The Effect of Asynchronism on the Total Capacity of Gaussian Multiple-Access Channels

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Abstract—**The degradation due to complete asynchronism (at the codeword and symbol levels) in the total capacity, maximum rate-sum, of white Gaussian multiple-access channels is investigated. It is shown that asynchronism reduces the total capacity of a K-user channel by at most a factor of K. Moreover, this bound is achieved, in asymptotically high signal-to-noise ratios, by the TDMA signalling strategy. When the signalling strategies are optimally designed to maximize the asynchronous total capacity under bandwidth constraints, we find that in a two-user channel 1) for a certain set of signal-to-noise ratios there is no degradation due to asynchronism, 2) for any bandwidth and signal-to-noise ratios the asynchronous total capacity is at least 88% of the synchronous total capacity, and 3) asynchronism has a vanishing small effect on total capacity for both low and high signal-to-noise ratios.**

Index Terms: **Multiple-access channels, asynchronous channels, code-division multiple-access, time-division multiple-access.**

I. Introduction

THE multiple-access channel is an information-theoretic model of communication systems where several independent users transmit simultaneously to a common receiver. Typical practical multiplexing strategies that fall into that category include TDMA, FDMA, and CDMA (i.e., time, frequency, and code-division multiple-access), depending on whether the users modulate pulse waveforms which are nonoverlapping in the time-domain, in the frequency-domain, or in neither domain, respectively.

A practically and theoretically important question is the degradation in achievable performance caused by the absence of synchronism among the transmitters. This has been investigated in [1], [2] for frame-asynchronous channels and in [3] for completely asynchronous channels (i.e., when synchronism is not assumed at either the codeword or the symbol level). The latter work [3] found a formula for the capacity region of the asynchronous two-user white Gaussian channel for linear modulation of arbitrary pulse waveforms. Using those results, in this paper we quantify the effect of asynchronism on the total capacity (maximum achievable sum of transmission rates) of the white Gaussian multiple-access channel. This measure allows a succinct comparison (cf. [4]–[6] in the case of feedback) of the overall system throughput with and without synchronism.

The comparison of capacities is carried out assuming that the transmitters and the receiver know whether the users maintain synchronism or not, and therefore the coding strategies can be chosen accordingly. Specifically, we do not investigate the effect of asynchronism on coding strategies designed for synchronous channels. Naturally, it is assumed that when choosing their codebooks the asynchronous transmitters have no knowledge of the offsets, or delays, between their data streams. Since reliable communication is required regardless of the actual offsets (in particular even in the case where the users happen to be perfectly aligned), it is clear that asynchronism cannot increase capacity. Notice that this conclusion would not be necessarily true, had we assumed that the transmitters have access to the value of the relative offsets prior to encoding their messages.

If all users modulate the same pulse, then it is known [3, corollary on p. 745] that the synchronous and asynchronous capacities coincide and are given by the Cover–Wyner pentagon. Even though such a modulation choice affords the nice property of complete insensitivity to asynchronism, it also achieves the lowest capacity for given signal-to-noise ratios, as it does not attempt to differentiate the users' pulses for multiplexing purposes.

A multiplexing strategy that does use different pulses for different users is TDMA. It is common that the pulse waveforms assigned to TDMA users are shifted nonoverlapping versions of a common pulse. In such case, we find in Section II that asynchronism degrades the total capacity of the K-user TDMA channel by a factor of K in asymptotically high signal-to-noise ratios. Moreover, we show that no other multiplexing strategy can suffer a higher degradation due to asynchronism. Naturally, symbol-asynchronism plays havoc with TDMA, a strategy predicated on the availability of a common clock for all transmitters; although nonzero transmission rates can still be achieved thanks to the use of appropriate channel coding strategies.

In Section II, we also investigate the effect of asynchronism in both high and low signal-to-noise ratio cases. We show that regardless of the multiplexing strategy, as the signal-to-noise ratios go to zero, asynchronism causes vanish-

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ing relative degradation in total capacity. This is due to the fact that in such an asymptotic scenario, the overwhelmingly dominant factor limiting performance is the background Gaussian noise rather than the interference from other users. On the other hand, as the signal-to-noise ratios go to infinity, we show that if appropriate pulse shapes (specified in Section II) are used, the effect of asynchronism on the total capacity also approaches zero. This is because as long as the pulse waveforms are *linearly independent* for all relative delays, the receiver can distinguish the users and the equivalent signal-to-noise ratio is an *nonzero* constant fraction of the actual signal-to-noise ratio. As the actual signal-to-noise ratios go to infinity, the effect of the nonzero constant factor vanishes since the capacity grows asymptotically as the logarithm of the signal-to-noise ratio.

