}_L & 2\Sigma^I \end{bmatrix}\right). \tag{51}$$

Hence

$$U_{-1} \equiv \widetilde{\mathbf{x}}_{-1}^T\widetilde{\mathbf{x}}_{-1}, \tag{52}$$

$$U_1 \equiv \widetilde{\mathbf{x}}_1^T\widetilde{\mathbf{x}}_1 \tag{53}$$

have, respectively, noncentral and central chi-square distributions with L degrees of freedom. Let

$$\widetilde{\mathbf{x}}'_{-1} \equiv \begin{bmatrix} \widetilde{\mathbf{x}}'^R_{-1} \\ \widetilde{\mathbf{x}}'^I_{-1} \end{bmatrix} = \begin{bmatrix} (\Lambda^R)^{-1/2}(\mathbf{V}^R)^T & \mathbf{0}_L \\ \mathbf{0}_L & (\Lambda^I)^{-1/2}(\mathbf{V}^I)^T \end{bmatrix}\begin{bmatrix} \widetilde{\mathbf{x}}_{-1}^R \\ \widetilde{\mathbf{x}}_{-1}^I \end{bmatrix}$$

and let $\mu^{R'} \equiv 2(\Lambda^R)^{-1/2}(\mathbf{V}^R)^T\mu^R$ and $\mu^{I'} \equiv 2(\Lambda^I)^{-1/2}(\mathbf{V}^I)^T\mu^I$ be the respective means of $\widetilde{\mathbf{x}}'^R_{-1}$ and $\widetilde{\mathbf{x}}'^I_{-1}$. The probability of error is

$$\begin{aligned} P(\sigma) &= \Pr[\text{decide "1" when "}\Leftrightarrow\text{1" sent}] && (54) \\ &= \Pr\left[\mathrm{Re}(\mathbf{x}^H[j]\mathbf{x}[j \Leftrightarrow 1]) > 0 | b[j] = \Leftrightarrow b[j \Leftrightarrow 1]\right] && (55) \\ &= \Pr\left[\widetilde{\mathbf{x}}_1^T\widetilde{\mathbf{x}}_1 \Leftrightarrow \widetilde{\mathbf{x}}_{-1}^T\widetilde{\mathbf{x}}_{-1} > 0 | b[j] = \Leftrightarrow b[j \Leftrightarrow 1]\right] && (56) \\ &= \int \Pr[U_{-1} < U_1 | U_{-1}] p(\widetilde{\mathbf{x}}_{-1})\, \mathrm{d}\widetilde{\mathbf{x}}_{-1} && (57) \\ &\leqslant \int \Pr[U_{-1} < \sum_{l=1}^{L} 2\sigma^2\lambda_l^M\alpha_l^2 | U_{-1}] p(\widetilde{\mathbf{x}}_{-1})\, \mathrm{d}\widetilde{\mathbf{x}}_{-1} && (58) \\ &= \int \left[\sum_{l=1}^{L} G_l^M\, \mathrm{e}^{-U_{-1}/4\sigma^2\lambda_l^M}\right] p(\widetilde{\mathbf{x}}_{-1})\mathrm{d}\widetilde{\mathbf{x}}_{-1} && (59) \end{aligned}$$

$$= \int \left[\sum_{l=1}^{L} G_l^M \left\{ \prod_{i=1}^{L} \frac{1}{\sqrt{2\pi}} \exp \left(\Leftrightarrow \frac{\lambda_i^R (x_{i,-1}'^R)^2}{2\sigma^2 \lambda_l^M} \Leftrightarrow \frac{1}{2\sigma^2} (x_{i,-1}'^R \Leftrightarrow \mu_i^{R'})^2 \right) \right.\right. \tag{60}$$

$$\left.\left. \frac{1}{\sqrt{2\pi}} \exp \left(\Leftrightarrow \frac{\lambda_i^I (x_{i,-1}'^I)^2}{2\sigma^2 \lambda_l^M} \Leftrightarrow \frac{1}{2\sigma^2} (x_{i,-1}'^I \Leftrightarrow \mu_i^{I'})^2 \right) \right\} \right] d\tilde{\mathbf{x}}_{-1}'$$

