Capacity of Root-Mean-Square Bandlimited Gaussian Multiuser Channels

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***Abstract*—Continuous-time additive white Gaussian noise channels with strictly time-limited and root-mean-square (RMS) bandlimited inputs are studied. RMS bandwidth is equal to the normalized second moment of the spectrum, which has proved to be a useful and analytically tractable measure of the bandwidth of strictly time-limited waveforms. The capacity of the single-user and two-user RMS-bandlimited channels are found in easy-to-compute parametric forms, and are compared to the classical formulas for the capacity of strictly bandlimited channels. In addition, channels are considered where the inputs are further constrained to be pulse amplitude modulated (PAM) waveforms. The capacity of the single-user RMS-bandlimited PAM channel is shown to coincide with Shannon's capacity formula for the strictly bandlimited channel. This shows that the laxer bandwidth constraint precisely offsets the PAM structural constraint, and illustrates a tradeoff between the time domain and frequency domain constraints. In the synchronous two-user channel, we find the pair of pulses that achieves the boundary of the capacity region, and show that the shapes of the optimal pulses depend not only on the bandwidth but also on the respective signal-to-noise ratios.**

***Index Terms*—Bandlimited communication, information rate, multiuser channels.**

I. Introduction

The capacity of the continuous-time strictly bandlimited white Gaussian channel is given by the celebrated Shannon formula [1], [2]

$$C_C = B\log\left[1+\frac{W}{N_0B}\right], \tag{1}$$

where $N_0/2$ is the noise spectral level, and W and B are the transmitter's maximum allowable power and bandwidth, respectively. The generalization of this result to the two-user Gaussian channel yields [3]

$$C_C = \left\{ (R_1,R_2):\ \begin{array}{l} 0\le R_1\le B\log\left[1+\dfrac{W_1}{N_0B}\right] \\ 0\le R_2\le B\log\left[1+\dfrac{W_2}{N_0B}\right] \\ R_1+R_2\le B\log\left[1+\dfrac{W_1}{N_0B}+\dfrac{W_2}{N_0B}\right] \end{array}\right\}. \tag{2}$$

These formulas are achieved by approximately time-limited input waveforms which have no frequency content beyond B and can be put as linear combinations of shifted sinc functions.

In this paper, we investigate the capacity achievable by strictly time-limited input waveforms which satisfy the root-mean-square (RMS) bandwidth constraint. A number of bandwidth measures for time-limited waveforms have been proposed in the literature [4]. Perhaps the most popular measures are the root-mean-square (RMS) and the fractional out-of-band-energy (FOBE) bandwidth measures. In this paper, we concentrate on the RMS bandwidth (we have obtained similar results for FOBE bandwidth [5]) introduced by Gabor [6] and used subsequently in (e.g., [7]–[14]).

A finite-energy signal $s(t)$ has RMS bandwidth B if its Fourier transform $S(f)$ satisfies

$$\frac{\int_{-\infty}^{\infty} f^2|S(f)|^2\,df}{\int_{-\infty}^{\infty}|S(f)|^2\,df} = B^2. \tag{3}$$

Thus, the RMS bandwidth of a signal is equal to the standard deviation of its spectrum, which is indeed a legitimate measure of its width. Also, via Parseval's theorem, the RMS bandwidth is equal to the square root of the energy of the derivative of the normalized signal. The RMS bandwidth has many desirable properties. The spec-

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trum of a signal with finite RMS bandwidth falls off at rate faster than $1/f^3$. The fixed duration signal with minimum RMS bandwidth is the truncated sinusoid [6] which is easy to generate. Also, the signal with minimum RMS bandwidth compares favorably [11], in terms of energy concentration, with the time-truncated prolate spheroidal wave functions. Notice that the RMS bandwidth of a waveform which is strictly bandlimited to B is less than B, and it approaches B when it concentrates all its energy in frequencies close to B.

A number of technical difficulties encountered in the derivation of the capacity of strictly bandlimited channels are sidestepped by working with the RMS bandwidth measure that is ideally suited for dealing with strictly time-limited waveforms, and is a more faithful model for the physical bandwidth constraint present in many real channels. Especially in wideband communication channels, such as spread spectrum and optical channels, where bandwidth is neither free nor strictly limited, RMS bandwidth is an appropriate useful alternative [14]–[17]. Also, it allows us to determine the sensitivity of the white Gaussian channel capacity to the bandwidth measure by comparison with the classical results for strictly bandlimited channels. For example, the single-user RMS-bandlimited channel capacity has the same behavior as (1) at low signal-to-noise ratios, whereas it grows asymptotically as the cubic root of the signal-to-noise ratio, rather than its logarithm. In the two-user case, the capacity region of the RMS-bandlimited channel is, as in the strictly bandlimited channel (2), a pentagon delimited by single-user capacities. However, unlike the strictly bandlimited channel, no input distribution achieves all rate pairs simultaneously.

But one of the main motivations for considering the RMS bandwidth constraints is the following. In digital communication systems such as Spread Spectrum it is customary to send codewords of symbols by letting each symbol modulate a fixed nonoverlapping time-limited pulse of duration T, which is equal to the inverse of the symbol rate. It is of interest to find the capacity achievable by specific modulation formats when the pulses are designed under bandwidth constraints. Hence, the need for bandwidth measures for strictly time-limited waveforms arises. In the second part of this paper, we carry out this analysis for the specific modulation format of pulse amplitude modulation (PAM). In this case, the input signals can be put as linear combinations of the shifts of a pulse of finite duration which is selected to maximize capacity under the RMS bandwidth constraint (instead of the overlapping sincs used in the strictly bandlimited channel). This formulation includes the synchronous code division multiple access channel and the pulses we obtain are the optimal signature waveforms in the sense of maximizing the capacity region.

We find that the capacity of the single-user RMS-bandlimited PAM channel coincides with the Shannon formula (1). Hence, relaxing from strictly bandlimited to RMS-bandlimited inputs, we can introduce the PAM structure in the time domain and still attain the same capacity. However, in the case of the synchronous multiple-access channel, we find that the effect of the laxer bandwidth constraint more than offsets the effect of the PAM structure and the capacity regions of the RMS-bandlimited PAM multiple-access channel is larger than that of the classical strictly bandlimited channel (2). Comparing those results to their counterparts without imposing the PAM structure, we conclude that the decease in capacity caused by this structural constraint is substantial at high signal-to-noise ratios.

II. RMS-Bandlimited Channel

In this section, we find the capacities of continuous-time white Gaussian channels where the input waveforms are power constrained and RMS-bandlimited.

The RMS-bandlimited two-user channel can be expressed as

$$y(t) = x_1(t) + x_2(t) + n(t), \tag{4}$$

where $x_1(t)$ and $x_2(t)$ are the codewords sent by user 1 and 2 respectively and $n(t)$ is white Gaussian noise with spectral density $N_0/2$. The power and the RMS bandwidth constraints require each codeword of the kth user to satisfy

$$\frac{1}{T}\int_0^T x_k^2(t)\,dt \le W_k, \qquad k = 1,2, \tag{5}$$

and

$$\frac{\int_{-\infty}^{\infty} f^2|X_k(f)|^2\,df}{\int_{-\infty}^{\infty} |X_k(f)^2|\,df} = \frac{1}{(2\pi)^2}\frac{\int_0^T\left[\frac{dx_k(t)}{dt}\right]^2 dt}{\int_0^T x_k^2(t)\,dt} \le B^2, \qquad k = 1,2, \tag{6}$$

where $X_k(f)$ is the Fourier transform of the strictly time-limited codeword $x_k(t)$. Since the sum of RMS-bandlimited signals is not, in general, RMS-bandlimited, the following constraint is needed to guarantee the transmitted signal to be RMS-bandlimited.

$$\frac{\int_{-\infty}^{\infty} f^2|X_1(f)+X_2(f)|^2\,df}{\int_{-\infty}^{\infty} |X_1(f)+X_2(f)|^2\,df} = \frac{1}{(2\pi)^2}\frac{\int_0^T\left[\frac{dx_1(t)}{dt}+\frac{dx_2(t)}{dt}\right]^2 dt}{\int_0^T [x_1(t)+x_2(t)]^2\,dt} \le B^2. \tag{7}$$

However, as we will see, the additional constraint in (7) does not reduce the capacity region.