Ideally, the system designer would like to design a system that combines the advantages of the two aforementioned multiplexing methods: the insensitivity to asynchronism achieved by common-pulse signalling and the orthogonality of synchronous TDMA that leads to the simultaneous achievability of single-user capacities by all users. Even though it is not possible to design pulses that are orthogonal for all possible relative delays, it is indeed possible to design pulses that approach that utopian situation as closely as desired provided that bandwidth is unlimited. For example, in an FDMA system with sufficiently separated frequencies or in a direct-sequence spread spectrum system with very large spreading factors, pulses are quasi-orthogonal for all relative delays. Therefore, if bandwidth is unlimited, total capacity is equal to the sum of single-user capacities even if the channel is asynchronous.

More interesting is the situation studied in Section III, where the effect of asynchronism is studied for signals designed under a bandwidth constraint. Since we assume that the encoders and the decoders know whether the transmitters are synchronized or not, different pulse waveforms can be designed accordingly. We define the root-mean-square (RMS) bandlimited synchronous (asynchronous) capacity as the maximum synchronous (asynchronous) capacity over all pulse waveforms satisfying RMS bandwidth constraint. The two-user RMS bandlimited synchronous capacity and the corresponding optimum time-limited pulse waveforms are found in [7]. It turns out that, in the two-user case, for a certain region of signal-to-noise ratios (which is exactly characterized in Section III) the RMS bandlimited asynchronous capacity coincides with the RMS bandlimited synchronous capacity. Outside that region, the asynchronous capacity is at least 88% of the synchronous capacity.

We also find that the RMS bandlimited total capacity is insensitive to asynchronism in both asymptotically high and asymptotically low signal-to-noise ratios. In asymptotically low signal-to-noise ratios, it is clear that the RMS bandlimited asynchronous capacity approaches the RMS bandlimited synchronous capacity since the effect of asynchronism vanishes regardless of the multiplexing strategy. However, in asymptotically high signal-to-noise ratios, it is not obvious that the effect of asynchronism also vanishes since the aforementioned appropriate pulses may not satisfy the RMS bandwidth constraint. In Section III, we demonstrate the existence of some simple pulses that cause asymptotically small relative asynchronous degradation in high signal-to-noise ratios and satisfy the RMS bandwidth constraint.

II. Channels with Fixed Pulse Shapes

We consider a multiple-access additive white Gaussian noise channel. In this channel, each user is assigned a fixed deterministic pulse which is time-limited to an interval $[0, T]$. The users transmit their information sequences through linear modulation of the assigned pulses and the modulated signals are superimposed and embedded in additive white Gaussian noise. The signal received by the receiver can be expressed as

$$y(t) = \sum_{i=1}^{n} \sum_{k=1}^{K} b_k(i) s_k(t - iT - \tau_k) + n(t), \quad (1)$$

where $s_k(t)$, τ_k, and $\{b_k(i)\}_{i=1}^{n}$ are the assigned pulse, relative delay, and the information symbols of the kth user, and $n(t)$ is white Gaussian noise with spectral density $N_0/2$. When all users are symbol-synchronous (i.e., all relative delays are the same), the channel is referred to as a synchronous channel; otherwise, the channel is referred to as an asynchronous channel. Notice that this model encompasses channels with linear modulation and multiplexing schemes such as TDMA, FDMA or CDMA.

We assume that both the transmitters and the receiver know the users' pulses and whether the users are synchronized or not; therefore, the corresponding coding strategies can be chosen accordingly. However, except in the synchronous channel where all relative delays are the same, the relative delays are known to the receiver (because it can acquire the timing of each signal), but are unknown to the transmitters. It is this fact that makes the asynchronous channel inferior to the synchronous channel in terms of channel capacity for a given set of assigned pulses.

In this section, we assume that the same set of pulses are used in both the synchronous and the asynchronous channels. Without loss of generality, we assume that $0 = \tau_1$ and $\tau_2, \cdots, \tau_K \in [0, T]$. We define, for every $k, l \in \{1, 2, \cdots, K\}$, the cross-correlations

$$\rho_{kl} \triangleq \begin{cases} \int_{-\infty}^{\infty} s_k(t - \tau_k) s_l(t - \tau_l)\, dt, & \text{if } \tau_k \le \tau_l, \\ \int_{-\infty}^{\infty} s_k(t + T - \tau_k) s_l(t - \tau_l)\, dt, & \text{if } \tau_l < \tau_k, \end{cases} \quad (2)$$

and the cross-correlation matrix $\boldsymbol{H}(w)$ is defined by

$$\{\boldsymbol{H}(w)\}_{kl} \triangleq \begin{cases} \rho_{kl} + \rho_{lk} e^{-jw}, & \text{if } \tau_k \le \tau_l, \\ \rho_{lk} + \rho_{kl} e^{jw}, & \text{if } \tau_l < \tau_k. \end{cases} \quad (3)$$