$$= \sum_{l=1}^{L} G_l^M \left\{ \prod_{i=1}^{L} \sqrt{\frac{\lambda_l^M}{\lambda_l^M + \lambda_i^R}} \exp \left[\frac{\Leftrightarrow (\mu_i^{R'})^2 \lambda_i^R}{4\sigma^2 (\lambda_i^R + \lambda_l^M)} \right] \right.$$

$$\left. \sqrt{\frac{\lambda_l^M}{\lambda_l^M + \lambda_i^I}} \exp \left[\frac{\Leftrightarrow (\mu_i^{I'})^2 \lambda_i^I}{4\sigma^2 (\lambda_i^I + \lambda_l^M)} \right] \right\} \tag{61}$$

$$= \sum_{l=1}^{L} G_l^M (\lambda_l^M)^L \left[\prod_{i=1}^{L} \sqrt{\frac{1}{(\lambda_l^M + \lambda_i^R)(\lambda_l^M + \lambda_i^I)}} \right. \tag{62}$$

$$\left. \exp \left(\frac{\Leftrightarrow 1}{4\sigma^2} \left\{ \frac{(\mu_{(i)}^{R'})^2 \lambda_i^R}{\lambda_i^R + \lambda_l^M} + \frac{(\mu_{(i)}^{I'})^2 \lambda_i^I}{\lambda_i^I + \lambda_l^M} \right\} \right) \right]$$

$$= \sum_{l=1}^{L} G_l^M (\lambda_l^M)^L \left[\prod_{i=1}^{L} \sqrt{\frac{1}{(\lambda_l^M + \lambda_i^R)(\lambda_l^M + \lambda_i^I)}} \right. \tag{63}$$

$$\exp \left(\frac{\Leftrightarrow 1}{2\sigma^2} \left\{ (\mu^R)^T \mathbf{V}^R \operatorname{diag} \left(\frac{2}{\lambda_1^R + \lambda_l^M} \cdots \frac{2}{\lambda_L^R + \lambda_l^M} \right) (\mathbf{V}^R)^T \mu^R \right.\right.$$

$$\left.\left.\left. + (\mu^I)^T \mathbf{V}^I \operatorname{diag} \left(\frac{2}{\lambda_1^I + \lambda_l^M} \cdots \frac{2}{\lambda_L^I + \lambda_l^M} \right) (\mathbf{V}^I)^T \mu^I \right\} \right) \right],$$

where

$$G_l^M \equiv \frac{(\lambda_l^M)^{L-1}}{\prod_{j \neq l} (\lambda_l^M \Leftrightarrow \lambda_j^M)},$$

where $\lambda_l^M \equiv \max(\lambda_l^R, \lambda_l^I)$, α_l^2 $(l = 1, \ldots, L)$ are independent zero-mean Gaussian random variables with unit variance, and where (58) follows from a geometric interpretation of the random variable U_1 (see [16], for example). Using the simple lower bound [16] based on the geometric mean $\lambda_l^G \equiv \sqrt{\lambda_l^R \lambda_l^I}$, we have that $\Pr(U_{-1} < U_1) \geqslant \Pr(U_{-1} < \sum_{l=0}^{L} \lambda_l^G \alpha_l^2)$. Hence we can obtain a lower bound on the error rate by replacing λ_l^M in (62) with λ_l^G:

$$P(\sigma) \geqslant \sum_{l=1}^{L} G_l^G (\lambda_l^G)^L \left[\prod_{i=1}^{L} \sqrt{\frac{1}{(\lambda_l^G + \lambda_i^R)(\lambda_l^G + \lambda_i^I)}} \right.$$

$$\exp \left(\frac{\Leftrightarrow 1}{2\sigma^2} \left\{ (\mu^R)^T \mathbf{V}^R \operatorname{diag} \left(\frac{2}{\lambda_1^R + \lambda_l^G} \cdots \frac{2}{\lambda_L^R + \lambda_l^G} \right) (\mathbf{V}^R)^T \mu^R \right.\right.$$

$$\left.\left.\left. + (\mu^I)^T \mathbf{V}^I \operatorname{diag} \left(\frac{2}{\lambda_1^I + \lambda_l^G} \cdots \frac{2}{\lambda_L^I + \lambda_l^G} \right) (\mathbf{V}^I)^T \mu^I \right\} \right) \right], \tag{64}$$