Theorem 1: The capacity region of the two-user RMS-bandlimited white Gaussian channel with noise power spectral density equal to $N_0/2$, and signal powers and RMS bandwidth less than or equal to W_1, W_2, and B

respectively, is given by the pentagon

$$C_U(B,\Lambda_1,\Lambda_2) = \left\{ (R_1,R_2): \begin{array}{c} 0 \le R_1 \le C_U(B,\Lambda_1) \\ 0 \le R_2 \le C_U(B,\Lambda_2) \\ R_1+R_2 \le C_U(B,\Lambda_1+\Lambda_2) \end{array} \right\}, \quad (8)$$

where $\Lambda_i = W_i/(BN_0)$ for $i=1,2$ and

$$C_U(B,\Lambda) = \sup_{\substack{\int_0^\infty S(w)\,dw \le \Lambda \\ \int_0^\infty S(w)w^2\,dw \le \Lambda \\ 0 \le S(w)}} B\int_0^\infty \log[1+S(w)]\,dw \quad (9)$$

Corollary 1: The capacity of the single-user RMS-bandlimited white Gaussian channel with noise power spectral density equal to $N_0/2$, and signal power and RMS bandwidth less than or equal to W and B respectively, is given by $C_U(B,\Lambda)$, where $\Lambda = W/(BN_0)$.

$$C_U(B,\Lambda_1,\Lambda_2) \subseteq \left\{ (R_1,R_2): \begin{array}{c} 0 \le R_1 \le g(B,\Lambda_1) \\ 0 \le R_2 \le g(B,\Lambda_2) \\ R_1+R_2 \le g(B,\Lambda_1+\Lambda_2) \end{array} \right\} \quad (14)$$

$$\subseteq \bigcup_{\substack{\int_0^\infty S_k(w)\,dw = \Lambda_k \\ \int_0^\infty S_k(w)w^2\,dw \le \Lambda_k \\ 0 \le S_k(w) \\ k=1,2.}} \left\{ (R_1,R_2): \begin{array}{c} 0 \le R_1 \le B\int_0^\infty \log[1+S_1(w)]\,dw \\ 0 \le R_2 \le B\int_0^\infty \log[1+S_2(w)]\,dw \\ R_1+R_2 \le B\int_0^\infty \log[1+S_1(w)+S_2(w)]\,dw \end{array} \right\} \quad (15)$$

$$\subseteq C_U(B,\Lambda_1,\Lambda_2). \quad (16)$$

Remark: Corollary 1 admits the following heuristic interpretation. Suppose that the frequency axis is partitioned into small intervals of length δ and each of the frequency bands is used to transmit information independently. If the frequency band $[\delta i, \delta i+\delta]$ is allocated signal power equal to δE_i, such that

$$\sum_i \delta E_i \le W \quad (10)$$

and

$$\sum_i \delta E_i (i\delta)^2 \le B^2 W, \quad (11)$$

then the capacity of the (strictly bandlimited) ith channel can be computed using the Shannon formula: $\delta\log[1+E_i/N_0]$. In the limit as $\delta \to 0$, the overall capacity becomes (cf. (9))

$$\sup_{\substack{\int_0^\infty E(\lambda)\,d\lambda \le W \\ \int_0^\infty E(\lambda)\lambda^2\,d\lambda \le B^2W}} \int_0^\infty \log\left[1+\frac{E(\lambda)}{N_0}\right] d\lambda. \quad (12)$$

However, it is important to note that this heuristic interpretation is not a proof and the capacity of a white Gaussian channel under other bandwidth constraints should not be taken on faith to have a similar expression as in (9). This is because under a strictly time-limited constraint, we can only consider approximately bandlimited channels which interfere with each other and their capacities cannot be simply added up.

Proof: Let us denote the capacity of the two-user RMS-bandlimited channel with bandwidth B and signal-to-noise ratios Λ_1 and Λ_2 as $C_U(B,\Lambda_1,\Lambda_2)$ and define

$$g(B,\Lambda) \triangleq \sup_{\substack{\int_0^\infty S(w)\,dw \le \Lambda \\ \int_0^\infty S(w)w^2\,dw \le \Lambda \\ 0 \le S(w)}} B\int_0^\infty \log[1+S(w)]\,dw. \quad (13)$$

We prove this theorem by showing the following circular set inclusions:

Since the sum of the user's signals is also RMS-bandlimited, to show (14), it is sufficient to show that the capacity of a single-user RMS-bandlimited channel with bandwidth B and signal-to-noise ratio Λ, denoted by $C_U(B,\Lambda)$, is upper bounded by $g(B,\Lambda)$.

To that end, let us consider the single-user version of (4),

$$y(t) = x(t) + n(t). \quad (17)$$

Since $\{\phi_j(t,T)\}_{j=0}^\infty$, defined as

$$\phi_j(t,T) \triangleq \begin{cases} \sqrt{\frac{2}{T}}\sin\frac{j\pi t}{T}, & \text{if } t \in (0,T), \\ 0, & \text{otherwise}, \end{cases} \quad (18)$$

forms a complete orthonormal basis [11] of all RMS-bandlimited signals that are time-limited to $(0,T)$, every codeword $x(t)$ of every (T,M,ϵ) code of the single-user channel (17) can be expressed as

$$x(t)=\sum_{j=1}^{\infty} x_j\phi_j(t,T). \tag{19}$$

Then, the power and the RMS bandwidth constraints become

$$\frac{1}{T}\sum_{j=1}^{\infty} x_j^2 \le W \tag{20}$$

and

$$\frac{1}{(2T)^2}\sum_{j=1}^{\infty} x_j^2 j^2 \le B^2 \sum_{j=1}^{\infty} x_j^2. \tag{21}$$

If we construct y_j, $j=1,2,\cdots$ by

$$y_j=\int_0^T y(t)\phi_j(t)\,dt \tag{22}$$

then, substituting (17) in (22), we have

$$y_j=x_j+n_j, \tag{23}$$

where n_j is white Gaussian noise with variance equal to $N_0/2$. It is important to note that $\{y_j\}_{j=1}^{\infty}$ are sufficient statistics for the transmitted messages. Therefore, the error probability does not increase if the decoder uses the observables $\{y_i\}_{i=1}^{\infty}$ in lieu of $y(t)$.

Since every codeword $\{x_i\}_{i=1}^{\infty}$ satisfying (20) and (21) also satisfies

$$\frac{1}{(2T)^2}\sum_{j=1}^{\infty} x_j^2 j^2 \le B^2 WT, \tag{24}$$

constraints (20) and (24) are laxer than constraints (20) and (21), and we have, from the general converse theorem of the single-user channel [18],

$$\begin{aligned} C_U(B,\Lambda) &\le \liminf_{T\to\infty} \sup_{\substack{\frac{1}{T}\sum_{j=1}^{\infty} E[X_j^2]\le W\\ \frac{1}{(2T)^2}\sum_{j=1}^{\infty} E[X_j^2]j^2\le B^2WT}} \frac{1}{T} I(X;Y)\\ &\le \liminf_{T\to\infty} \sup_{\substack{\frac{1}{T}\sum_{j=1}^{\infty} E[X_j^2]\le W\\ \frac{1}{T^3}\sum_{j=1}^{\infty} E[X_j^2]j^2\le (2B)^2W}} \frac{1}{T}\sum_{j=1}^{\infty} I(X_j,Y_j), \end{aligned} \tag{25}$$

where the last inequality follows from the memorylessness of the channel. The constraints are in terms of the variances of the X_j's and, given the variance, each term $I(X_j;Y_j)$, in the summation is maximized when X_j is Gaussian distributed. Therefore, we have

$$\begin{aligned} C_U(B,\Lambda) &\le \liminf_{T\to\infty} \sup_{\substack{\frac{1}{T}\sum_{j=1}^{\infty}\sigma_j^2\le W\\ \frac{1}{T^3}\sum_{j=1}^{\infty}\sigma_j^2 j^2\le (2B)^2W}} \frac{1}{2T}\sum_{j=1}^{\infty}\log\left[1+\frac{2\sigma_j^2}{N_0}\right]\\ &= \liminf_{T\to\infty} \sup_{\substack{(\lambda_1,\lambda_2,\cdots):\\ \frac{1}{2BT}\sum_{j=1}^{\infty}\lambda_j\le\Lambda\\ \frac{1}{(2BT)^3}\sum_{j=1}^{\infty}\lambda_j j^2\le\Lambda\\ 0\le\lambda_j\ j=1,2,\cdots,\infty}} \frac{1}{2T}\sum_{j=1}^{\infty}\log\left[1+\lambda_j\right]. \end{aligned} \tag{26}$$

Finally, we obtain the upper bound (14) from (26) by the following Lemma which is proved in the Appendix using elementary functional analysis techniques.