If one rearranges the indexing by a permutation matrix $\boldsymbol{P}$

such that $\tau_{k_1} \le \tau_{k_2} \le \cdots \le \tau_{k_K}$, then

$$P^T H(w) P = \begin{bmatrix} 1 & \rho_{k_1k_2} + \rho_{k_2k_1} e^{-jw} & \cdots & \rho_{k_1k_K} + \rho_{k_Kk_1} e^{-jw} \\ \rho_{k_1k_2} + \rho_{k_2k_1} e^{jw} & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \rho_{k_{K-1}k_K} + \rho_{k_Kk_{K-1}} e^{-jw} \\ \rho_{k_1k_K} + \rho_{k_Kk_1} e^{jw} & \cdots & \rho_{k_{K-1}k_K} + \rho_{k_Kk_{K-1}} e^{jw} & 1 \end{bmatrix}. \tag{4}$$

Also, we define

$$S'_k \triangleq \frac{2W_k T}{N_0}, \qquad k = 1, 2, \cdots, K, \tag{5}$$

where W_k is the maximum power allowed by the kth user, and $\Sigma' \triangleq \text{diag}\,[S'_1 \cdots S'_K]$[1]. Let C_S and C_A denote the total capacities of the synchronous and asynchronous channels in (1), respectively. Then, it was found in [3], [8] that

$$C_S = \frac{1}{2} \log\left[\det\left(I_K + \Sigma' H\right)\right] \tag{6}$$

and

$$C_A = \sup_{\substack{S_k(w) \ge 0\ w \in [-\pi, \pi] \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} S_k(w)\,dw = S'_h \\ k = 1, 2, \cdots, K.}} \inf_{\tau_2, \cdots, \tau_K \in [0, T]} \frac{1}{4\pi} \cdot \int_{-\pi}^{\pi} \log\left[\det\left(I_K + \Sigma(w) H(w)\right)\right] dw \tag{7}$$

in information unit per symbol, where $H = H(w)\big|_{\tau_1 = \tau_2 = \cdots = \tau_K}$, and $\Sigma(w) = \text{diag}\,[S_1(w) \cdots S_K(w)]$.

Since the total capacities of the synchronous and the asynchronous channels are known, for a given set of pulses, one can calculate their total capacities and assess the decrease in total capacity due to symbol-asynchronism. However, the computation for the asynchronous total capacity can be very involved because the supremum is taken over an infinite dimensional space. Hence, it is of interest to obtain an easy-to-compute bound to the asynchronous total capacity which, in turn, gives a bound to the effect of symbol-asynchronism.

In the following theorem, we give lower bounds to the asynchronous total capacity and the ratio of the asynchronous to the synchronous total capacity. Also, it shows that TDMA[2] achieves both lower bounds and thus is one of the signalling strategies most sensitive to symbol asynchronism. These bounds lead to the conclusion in Corollary 1 that the asynchronous total capacity is at least $1/K$ of the synchronous total capacity. In asymptotically high signal-to-noise ratios, TDMA achieves this lower bound.

[1] diag $[x_1, \cdots, x_k]$ denotes a $k \times k$ diagonal matrix with the jjth diagonal entry equal to x_j.

[2] Here, TDMA corresponds to the case where the users are assigned equal length nonoverlapping time slots and modulate identical pulse shapes.

Theorem 1: For any pulses, we have

$$\frac{1}{2} \log\left[1 + \sum_{k=1}^{K} S'_k\right] \le C_A \tag{8}$$

with equality, if and only if the pulses are shifted versions of a common pulse modulo sign (i.e., $s_k(t) = \pm f(t - \zeta_k)$, $k = 1, \cdots, K$, for some $\zeta_1, \cdots, \zeta_K \in [0, T)$).

Also, for any pulses,

$$\frac{\log\left[1 + \sum_{k=1}^{K} S'_k\right]}{\sum_{k=1}^{K} \log\left[1 + S'_k\right]} \le \frac{C_A}{C_S} \tag{9}$$

with equality, if and only if the pulses are shifted versions of a common pulse (modulo sign) and form an orthonormal set in the $L_2[0, T]$.

Corollary 1: For any pulses and signal-to-noise ratios, $S'_1, \cdots, S'_K > 0$, we have

$$\frac{1}{K} < \frac{C_A}{C_S}. \tag{10}$$

Moreover, let $S'_k = \alpha_k S' > 0$ for $k = 1, \cdots, K$ such that $\sum_{k=1}^{K} \alpha_k = 1$. Then, if the pulses are shifted versions of a common pulse (modulo sign) and are linearly independent in $L_2[0, T]$, we have

$$\lim_{S' \to \infty} \frac{C_A}{C_S} = \frac{1}{K}. \tag{11}$$

Proof: We need to use the following lemma, which proved in the Appendix.

