Comments on "Anomalous Behavior of Receiver Output SNR as a Predictor of Signal Detection Performance Exemplified for Quadratic Receivers and Incoherent Fading Gaussian Channels"

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***Abstract*—A previously published derivation of an optimal quadratic receiver with respect to a generalized signal-to-noise ratio is corrected, and a different performance measure is proposed for which a general analytical solution exists.**

In the above paper,[1] several particular cases of quadratic receivers for the detection of Gaussian signals in additive Gaussian noise are considered. It is found that even though the maximum signal-to-noise ratio (SNR) and the minimum probability of error receivers coincide, the asymptotic behavior of both performance measures with respect to some parameters are different. Furthermore, an optimum quadratic receiver with respect to a generalized output signal-to-noise ratio (GSNR) is obtained for the white noise case. In this correspondence it is shown that although the derivation of such an optimum receiver in Gardner's paper[1] is incorrect, its particularization for the cases that illustrate the aforementioned behavior differences between SNR and probability of error turns out to be valid. Moreover, we propose a different GSNR for which an analytical solution of the optimum receiver is obtainable.

We suppose that the detection of the zero-mean stochastic signal embedded in additive independent zero-mean noise is based on a real quadratic form, X, of the input with an operator H (self-adjoint without loss of generality) mapping a separable Hilbert space $\mathcal{H}$ into itself. Denoting by X_{sn}, X_{ns}, X_{ss}, X_{nn}, the signal-cross-noise term, etc., the GSNR[1] is

$$\mathrm{GSNR}(X) = \frac{[m(X_{ss})]^2}{\sigma^2(X_{nn}) + \alpha[\sigma^2(X_{ss}) + \sigma^2(X_{sn}+X_{ns})]}. \tag{1}$$

Assuming that the signal and noise are Gaussian processes, the terms in (1) were derived[1] in terms of H and the (nonnegative) signal covariance operator K_s; however, the calculation of the variance of the signal-cross-signal term, $\sigma^2(X_{ss})$, is incorrect. Restricting our attention to real self-adjoint operators note that, even though $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, the operators do not commute in general and therefore they can be rotated but not permuted otherwise under the trace functional. Applying this to the case in question

$$\begin{aligned}\sigma^2(X_{ss}) &= \int_0^T\int_0^T\int_0^T\int_0^T h(t_1,t_2)h(\tau_1,\tau_2)[k_s(t_1,\tau_1)k_s(t_2,\tau_2) \\ &\quad + k_s(t_1,\tau_2)k_2(t_2,\tau_1)]\, dt_1\, dt_2\, d\tau_1\, d\tau_2 \\ &= 2\,\mathrm{tr}\{HK_sHK_s\},\end{aligned} \tag{2}$$

rather than $2\,\mathrm{tr}\{H^2K_s^2\}$. Consequently expressions (A.7) and (A.8)[1] do not follow. Without adding any further complexity it is possible to analyze the problem in which the zero-mean noise is not necessarily white. Denoting the noise covariance (positive) operator by K_n, (1) becomes

$$\mathrm{GNSR} = \frac{\frac{1}{2}\mathrm{tr}^2\{HK_s\}}{\mathrm{tr}\{HK_nHK_n\} + \alpha[\mathrm{tr}\{HK_sHK_s + 2HK_nHK_s\}]}. \tag{3}$$

Note that this expression holds for a generic Hilbert space $\mathcal{H}$. Its maximization with respect to H can be accomplished in particular cases such as the following.

i) GSNR = *deflection* ($\alpha = 0$); $H_{\mathrm{opt}} = K_n^{-1}K_sK_n^{-1}$, (see [1]).
ii) GSNR = *complementary deflection* ($\alpha = 1$); $H_{\mathrm{opt}} = (K_n + K_s)^{-1}K_s(K_n+K_s)^{-1}$.
iii) All the nonzero eigenvalues of K_s are equal (to $N_0\lambda$), and the noise is white, i.e., $K_n = N_0 I$.

Case iii) is considered by Gardner[1] for slow and fast fading channels.