Lemma 1:

$$\lim_{T_0\to\infty} \sup_{\substack{(\lambda_1,\lambda_2,\cdots):\\ \frac{1}{2BT_0}\sum_{j=1}^{\infty}\lambda_j=\Lambda\\ \frac{1}{(2BT_0)^3}\sum_{j=1}^{\infty}\lambda_j j^2\le\Lambda\\ 0\le\lambda_j\ j=1,2,\cdots,\infty}} \frac{1}{2T_0}\sum_{j=1}^{\infty}\log\left[1+\lambda_j\right] = \sup_{\substack{\int_0^\infty S(w)\,dw=\Lambda\\ \int_0^\infty S(w)w^2\,dw\le\Lambda\\ 0\le S(w)}} B\int_0^\infty \log\left[1+S(w)\right]dw. \tag{27}$$

Now, we show the lower bound (16) using a sequence of discrete-time two-user vector channels. For each real number $T_0>1$ and positive integer J, we define the discrete-time vector channel as

$$\boldsymbol{y}_i=\boldsymbol{x}_{1i}+\boldsymbol{x}_{2i}+\boldsymbol{n}_i, \tag{28}$$

where $\boldsymbol{y}_i$, $\boldsymbol{x}_{1i}$, $\boldsymbol{x}_{2i}$, $\boldsymbol{n}_i \in \boldsymbol{R}^J$, and $\{\boldsymbol{n}_i\}$ is white vector Gaussian noise with covariance matrix, $(N_0/2)\boldsymbol{I}_J$. Each codeword of the kth user, $\boldsymbol{x}_k=(\boldsymbol{x}_{k1},\cdots,\boldsymbol{x}_{kN})$, is constrained to satisfy

$$\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{x}_{ki}^T\boldsymbol{x}_{ki}=W_kT_0, \qquad k=1,2, \tag{29}$$

$$\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{x}_{ki}^T\Pi\boldsymbol{x}_{ki}\le(2BT_0)^2W_kT_0, \qquad k=1,2, \tag{30}$$

and

$$\frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{x}_{1i}+\boldsymbol{x}_{2i})^T\Pi(\boldsymbol{x}_{1i}+\boldsymbol{x}_{2i})\le(2BT_0)^2(W_1+W_2)T_0, \tag{31}$$

where Π is a diagonal matrix with the jth diagonal entry $\Pi_{jj}=j^2$.

The capacity region of the discrete-time channel with additive inputs, denoted by $C_D(J,T_0)$, is equal to [18]

$$C_D(J,T_0) = \text{convex closure of} \bigcup_{\substack{E(X_k^T X_k) = W_k T_0 \\ E(X_k^T \Pi X_k) \le (2BT_0)^2 W_k T_0 \\ E((X_1+X_2)^T \Pi (X_1+X_2)) \le (2BT_0)^2 (W_1+W_2) T_0 \\ k=1,2.}} \left\{ (R_1,R_2):\ \begin{matrix} 0 \le R_1 \le I(X_1;Y|X_2) \\ 0 \le R_2 \le I(X_2;Y|X_1) \\ R_1+R_2 \le I(X_1,X_2;Y) \end{matrix} \right\}. \tag{32}$$

Since X_1 and X_2 are independent, the third constraint is redundant, and the capacities, with and without the RMS bandwidth constraint on the sum of the codewords (7), are the same. Moreover, all three mutual informations can be simultaneously maximized by independent Gaussian input distributions. Therefore, we have, after applying Hadamard's inequality,

$$C_D(J,T_0) = \bigcup_{\substack{(\lambda_{k1},\lambda_{k2},\cdots,\lambda_{kJ}): \\ \frac{1}{2BT_0}\sum_{j=1}^{J}\lambda_{kj} = \Lambda_k \\ \frac{1}{(2BT_0)^3}\sum_{j=1}^{J}\lambda_{kj} j^2 \le \Lambda_k \\ 0 \le \lambda_{kj}\ j=1,2,\cdots,J}} \left\{ (R_1,R_2):\ \begin{matrix} 0 \le R_1 \le \frac{1}{2}\sum_{j=1}^{J}\log\left[1+\lambda_{1j}\right] \\ 0 \le R_2 \le \frac{1}{2}\sum_{j=1}^{J}\log\left[1+\lambda_{2j}\right] \\ R_1 = R_2 \le \frac{1}{2}\sum_{j=1}^{J}\log\left[1+\lambda_{1j}+\lambda_{2j}\right] \end{matrix} \right\}, \tag{33}$$

which is closed and convex.

Now, we proceed to show that for each J and T_0, $(R_1,R_2) \in C_D(J,T_0)$ implies that $(R_1/T_0, R_2/T_0) \in C_U(B,\Lambda_1,\Lambda_2)$.

Let us fix $J, \epsilon, \delta > 0$, $T_0 > 1$ and $(R_1,R_2) \in C_D(J,T_0)$. From the definition of capacity, there exists an N_0 such that for each $N \ge N_0$, there exists an (N, M_1, M_2, ϵ) code (i.e., a code with blocklength N, M_i codewords for user i and average probability of error less than ϵ) satisfying $R_k - \delta \le (\log M_k)/N$ for $k=1,2$.

Let us define $T' = \max\{R_1/\delta, R_2/\delta, N_0 T_0\}$. Then, for each $T \ge T'$, $T \in [N_T T_0, (N_T+1)T_0)$ for some $N_T \ge N_0$ and there is a corresponding $(N_T, M_1, M_2, \epsilon)$ satisfying

$$R_k - \delta \le \frac{\log M_k}{N_T}, \qquad k=1,2. \tag{34}$$

For each codeword $(x_{k1},\cdots,x_{kN_T})$, of the kth user in the $(N_T, M_1, M_2, \epsilon)$ code, we define

$$x_k(t) = \sum_{i=1}^{N_T} x_{ki}^T \Phi(t-(i-1)T_0, T_0), \tag{35}$$

where $\Phi(t,T_0) = [\phi_1(t,T_0) \cdots \phi_J(t,T_0)]^T$ and $\phi_j(t,T_0)$ is defined by (18). Since $\{\phi_j(t,T_0)\}$ forms an orthonormal set, it is easy to check, using (29)–(31) and (35), that $x_k(t)$ satisfies the power of RMS bandwidth constraints (5)–(7). Therefore, $x_k(t)$ is a codeword of the kth user in the RMS-bandlimited channel. We decode the output $y(t)$ by forming y_i from

$$y_i = \int_{(i-1)T_0}^{iT_0} y(t)\Phi(t-(i-1)T_0, T_0)\,dt, \tag{36}$$

and apply the decoder of the $(N_T, M_1, M_2, \epsilon)$ code on y_i.

Since (36) can be written, using (4) and (35), as

$$y_i = x_{1i} + x_{2i} + n_i, \tag{37}$$

where

$$n_i = \int_{(i-1)T_0}^{iT_0} n(t)\Phi(t-(i-1)T_0, T_0)\,dt, \tag{38}$$

which can be shown to be white Gaussian with covariance, $(N_0/2)I_J$, the error probability is the same as the error probability of the discrete-time channel.

Hence, for each $T \ge T'$, using the $(N_T, M_1, M_2, \epsilon)$ code, we have constructed a (T, M_1, M_2, ϵ) code for the RMS-bandlimited channel. Following from (34), the rate of the (T, M_1, M_2, ϵ) code satisfies

$$\begin{aligned} \frac{\log M_k}{T} &\ge (R_k - \delta)\frac{N_T}{T} \\ &\ge \frac{R_k N_T}{(N_T+1)T_0} - \frac{\delta}{T_0} \\ &\ge \frac{R_k}{T_0}\left(1 - \frac{T_0}{T'}\right) - \delta \\ &\ge \frac{R_k}{T_0} - 2\delta, \qquad \text{for } k=1,2. \end{aligned} \tag{39}$$

Therefore, $(R_1/T_0, R_2/T_0) \in C_U(B,\Lambda_1,\Lambda_2)$.