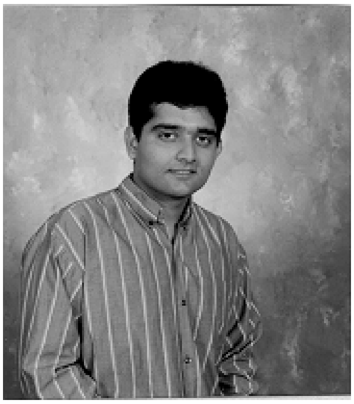
where

$$G_l^G \equiv \frac{(\lambda_l^G)^{L-1}}{\prod_{j\neq l}(\lambda_l^G \Leftrightarrow \lambda_j^G)}.$$

The more similar λ_l^R and λ_l^I are, the tighter both lower and upper bounds will be, and if $\lambda_l^R = \lambda_l^I$ for all l (as in the case of the Dm detector), the upper and lower bounds will be equal and we will have an exact expression for $P(\sigma)$.

References

1. S. Verdú, "Adaptive multiuser detection," in: Proceedings of the IEEE International Symposium on Spread Spectrum Theory and Applications, pp. 43–50, Oulu, Finland, 1994.
2. A. Duel-Hallen, J. Holtzman and Z. Zvonar, "Multiuser detection for CDMA systems," IEEE Personal Communications Magazine, Vol. 2, No. 2, pp. 46–58, 1995.
3. S. Verdú, "Minimum probability of error for asynchronous Gaussian multiple-access channels," IEEE Transactions on Information Theory, Vol. 32, No. 1, pp. 85–96, 1986.
4. R. Lupas and S. Verdú, "Linear multiuser detectors for synchronous CDMA channels," IEEE Transactions on Information Theory, Vol. 35, No. 1, pp. 123–136, 1989.
5. M.K. Varanasi and B. Aazhang, "Optimally near-far resistant multiuser detection in differentially coherent synchronous channels," IEEE Transactions on Information Theory, Vol. 37, No. 4, pp. 1006–1018, 1991.
6. J. Proakis, *Digital Communications*, Cambridge: McGraw Hill, 3rd edn, 1995.
7. Z. Zvonar and D. Brady, "Optimum detection in asynchronous multiple-access multipath Rayleigh fading channels," in: Proceedings of the 26th Annual Conference on Information Sciences and Systems, Princeton University, pp. 826–831, 1992.
8. H. Huang and S. Schwartz, "A comparative analysis of linear multiuser detectors for fading multipath channels," in Proceedings of the 1994 IEEE GLOBECOM Conference, pp. 11–15, San Francisco, CA.
9. A. Klein and P.W. Baier, "Linear unbiased data estimation in mobile radio systems applying CDMA," IEEE Selected Areas in Communications, Vol. 11, No. 7, pp. 1058–1066, 1993.
10. Z. Zvonar and D. Brady, "Suboptimum multiuser detector for frequency-selective Rayleigh fading synchronous CDMA channels," IEEE Transactions on Communications, Vol. 43, No. 2/3/4, pp. 154–157, 1995.
11. U. Madhow and M. Honig, "MMSE interference suppression for direct-sequence spread spectrum CDMA," IEEE Transactions on Communications, Vol. 42, pp. 3178–3188, 1994.
12. M. Honig, U. Madhow and S. Verdú, "Blind adaptive multiuser detection," IEEE Transactions on Information Theory, Vol. 41, No. 4, pp. 944–960, 1995.
13. Z. Zvonar and D. Brady, "Coherent and differentially coherent multiuser detectors for asynchronous CDMA frequency-selective channels," in: Proceedings of the 1992 IEEE MILCOM Conference, pp. 442–446, San Diego, CA.
14. R. Horn and C. Johnson, *Matrix Analysis*, Cambridge: Cambridge University Press, 1985.
15. H.V. Poor and S. Verdú, "Probability of error in MMSE multiuser detection," IEEE Transactions on Information Theory, April 1997.
16. A.M. Mathai and S.B. Provost, *Quadratic Forms in Random Variables*, New York: M. Kekker, 1992.



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