It is possible to show that the GSNR is maximized for any nonnegative α by $H_{\mathrm{opt}} = K_s$, and thus (A.9)–(A.11) derived from a wrong general expression for H_{opt} are still valid. In order to prove this result, let $\mathcal{S} \subset \mathcal{H}$ denote the eigenspace of K_s with nonzero eigenvalues, and define the operators H_s and D by

$$H_s x = \begin{cases} Hx, & x \in \mathcal{S},\ Hx \in \mathcal{S}, \\ 0, & \text{otherwise,}\end{cases} \tag{4}$$

and $D = H^2 - H_s^2$. Since HK_s has the same eigenvalues as H_sK_s, (3) results in

$$\mathrm{GSNR} = \frac{\frac{1}{2}\mathrm{tr}^2\{H_sK_s\}}{N_0^2\,\mathrm{tr}\{H_s^2\} + N_0^2\,\mathrm{tr}\{D\} + \alpha\,\mathrm{tr}\{(H_sK_s)^2\} + 2\alpha N_0[\mathrm{tr}\{H_s^2K_s\} + \mathrm{tr}\{DK_s\}]}. \tag{5}$$

Considering that D and K_s are nonnegative operators and $\alpha \geq 0$, the right side of (5) is maximized with $D = 0$; i.e., $H = H_s$. Taking into account that all nonzero eigenvalues of K_s are equal to $N_0\lambda$, (5) reduces to

$$\mathrm{GSNR} = \frac{1}{2N_0^2[1+\alpha(\lambda^2+2\lambda)]}\,\frac{\mathrm{tr}^2\{H_sK_s\}}{\mathrm{tr}\{H_s^2\}}, \tag{6}$$

which is maximized (via the Schwarz inequality) by $H_s = K_s$. Thus, $H_{\mathrm{opt}} = K_s$, as was to be shown.

A drawback of the above generalized signal-to-noise ratio (1) is that it does not appear to lead to analytical solutions of H_{opt} for arbitrary values of α and arbitrary signal and noise operators. Another generalized signal-to-noise ratio not suffering from this shortcoming is

$$\mathrm{GSNR}^*(X) = \frac{[m(X_{ss})]^2}{\alpha_0^2\sigma^2(X_{nn}) + \alpha_0\alpha_1\sigma^2(X_{sn}+X_{ns}) + \alpha_1^2\sigma^2(X_{ss})}, \tag{7}$$

whose parameters α_0, α_1 provide the desired flexibility in weighting the signal/noise-cross-signal/noise terms in the output of the receiver. Note that this performance measure is equivalent to the *complementary deflection* (see [2]) when the input to the receiver is $\alpha_0^{1/2}n + \alpha_1^{1/2}s$. By using this GSNR instead of (1), in which the weighting is performed on the variances of X given the signal present/absent hypotheses, it is possible to obtain through the Schwarz inequality an explicit expression for the optimum receiver; namely,

$$H_{\mathrm{opt}} = (\alpha_0K_n + \alpha_1K_s)^{-1}K_s(\alpha_0K_n+\alpha_1K_s)^{-1}, \tag{8}$$

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[1] W. A. Gardner, *IEEE Trans. Inform. Theory*, vol. IT-25, no. 6, pp. 743–745, Nov. 1979.

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that gives, in the three particular cases considered above, the same solution as the optimizing filter for the original GSNR.

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Reverse-Time Diffusion Processes

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***Abstract*—Given a diffusion process on $\mathbb{R}^n$ described by a stochastic differential equation forward in time, we develop a corresponding stochastic differential equation in reverse time which yields the same sample paths. This stochastic differential equation can be used in problems of estimation and smoothing.**

I. INTRODUCTION

A diffusion process is said to be modeled by a forward Markovian model, when it is obtained from an Ito differential equation as

$$dx_t = f(x_t, t)\,dt + g(x_t, t)\,dw_t, \qquad t \in [0, T), \tag{1}$$

where w_t is a standard m-dimensional Wiener process, independent of the random initial condition $x_0 \in \mathbb{R}^n$, and $x_t \in \mathbb{R}^n$ is the diffusion process. The functions f and g are assumed sufficiently smooth to guarantee global existence and uniqueness of the x_t process specified by [1, eq. (1)].

In this correspondence, we address the problem of obtaining a reverse Markovian model for the diffusion process x_t. This problem has been studied by several authors when the function f is linear in x and g is independent of x [2], [5], [6]. In many of these works, the emphasis has been in obtaining a reverse model for a process whose second-order statistics are consistent with (1). Verghese and Kailath [6] have extended these approaches to provide a sample-path identification between the process given by (1) and the reverse-time model.