Since J and T_0 are arbitrarily chosen and $C_D(J,T_0)$ is monotonically increasing in J, we have,

$$C_U(B,\Lambda_1,\Lambda_2) \supseteq \limsup_{T_0\to\infty} \bigcup_{\substack{(\lambda_{k1},\lambda_{k2},\cdots): \\ \frac{1}{2BT_0}\sum_{j=1}^{\infty}\lambda_{kj}=\Lambda_k \\ \frac{1}{(2BT_0)^3}\sum_{j=1}^{\infty}\lambda_{kj}j^2\le\Lambda_k \\ 0\le\lambda_{kj}\; j=1,2,\cdots,\infty \\ k=1,2.}} \left\{ (R_1,R_2):\; \begin{array}{l} 0\le R_1\le \frac{1}{2T_0}\sum_{j=1}^{\infty}\log\left[1+\lambda_{1j}\right] \\ 0\le R_2\le \frac{1}{2T_0}\sum_{j=1}^{\infty}\log\left[1+\lambda_{2j}\right] \\ R_1+R_2\le \frac{1}{2T_0}\sum_{j=1}^{\infty}\log\left[1+\lambda_{1j}+\lambda_{2j}\right] \end{array} \right\}. \quad (40)$$

Then, (16) follows immediately by applying Lemma 1.

Now, we finish the proof of the theorem by showing the set inclusion of (15). Since (15) is convex, it is sufficient to show that it contains the points $(g(B,\Lambda_1), g(B,\Lambda_1+\Lambda_2) - g(B,\Lambda_1))$ and $(g(B,\Lambda_1+\Lambda_2)-g(B,\Lambda_2), g(B,\Lambda_2))$. By symmetry, we need only to show that one point, say $(g(B,\Lambda_1), g(B,\Lambda_1+\Lambda_2)-g(B,\Lambda_1))$, is contained in (15). This is equivalent to showing that there exists $S_1(w)$ and $S_2(w)$ such that

$$B\int_0^{\infty}\log\left[1+S_1(w)\right]dw = g(B,\Lambda_1), \quad (41)$$

$$B\int_0^{\infty}\log\left[1+S_1(w)+S_2(w)\right]dw = g(B,\Lambda_1,\Lambda_2). \quad (42)$$

It can be shown, using functional optimization techniques, that the $S_1(w), S_2(w)$ satisfying (41) and (42) have the forms

$$S_1(w)=\begin{cases} \dfrac{1}{a_1+b_1w^2}-1, & \text{if } 0\le w\le\Gamma_1 \\ 0, & \text{otherwise,}\end{cases} \quad (43)$$

and

$$S_1(w)+S_2(w)=\begin{cases} \dfrac{1}{a_T+b_Tw^2}-1, & \text{if } 0\le w\le\Gamma_T, \\ 0, & \text{otherwise,}\end{cases} \quad (44)$$

where a_1, b_1, Γ_1 and a_T, b_T, Γ_T are constants depending on the signal-to-noise ratios Λ_1 and $\Lambda_1+\Lambda_2$, respectively. Therefore, there exists $(S_1(w), S_2(w))$ satisfying (41) and (42) if $\Gamma_1\le\Gamma_T$ and

$$\frac{1}{a_1+b_1w^2}-\frac{1}{a_T+b_Tw^2}\le 0, \qquad \forall w\in[0,\Gamma_1]. \quad (45)$$

Now we show that $a_T\le a_1$ and $\Gamma_1\le\Gamma_T$ are sufficient conditions for (45). Since $a_1+b_1\Gamma_1^2=a_T+b_T\Gamma_T^2=1$ and $\Gamma_1\le\Gamma_T$, we have $(a_1+b_1\Gamma_1^2)^{-1}\le(a_T+b_T\Gamma_1^2)^{-1}$. Also, the left-hand side of (45) is a monotonic function, and is less than 0 when either $w=0$ or $w=\Gamma_1$; therefore, (45) must hold for all $w\in[0,\Gamma_1]$.

Since signal-to-noise ratios Λ_1 and Λ_2 are arbitrary positive numbers, in order to complete the proof of the theorem, we need to show that $0\le d\Gamma/d\Lambda$ and $da/d\Lambda\le 0$ or, equivalently, $d\Gamma/da\le 0$ and $d\Lambda/da\le 0$. Obtaining the relationships between a, b, and Γ using functional optimization techniques and expressing Λ and Γ in terms of a, we have

$$\Lambda=\frac{\left[\arctan\left[\sqrt{\dfrac{1-a}{a}}\right]-\sqrt{a(1-a)}\right]^{3/2}}{\sqrt{\dfrac{1}{3}a(a+2)\sqrt{a(1-a)}-a^2\arctan\left[\sqrt{\dfrac{1-a}{a}}\right]}} \quad (46)$$

and

$$\Gamma=\left\{\frac{(1-a)\left[\arctan\left[\sqrt{\dfrac{1-a}{a}}\right]-\sqrt{a(1-a)}\right]}{\dfrac{1}{3}(a+2)\sqrt{a(1-a)}-a\arctan\left[\sqrt{\dfrac{1-a}{a}}\right]}\right\}^{1/2}. \quad (47)$$

After taking the derivatives and some tedious calculations, one can show that the derivatives, $d\Lambda/da$ and $d\Gamma/da$, are both negative, and the proof is completed. □

In the previous proof, it appears that it would be problematic to apply the Karhunen–Loeve expansion, the usual tool in dealing with waveform channels (e.g., [19, Ch. 8]), since the role of the integral operator therein would be taken by the inverse Fourier transform of the function f^2, $-\infty<f<\infty$, which does not exist.

Notice that although the capacity region is a pentagon, no $(S_1(w), S_2(w))$ maximizes the three constraints simultaneously in contrast to the strictly bandlimited case.

The following result solves parametrically the optimization problem in (9) and gives a convenient, easy-to-compute expression for $C_U(B,\Lambda)$.

Theorem 2: The capacity of the single-user RMS-bandlimited while Gaussian channel with noise power spectral density equal to $N_0/2$ and signal power and RMS bandwidth less than or equal to W and B, respectively, is given

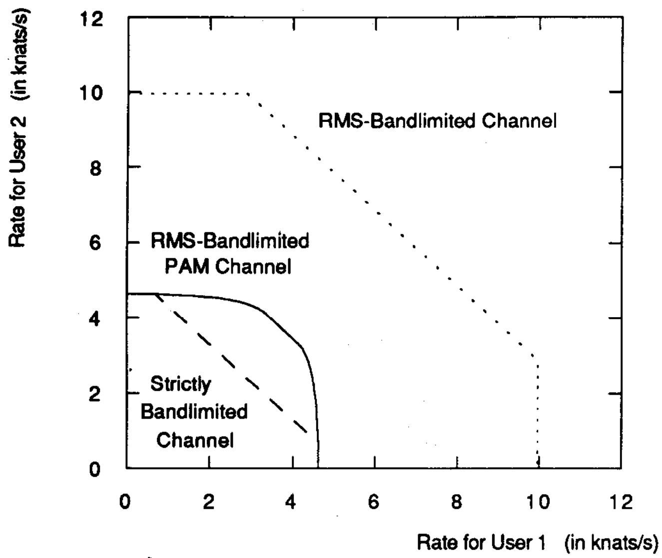

Fig. 1. Capacity regions of channels under different constraints with SNR = 20 db and $B = 1$ khz.

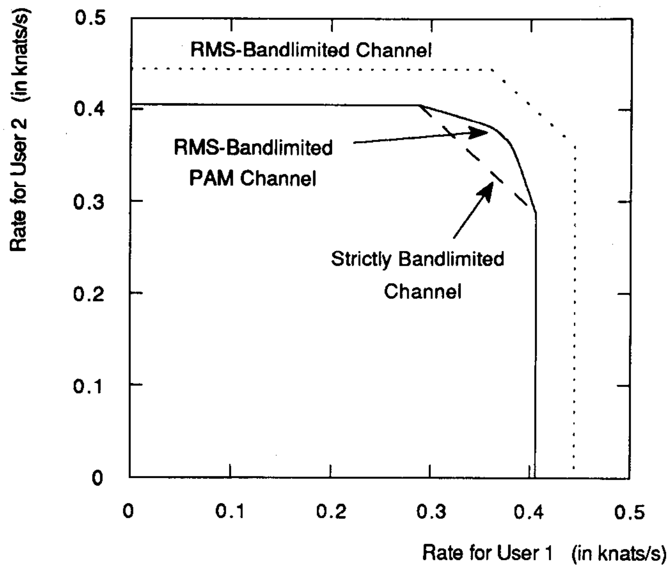

Fig. 2. Capacity regions of channels under different constraints with SNR = −3 db and $B = 1$ khz.

by

$$C_U(B,\Lambda) = B\Lambda\left[1 - \frac{(\Gamma^{*2}-3)\Lambda}{\frac{2}{3}\Gamma^{*3} - \Lambda + \Gamma^{*2}\Lambda}\right]\log e, \quad (48)$$

where Γ^* is the unique solution of the equation

$$0 = f(\Lambda,\Gamma) \triangleq \frac{\Gamma^2\Lambda + \frac{2}{3}\Gamma^3 - \Lambda}{\sqrt{\Lambda\left(\frac{2}{3}\Gamma^3 - \Lambda\right)}}\arctan\left[\sqrt{\frac{\Lambda\Gamma^2}{\frac{2}{3}\Gamma^3 - \Lambda}}\right] - \Lambda - \Gamma. \quad (49)$$

Moreover, $f(\Lambda,\Gamma) > 0$ for $\Gamma \in ((3\Lambda/2)^{1/3}, \Gamma^*)$, and $f(\Lambda,\Gamma) < 0$ for $\Gamma \in (\Gamma^*, \infty)$.

Proof: The proof is a straightforward application of functional optimization techniques. □

Remarks:

- This parametric solution involves only one parameter that is a *unique* solution of $f(\Lambda,\Gamma)$. Since the solution is unique and the signs of $f(\Lambda,\Gamma)$ are known for $\Gamma > \Gamma^*$ and $\Gamma < \Gamma^*$, Γ^* can be computed efficiently.
- Although no operational significance had been shown before for (9), a slightly different parametric solution to the functional optimization therein was obtained in [20]–[22]. In contrast to Theorem 2, that solution involves two simultaneous nonlinear equations in two unknowns whose existence and uniqueness are not elucidated in [20]–[22].

In Figs. 1 and 2, we plot the capacity regions of the RMS-bandlimited channels along with the capacity region of the strictly bandlimited channel for different signal-to-noise ratios. In the high signal-to-noise ratio region, there is a significant gap between the total capacities of the strictly bandlimited channel and the RMS-bandlimited channel. In the low signal-to-noise ratio case, the capacity regions are very close to each other in ratio and resemble rectangles. This is because the underlying noise is the major channel impairment at low signal-to-noise ratios, and the difference in bandwidth constraints has virtually no effect on the capacity unless the signal-to-noise ratio is moderately high.

From Theorem 2 or the results in [20], [22], it follows immediately that

$$\lim_{\Lambda\to 0}\frac{C_U(B,\Lambda)}{\Lambda} = B\log e \quad (50)$$

$$\lim_{\Lambda\to\infty}\frac{C_U(B,\Lambda)}{(\Lambda)^{1/3}} = (12)^{1/3}B\log e = 2.29B\log e \quad (51)$$

$$\lim_{B\to\infty} C_U\left(B, \frac{W}{N_0 B}\right) = \frac{W}{N_0}\log e, \quad (52)$$

which show that the low SNR behavior and the wideband behavior of C_U and C_C coincide, whereas C_U grows with the cubic root of the SNR asymptotically.

III. RMS-Bandlimited PAM Channel

In this section, we consider the RMS-bandlimited PAM channel in which the input is constrained to be a pulse amplitude modulated signal using an RMS-bandlimited finite duration pulse. We find the capacity region for the two-user RMS-bandlimited PAM channel and the pulses (or *signature waveforms*) which achieve the region boundary. Since linear combinations of nonoverlapping shifts of a pulse have the same RMS bandwidth as the pulse itself, the transmitted signal is also RMS-bandlimited and the single-user RMS-bandlimited PAM channel is a special case of the single-user RMS-bandlimited channels discussed in the last section. In the two-user channel, the sum of the users' codewords is not necessary RMS-bandlimited. Therefore, the two-user RMS-bandlimited PAM channel is not a special case of the two-user RMS-bandlimited channel. However, since the users are independent, it can be justified by noting that the power spectral density of the transmitted signal can be shown to equal to a weighted sum of the power spectral densities of the users' pulses.

The RMS-bandlimited PAM Gaussian channel differs from the classical strictly bandlimited Gaussian channel in that the allowable input signals 1) are more structured (PAM) and 2) are not strictly bandlimited. The fact that the allowable input signals are linear combinations of nonoverlapping pulses makes them easier to generate than the approximations to the strictly bandlimited signals. It turns out that, in the single-user case, the effect of the laxer bandwidth measure cancels the effect of the additional structure imposed on the signals in the time domain, and the capacity of the channel coincides with the celebrated Shannon formula.

In the RMS-bandlimited PAM multiple-access channel, the kth user is assigned a fixed deterministic pulse (or signature waveform), $s_k(t)$, which is time-limited to $[0,T]$ and is modulated by the information stream. The transmitted signals are superimposed and corrupted by an additive white Gaussian noise. Assuming that the transmitters are symbol-synchronous, the two-user RMS-bandlimited PAM channel can be expressed as

$$y(t)=\sum_{i=1}^{n} b_1(i)s_1(t-iT)+b_2(i)s_2(t-iT)+n(t), \tag{53}$$

where $n(t)$ is white Gaussian noise with spectral density $N_0/2$ and $\{b_k(i)\}$ is the symbol stream transmitted by the kth user. The input waveform of the kth user is power-constrained by W_k while all the signature waveforms are RMS-bandlimited by B. Assuming that, without loss of generality, the signature waveforms have unit energy, the power constraints become

$$\frac{1}{nT}\int_T^{(n+1)T}\left[\sum_{i=1}^{n} b_k(i)s_k(t-iT)\right]^2 dt = \frac{1}{nT}\sum_{i=1}^{n} b_k^2(i)\le W_k, \quad k=1,2, \tag{54}$$

and the RMS bandwidth constraints become

$$\int_{-\infty}^{\infty} f^2|S_k(f)|^2\,df=\frac{1}{(2\pi)^2}\int_0^T\left[\frac{d}{dt}s_k(t)\right]^2 dt\le B^2, \quad k=1,2, \tag{55}$$

where $S_k(f)$ is the spectrum of $s_k(t)$.

We first consider the situation when the symbol period and the signature waveforms are fixed, and then we will optimize the capacity regions with respect to those degrees of freedom. It is easy to see that if $s_1(t)=s_2(t)$, the capacity region is equal to the Cover–Wyner pentagon [23], [24]:

$$C_W=\left\{(R_1,R_2):\ \begin{array}{l} 0\le R_1\le \frac{1}{2T}\log\left[1+\frac{2W_1T}{N_0}\right]\\ 0\le R_2\le \frac{1}{2T}\log\left[1+\frac{2W_2T}{N_0}\right]\\ R_1+R_2\le \frac{1}{2T}\log\left[1+\frac{2W_1T+2W_2T}{N_0}\right]\end{array}\right\} \tag{56}$$

in information units per second. (This result remains true even if the users are completely asynchronous [25].) When the signature waveforms are not necessarily identical, the Cover–Wyner pentagon generalizes to [25], [26]

$$C_V=\left\{(R_1,R_2):\ \begin{array}{l} 0\le R_1\le \frac{1}{2T}\log\left[1+\frac{2W_1T}{N_0}\right]\\ 0\le R_2\le \frac{1}{2T}\log\left[1+\frac{2W_2T}{N_0}\right]\\ R_1+R_2\le \frac{1}{2T}\log\left[1+\frac{2W_1T+2W_2T}{N_0}+\frac{4W_1W_2T^2}{N_0^2}(1-\rho^2)\right]\end{array}\right\} \tag{57}$$

in information units per second, where $\rho=\int_0^T s_1(t)s_2(t)\,dt$ is the cross-correlation between the signature waveforms. Notice that, for fixed T, the capacity region, C_V, is monotonically decreasing in the absolute cross-correlation $|\rho|$ and is maximized when $\rho=0$ (i.e., orthogonal signature waveforms.) However, under the bandwidth constraints, orthogonality between the signature waveforms cannot always be achieved for the given value of T. Hence, we will find the capacity region for the two-user RMS-bandlimited PAM channel in two stages.