When f is a nonlinear time-invariant function, and g is constant, Stratonovich [11] and Anderson [2] have obtained a reverse Markov model for a process whose transition probabilities are consistent with (1). In this correspondence, we follow the approach of Verghese and Kailath [6] and derive a reverse-time Markovian realization of the original process x_t. This realization extends the work of Stratonovich and Anderson to general diffusions; in addition, it provides a direct technique for interpreting the reverse-time model sample paths in terms of the original sample paths.

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II. DERIVATION

Here, we will make the following assumptions concerning the model of (1):

A1) the functions f_i, g_{ij} are bounded in $R^n \times [0, T]$ and uniformly Lipschitz continuous in (x, t) on compact subsets of $R^n \times [0, T]$;

A2) the functions $(g, g^T)_{ij}$ are uniformly Holder continuous in x;

A3) there exists $b > 0$ such that

$$\sum_{i,j=1}^{n} (g, g^T)_{ij}\xi_i\xi_j \geq b\,|\xi|^2$$

for all $\xi \in \mathbb{R}^n$, all $(x, t) \in \mathbb{R}^n \times [0, T]$.

Under assumptions A1)–A3), (1) describes a diffusion process whose transition probability distribution has a density, by [1, theorem 5.4]. Hence, the distribution of x_t will also have a density, denoted by $p(t, x)$. We will make the additional assumptions:

A4) the functions $\partial f_i/\partial x_i$, $\partial g_{ik}/\partial x_i$, $\partial^2 g_{ik}/\partial x_i \partial x_j$ satisfy A1), A2).

A5) Assumption (A1) holds uniformly in $\mathbb{R}^n \times [0, T]$.

Assumption A4) guarantees that the Fokker–Planck equation is well-defined, and has a unique solution described by $p(t, x)$, which is *continuously differentiable* in x. Assumption A5) is used to guarantee *pathwise uniqueness* of the solutions of (1).

As notation, denote the infinitesimal generator of x_t by

$$L_t h(x) = \sum_{i=1}^{n} f_i(x, t)\frac{\partial h}{\partial x_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} g_{ik}(x, t) g_{jk}(x, t)\frac{\partial^2 h}{\partial x_i \partial x_j}.$$

This operator is uniformly elliptic by A3).

Equation (1) can be written in integral form as

$$x_t = x_0 + \int_0^t f(x_s, s)\,ds + \int_0^t g(x_s, s)\,dw_s, \tag{2}$$

where the second integral is an Ito integral. Consider the sigma fields F_t, G_t, defined as

$$\begin{aligned} F_t &\triangleq \sigma\{x_0, w_s, 0 \leq s \leq t\} = \sigma\{x_s, 0 \leq s \leq t\}, \\ G_t &\triangleq \sigma\{x_t, x_s, t \leq s \leq T\}. \end{aligned} \tag{3}$$

By the properties of Ito integrals, $(\int_0^t g(x_s, s)\,dw_s)$ is an F_t martingale. This property reflects the independence between x_0 and the Wiener process w_t.

In order to obtain a reverse-time Markovian model for x_t, it is necessary and sufficient to write x_t in integral form, as

$$x_t = x_T + \int_T^t \bar{f}(x_s, s)\,ds + \int_T^t \bar{g}(x_s, s)\,d\bar{w}_s, \tag{4}$$

where $(\bar{w}_s, G_s)$ is a standard Brownian motion, independent of x_T, and $\int_T^t \bar{g}(x_s, s)\,d\bar{w}_s$ is a G_t martingale, representing a backward Ito integral.

Rewrite (2) as

$$x_t = x_T + \int_T^t -f(x_s, s)\,d(-s) + \int_T^t g(x_s, s)\,dw_s. \tag{5}$$

Although (4) and (5) appear similar, there is one important difference. The last term, $\int_T^t g(x_s, s)\,dw_s$ is not a G_t martingale, because w_s and x_T are correlated. However, it is a semimartingale; it can be decomposed uniquely as the sum of a predictable process and a martingale [8], the Doob–Meyer decomposition.

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