1) Fix T, and find $\rho^*(TB)$, the minimum absolute cross-correlation $|\rho|$ achievable under the time-bandwidth constraint. Then, the capacity region for fixed T is given by C_V in (57) evaluated at $\rho=\rho^*(TB)$. This is because C_V depends on the signature waveforms only through the rate-sum constraint which is monotonically decreasing in $|\rho|$.
2) Take the union of the capacity regions found in the first stage over all T. Note that there is a minimum value of T below which the time-bandwidth product is so small that no waveform can be found to satisfy

the bandwidth constraint. Also, there is a maximum value of T above which the allowed time-bandwidth product is so large that orthogonal signals can be assigned to both users, and therefore the capacity region decreases with T beyond that maximum value of T.

Theorem 3: If $TB \geq 0.5$, then the minimum absolute cross-correlation $\rho^*(TB)$ between any two unit-energy signals of duration T and RMS bandwidth less than or equal to B is

$$\rho^*(TB) = \max\left\{0, \frac{1}{3}\left[5-8(TB)^2\right]\right\}, \tag{58}$$

and is achieved by the signature waveforms

$$s_1(t) = \sqrt{\frac{1+\rho^*(TB)}{T}}\sin\frac{\pi t}{T} + \sqrt{\frac{1-\rho^*(TB)}{T}}\sin\frac{2\pi t}{T} \tag{59}$$

$$s_2(t) = \sqrt{\frac{1+\rho^*(TB)}{T}}\sin\frac{\pi t}{T} - \sqrt{\frac{1-\rho^*(TB)}{T}}\sin\frac{2\pi t}{T}. \tag{60}$$

If $TB < 0.5$, then there exists no signal of duration T and RMS bandwidth less than or equal to B.

Proof: It is known [6] that the unique signal of duration T with minimum RMS bandwidth $1/(2T)$ is $\phi_1(t,T)$ defined in (18). Therefore, the theorem follows immediately when $TB < 0.5$ and $TB = 0.5$.

If $TB > 0.5$, let $s_1(t)$, $s_2(t)$ be any two distinct unit-energy signals with duration T and RMS bandwidth B. Using the same complete orthonormal set $\{\phi_i(t,T)\}_{i=1}^{\infty}$, we denote the vector $\boldsymbol{\Phi}(t,T) = [\phi_1(t,T)\phi_2(t,T)\cdots]^T$, and express $s_1(t)$ and $s_2(t)$ as

$$s_k(t) = \boldsymbol{a}_k^T\boldsymbol{\Phi}(t,T), \qquad k=1,2. \tag{61}$$

From the unit energy assumption, we can write the cross-correlation matrix $\boldsymbol{H}$ as

$$\boldsymbol{H} \triangleq AA^T \triangleq \begin{bmatrix} \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \end{bmatrix}\begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \tag{62}$$

where ρ denotes the crosscorrelation between $s_1(t)$ and $s_2(t)$.

We find the minimum value of $|\rho|$ by first giving a lower bound on the cross-correlation and then showing that the lower bound is achievable. Let B_a be the minimum of the average RMS bandwidth of M equal energy signals of duration T and correlation matrix, $\boldsymbol{H}$. B_a was found by Nuttall [11]:

$$B_a^2 = \frac{1}{(2T)^2}\frac{1}{M}\sum_{l=1}^{r}\mu_l l^2, \tag{63}$$

where each μ_l is the positive eigenvalue of $\boldsymbol{H}$ with $\mu_i \leq \mu_j$ for $j \leq i$, and r is the rank of $\boldsymbol{H}$.

Applying this result with $M=2$, $r=2$ (since $s_1(t) \neq s_2(t)$ implies $\rho \neq 1$) and the correlation matrix $\boldsymbol{H}$ in (62), we have

$$\frac{1}{2(2T)^2}\left[(1+|\rho|)+4(1-|\rho|)\right] = B_a^2 \leq B^2, \tag{64}$$

because the average RMS bandwidth B_a is always less than the maximum RMS bandwidth, B. After rearrangement, (64) becomes

$$\frac{1}{3}\left[5-8(TB)^2\right] \leq |\rho|. \tag{65}$$

Since $s_1(t)$ and $s_2(t)$ are arbitrarily chosen, we have the following lower bound on the absolute cross-correlation:

$$\max\left\{0, \frac{1}{3}\left[5-8(TB)^2\right]\right\} \leq |\rho|. \tag{66}$$

We now show a signal pair that achieves this lower bound. Using the representation in (61), we can express the RMS bandwidth constraint, via (55), as

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\left[\frac{d}{dt}s_k(t)\right]^2 dt = \frac{1}{(2T)^2}\boldsymbol{a}_k^T\boldsymbol{\Pi}\boldsymbol{a}_k \leq B^2, \qquad k=1,2. \tag{67}$$

where $\boldsymbol{\Pi}$ is a diagonal matrix with the jth diagonal entry $\boldsymbol{\Pi}_{jj} = j^2$. Prompted by the fact that the functions $f(t)$ and $f(T-t)$ have the same magnitude spectrum, we consider signature waveforms which are mirror images of each other about $T/2$. Also, we note that $\sin(\pi t/T)$ is even about $T/2$ while $\sin(2\pi t/T)$ is odd about $T/2$. Therefore, we select $s_1(t)$ and $s_2(t)$ such that the matrix A has the form

$$A = \begin{bmatrix} \alpha & \sqrt{1-\alpha^2} & 0 & \cdots \\ \alpha & -\sqrt{1-\alpha^2} & 0 & \cdots \end{bmatrix} \tag{68}$$

for some $0 \leq \alpha \leq 1$ to be specified next.

Accordingly to (62), we have $\rho = 2\alpha^2 - 1$ and (67) constrains the choice of α to

$$\frac{4-4(TB)^2}{3} \leq \alpha^2. \tag{69}$$

The value of α^2 that minimizes $|2\alpha^2-1|$ and is consistent with (69) is equal to

$$\alpha^2 = \max\left\{\frac{1}{2}, \frac{4-4(TB)^2}{3}\right\} = \frac{1}{2}\left[\rho^*(TB)+1\right], \tag{70}$$

which, upon substitution on (68), results in a choice of signals which satisfies (66) with equality.

Finally, particularizing the matrix A to the value of α^2 in (70), we obtain the optimal signature waveforms in the theorem. □

Theorem 4: The capacity region of the two-user PAM white Gaussian multiple-access channel, with noise power spectral density equal to $N_0/2$, signal powers and RMS bandwidth less than or equal to W_1, W_2, and B, respectively, is given by

$$C_S(B,\Lambda_1,\Lambda_2) = \bigcup_{1\le\gamma\le\sqrt{\frac{5}{2}}} \left\{ (R_1,R_2): \begin{array}{l} 0 \le R_1 \le \dfrac{B}{\gamma}\log[1+\Lambda_1\gamma] \\ 0 \le R_2 \le \dfrac{B}{\gamma}\log[1+\Lambda_2\gamma] \\ R_1+R_2 \le \dfrac{B}{\gamma}\log\left[1+(\Lambda_1+\Lambda_2)\gamma+\Lambda_1\Lambda_2\gamma^2\left(1-\dfrac{4}{9}\left(\dfrac{5}{2}-\gamma^2\right)^2\right)\right] \end{array} \right\}, \quad (71)$$

where $\Lambda_k = W_k/(BN_0)$.

Corollary 2: The capacity of the single-user PAM white Gaussian channel, with noise power spectral density equal to $N_0/2$, signal power and RMS bandwidth less than or equal to W and B, respectively, is given by

$$C_S(B,\Lambda) = B\log\left[1+\frac{W}{BN_0}\right] = B\log[1+\Lambda], \quad (72)$$

where $\Lambda = W/(BN_0)$.

Remark: The capacity of the single-user RMS-bandlimited PAM channel coincides with the strictly bandlimited channel capacity. However, they are two very different channels: the classical channel has a stricter frequency domain constraint while the PAM channel has a stricter time domain constraint.

Proof: Recall that the capacity region $C_S(B,\Lambda_1,\Lambda_2)$ is the union of C_V evaluated at $\rho^*(TB)$ over T. We proceed to find the range of T of interest. From the last theorem, if $TB < 0.5$, no signature waveforms can be found to satisfy the constraints and the capacity region is an empty set. Also, if $TB \ge \sqrt{5/8}$, $\rho^*(TB) = 0$, and the capacity region for fixed T is a pentagon that is monotonically decreasing in T. Therefore, the range of T in interest is the interval $[1/(2B), 1/(2B)\sqrt{5/2}]$. Denoting $2TB$ by γ, and substituting γ into C_V, we have, after taking the union, $C_S(B,\Lambda_1,\Lambda_2)$ in the theorem. □

Fig. 3 compares the capacities of the RMS-bandlimited channels with and without the PAM structure. It indicates that the PAM structural (with nonoverlapping pulses) constraint has little effect on the capacity at low signal-to-noise ratios, but decreases the capacity severely at high signal-to-noise ratios.

Fig. 4 shows the capacity regions on the RMS-bandlimited PAM channel, $C_S(B,\Lambda_1,\Lambda_2)$, and the strictly bandlimited channel, $C_C(B,\Lambda_1,\Lambda_2)$, in (2). In contrast to the single-user case where they coincide, $C_C(B,\Lambda_1,\Lambda_2)$ is a subset of $C_S(B,\Lambda_1,\Lambda_2)$. It can also be seen from (71) and (2) that $C_C(B,\Lambda_1,\Lambda_2)$ is the pentagon inside the union in (71) when $\gamma = 1$. However, by increasing γ, we tradeoff the decrease in the single-user rate by the increase in the rate sum, such that the union gives a larger capacity region, $C_S(B,\Lambda_1,\Lambda_2)$. This indicates that, in the two-user case, the laxer bandwidth constraint more than offsets the additional structure (PAM) in the time domain.

At first glance, it seems that there is a conflict between Theorem 4 and Corollary 2 since the total capacity of $C_S(B,\Lambda_1,\Lambda_2)$ is larger than the capacity of the single-user RMS-bandlimited PAM channel with signal-to-noise ratio equal to $\Lambda_1+\Lambda_2$. However, the combined signal transmitted over the channel in the two-user case is a sum of two PAM signals and, in general, it cannot be viewed as a PAM signal since the signals in different time slots need not have the same shape.

The set of values of γ that achieves boundary points of $C_S(B,\Lambda_1,\Lambda_2)$ is a subset of $[1,\sqrt{5/2}]$, which depends on the signal-to-noise ratios, Λ_1 and Λ_2. According to Fig. 4, the boundary points in the segments AB and EF are achieved by $\gamma = 1$, while those in the segment CD are achieved by some $\gamma_{\max}$ in $[1,\sqrt{5/2}]$ depending on the signal-to-noise ratio. The boundary points in BC and DE are achieved by $1 \le \gamma \le \gamma_{\max}$.

Figs. 1 and 2 show the strictly bandlimited channel and the RMS bandlimited channels with and without the PAM structure under different signal-to-noise ratios. The observation that the PAM constraint reduces substantially the capacity at moderately high signal-to-noise ratios suggests the following conclusion. Considering that the PAM signal is formed by sending a signal from a one dimensional space every T seconds, the total capacities of the RMS-bandlimited channels with and without the PAM structure can be viewed as the single-user capacities of channels with a signal space of two and infinite dimensions, respectively. Hence, the gap in Fig. 1 suggests that increasing the number of degrees of freedom (or dimensions) of the signal set raises the capacity quite significantly. Hence, under the RMS bandwidth constraint, it is advantageous to design signal sets with many dimensions.

Finally, Figs. 5 and 6 show the signature waveforms which achieve the boundary points of the capacity region for two different time-bandwidth products. The signature waveforms are mirror images of each other and as γ increases, they become more asymmetric so as to decrease the cross-correlation while maintaining the same RMS bandwidth.

IV. Conclusion

In this paper, we have found the capacities (capacity regions) of the single-user (two-user) RMS-bandlimited Gaussian channels with and without the PAM structure. The single-user capacity of the RMS-bandlimited PAM channel coincides with the Shannon formula. This illustrates a tradeoff between the time domain and the frequency domain constraints. However, in the two-user

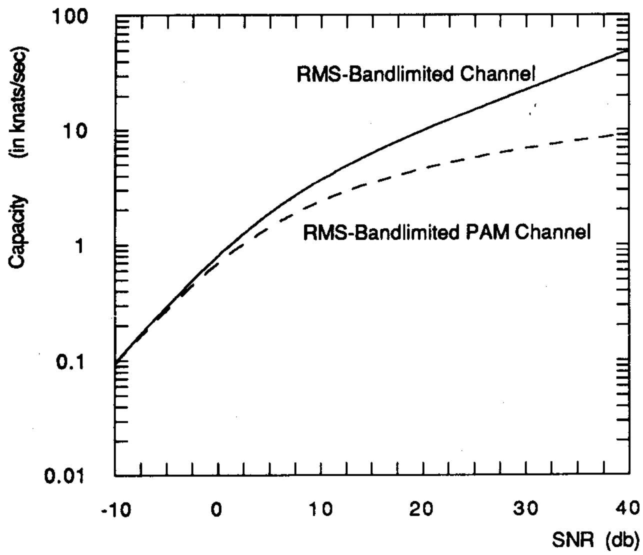

Fig. 3. Capacities of RMS-bandlimited and RMS-bandlimited PAM channels with $B = 1$ khz.

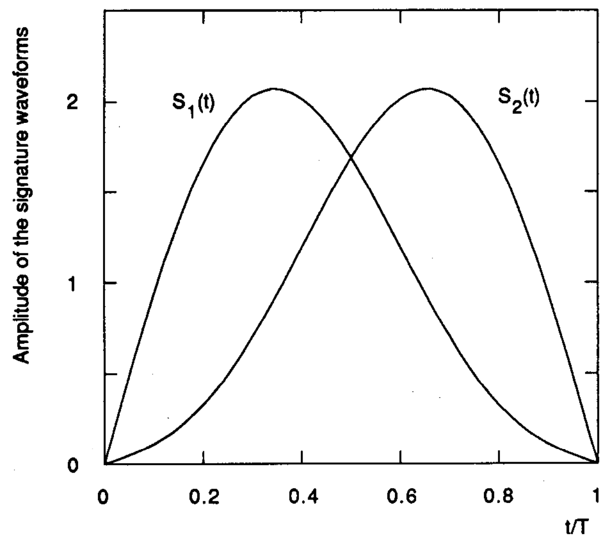

Fig. 5. Signature waveforms for two-user RMS-bandlimited PAM channel, $TB = 0.6$.

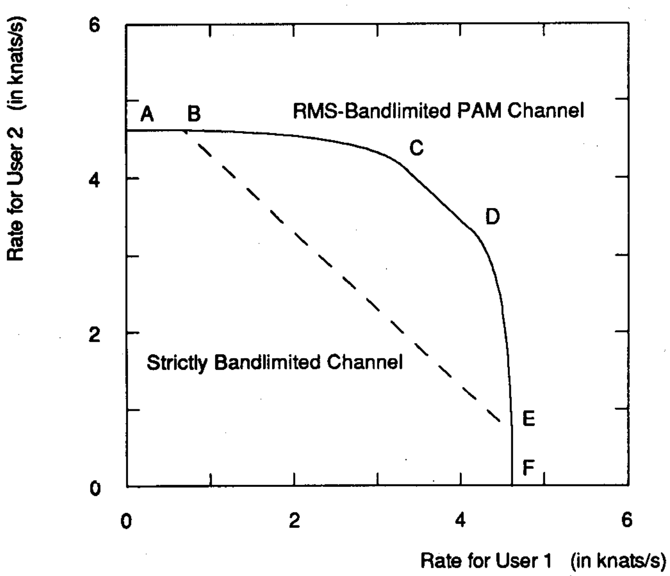

Fig. 4. Capacity regions of RMS-bandlimited PAM channel and strictly bandlimited channel with SNR = 20 db and $B = 1$ khz.

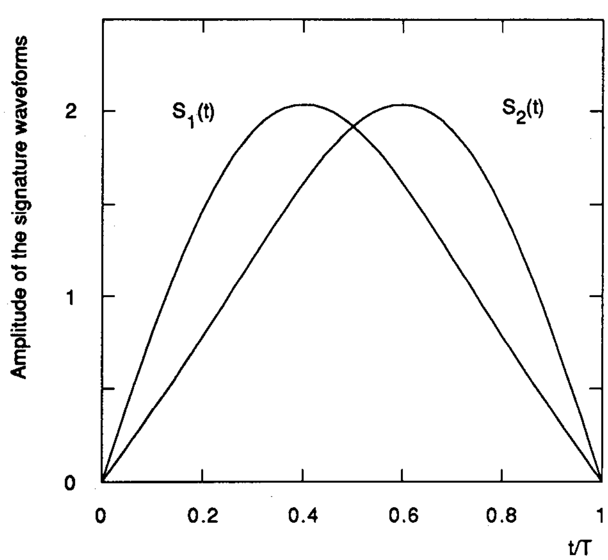

Fig. 6. Signature waveforms for two-user RMS-bandlimited PAM channel, $TB = 0.525$.

case, the laxer RMS bandwidth constraint more than offsets the PAM structure in the time domain, and the capacity region of the RMS-bandlimited PAM channel is larger than the strictly bandlimited channel capacity region.

We also consider the RMS-bandlimited channel without the PAM constraint. This channel can be viewed as the classical channel with the strictly bandlimited constraint replaced by the RMS-bandlimited constraint. The two-user capacity region is found to be a pentagon and its evaluation boils down to the computation of single-user capacity; unlike the strictly bandlimited case, no input distribution achieves all rate pairs simultaneously.

At high signal-to-noise ratios, the PAM structure decreases the capacity significantly. On the other hand, comparing to the strictly bandlimited channel, the capacity of the bandlimited channel is very sensitive to the bandwidth definitions. In particular, the RMS-bandlimited channel capacity admits an asymptotic growth rate proportional to the cubic root of the signal-to-noise ratio, as compared to the logarithmic growth rate in the strictly bandlimited case.

At small signal-to-noise ratios, the capacities of all these channels are almost the same. This is because the effect of the noise dominates and it emerges as the main factor in determining the capacities.

Appendix

Proof of Lemma 1: Note that, for each T_0,

$$\sup_{\substack{(\lambda_1,\lambda_2,\cdots): \\ \frac{1}{2BT_0}\sum_{j=1}^{\infty}\lambda_j=\Lambda \\ \frac{1}{(2BT_0)^3}\sum_{j=1}^{\infty}\lambda_j j^2\le\Lambda \\ 0\le\lambda_j\ j=1,2,\cdots,\infty}} \frac{1}{2T_0}\sum_{j=1}^{\infty}\log\left[1+\lambda_j\right] = \sup_{\substack{\int_0^\infty S(w)\,dw=\Lambda \\ \int_0^\infty S(w)G(T_0,w)\,dw\le\Lambda \\ 0\le S(w)}} B\int_0^\infty \log\left[1+S(w)\right]dw, \tag{A.1}$$

where $G(T_0,w)=[j/(2BT_0)]^2$ if $w\in((j-1)/(2BT_0),j/(2BT_0)]$. This is a consequence of the fact that

$$M(\lambda_j)\triangleq \max_{\substack{\int_{\frac{j-1}{2BT_0}}^{\frac{j}{2BT_0}} S(w)\,dw=\frac{\lambda_j}{2BT_0} \\ 0\le S(w)}} B\int_{\frac{j-1}{2BT_0}}^{\frac{j}{2BT_0}}\log\left[1+S(w)\right]dw \tag{A.2}$$

is maximized when $S(w)$ is constant over $(j-1/2BT_0,\ j/2BT_0]$.

So, to complete the proof, it is sufficient to show

$$\lim_{T_0\to\infty}\sup_{\substack{\int_0^\infty S(w)\,dw=\Lambda \\ \int_0^\infty S(w)G(T_0,w)\,dw\le\Lambda \\ 0\le S(w)}} B\int_0^\infty\log\left[1+S(w)\right]dw = \sup_{\substack{\int_0^\infty S(w)\,dw=\Lambda \\ \int_0^\infty S(w)w^2\,dw\le\Lambda \\ 0\le S(w)}} B\int_0^\infty\log\left[1+S(w)\right]dw. \tag{A.3}$$

Let $A(T_0)$ and A be the sets of $S(w)$ satisfying the constraints in the optimization problem in the left-hand side and right-hand side of (A.3) respectively. Since $w^2\le G(T_0,w)$ for all w, it is clear that for all T_0, $A(T_0)\subset A$, and

$$\lim_{T_0\to\infty}\sup_{\substack{\int_0^\infty S(w)\,dw=\Lambda \\ \int_0^\infty S(w)G(T_0,w)\,dw\le\Lambda \\ 0\le S(w)}} B\int_0^\infty\log\left[1+S(w)\right]dw \le \sup_{\substack{\int_0^\infty S(w)\,dw=\Lambda \\ \int_0^\infty S(w)w^2\,dw\le\Lambda \\ 0\le S(w)}} B\int_0^\infty\log\left[1+S(w)\right]dw. \tag{A.4}$$

Notice that for any $S(w)\in A$,

$$\lim_{T_0\to\infty}\int_0^\infty S(w)G(T_0,w)\,dw=\int_0^\infty S(w)w^2\,dw, \tag{A.5}$$

because

$$\left|\int_0^\infty S(w)G(T_0,w)\,dw-\int_0^\infty S(w)w^2\,dw\right|$$

$$=\int_0^\infty S(w)G(T_0,w)\,dw-\int_0^\infty S(w)w^2\,dw \tag{A.6}$$

$$\le\sum_{j=1}^{\infty}\int_{\frac{j-1}{2BT_0}}^{\frac{j}{2BT_0}}S(w)\frac{j^2-(j-1)^2}{(2BT_0)^2}\,dw \tag{A.7}$$

$$\le\frac{1}{BT_0}\int_0^\infty S(w)w\,dw+\frac{1}{(2BT_0)^2}\int_0^\infty S(w)\,dw, \tag{A.8}$$

which goes to zero as $T_0\to\infty$.

For every nonzero element, $S(w)\in A$, since w^2 can be arbitrarily small in $(0,\infty)$, there exists a $S'(w)\in A$ such that

$$\int_0^\infty S'(w)w^2\,dw<\int_0^\infty S(w)w^2\,dw. \tag{A.9}$$

For each T_0, we define

$$\alpha_{T_0}\triangleq\frac{\int_0^\infty S(w)w^2\,dw-\int_0^\infty S'(w)G(T_0,w)\,dw}{\int_0^\infty S(w)G(T_0,w)\,dw-\int_0^\infty S'(w)G(T_0,w)\,dw} \tag{A.10}$$

and let

$$S_{T_0}(w)\triangleq\alpha_{T_0}S(w)+(1-\alpha_{T_0})S'(w). \tag{A.11}$$

Particularizing $S'(w)$ in (A.5), it is easy to see that, using (A.9), for T_0 large enough, $0<\alpha_{T_0}\le 1$ and $0\le S_{T_0}(w)$. Moreover, as $T_0\to\infty$, $\alpha_{T_0}\to 1$, $S_{T_0}(w)\to S(w)$ and

$$\lim_{T_0\to\infty}B\int_0^\infty\log\left[1+S_{T_0}(w)\right]dw=B\int_0^\infty\log\left[1+S(w)\right]dw. \tag{A.12}$$

From the definition of $S_{T_0}(w)$, it is easy to check that, for T_0 large enough, $S_{T_0}(w) \in A(T_0)$. Since $S(w)$ is arbitrarily chosen, we have

$$\sup_{\substack{\int_0^\infty S(w)\,dw = \Lambda \\ \int_0^\infty S(w) w^2\,dw \le \Lambda \\ 0 \le S(w)}} B\int_0^\infty \log[1+S(w)]\,dw \le \lim_{T_0\to\infty} \sup_{\substack{\int_0^\infty S(w)\,dw = \Lambda \\ \int_0^\infty S(w) G(T_0,w)\,dw \le \Lambda \\ 0 \le S(w)}} B\int_0^\infty \log[1+S(w)]\,dw. \tag{A.13}$$

Combining (A.4) and (A.13), we complete the proof of (A.3) and Lemma 1. □

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