Lemma 1: For any pulses, $H(w)$ is an nonnegative definite matrix for all relative delays, $\tau_2, \cdots, \tau_K \in [0, T]$, and $w \in [0, 2\pi]$.

If the pulses are such that any collections of K length-T/K-interval truncations, one from each pulse, are linearly independent, i.e., they satisfy the following condition.

Condition 1: For any $\zeta_1, \cdots, \zeta_K \in [0, (1 - 1/K)T]$, $\{\hat{s}_k(t)\}_{k=1}^{K}$ forms a linearly independent set in $L_2[0, T/K]$, where

$$\hat{s}_k(t) \triangleq \begin{cases} s_k(t + \zeta_k), & t \in \left[0, \dfrac{T}{K}\right], \\ 0, & \text{otherwise}, \end{cases} \tag{12}$$

then there exists an $\lambda > 0$ such that $H(w) - \lambda I$ is positive definite for all relative delays, $\tau_2, \cdots, \tau_K \in [0, T]$, and $w \in [0, 2\pi]$.

Note that for any nonnegative definite matrix A, $\det(I + A) \geq 1 + \operatorname{tr}(A)$ with equality, if and only if A has rank one. Therefore, by the first part of Lemma 1 and the definition of C_A in (7), we have

$$C_A \geq \sup_{\substack{S_k(w) \geq 0 \; w \in [-\pi, \pi] \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} S_k(w)\,dw = S'_k \\ k = 1, 2, \cdots, K}} \frac{1}{4\pi} \int_{-\pi}^{\pi} \log\left[1 + \sum_{k=1}^{K} S_k(w)\right] dw \tag{13}$$

$$= \frac{1}{2} \log\left[1 + \sum_{k=1}^{K} S'_k\right], \tag{14}$$

where the last equality is due to the concavity of the log function. Moreover, the inequality in (13) is satisfied with equality, if and only if $H(w)$ has rank one for all $w \in [-\pi, \pi]$. However, note that any Hermitian matrix, A, that has rank one and identical diagonal entries, $A_{ii} = a$, must have off-diagonal entries equal to $\pm a$. Therefore, since $\{H(w)\}_{kk} = 1$ for all $k = 1, \cdots, K$, (13) is satisfied with equality, if and only if $\{H(w)\}_{kl} = \pm 1$, or, equivalently,

$$\rho_{kl} \triangleq \begin{cases} \pm 1, & \text{if } \tau_k \leq \tau_l, \\ 0, & \text{if } \tau_l < \tau_k. \end{cases} \tag{15}$$

Finally, it is easy to check that this condition is equivalent to the case where the pulses are shifted versions of a common pulse (modulo sign).

Applying the Hadamard inequality to (6), we have

$$C_S \leq \sum_{k=1}^{K} \frac{1}{2} \log\left[1 + S'_k\right], \tag{16}$$

with equality, if and only if $\{s_k(t)\}_{k=1}^{K}$ form an orthonormal set. Combining (16) with (8), we have (9) and the desired necessary and sufficient conditions for equality to hold.

Now, we proceed to prove Corollary 1. Since $\sum_{k=1}^{K} \log[1 + S'_k]$ is a symmetric concave function in $\{S'_k\}_{k=1}^{K}$, it follows (cf. [5]) that

$$\sum_{k=1}^{K} \log\left[1 + S'_k\right] \leq K \log\left[1 + \frac{1}{K} \sum_{k=1}^{K} S'_k\right]. \tag{17}$$

Substituting (17) into (9), we have (10) from the monotonicity of the logarithm.

If $S'_k = \alpha_k S' > 0$ for $k = 1, \cdots, K$ and the pulses are linearly independent in $L_2[0, T]$, then it is easy to check that

$$\lim_{S' \to \infty} \frac{C_S}{\log S'} = \frac{K}{2}. \tag{18}$$

Moreover, if the pulses are shifted versions of a common pulse (modulo sign), it follows from (8) that

$$\frac{1}{2} = \lim_{S' \to \infty} \frac{C_A}{\log S'} \tag{19}$$

Then, combining (18) and (19), we have the desired result. □

Next, we show that, as the signal-to-noise ratios go to zero, asynchronism has vanishing effect on the total capacity regardless of the pulses used. This is because the background Gaussian noise, rather than the inter-user interference, is the determining factor of the total capacity. As the signal-to-noise ratios go to infinity, we find that the same result holds if the pulses satisfy Condition 1 in Lemma 1. As long as Condition 1 in Lemma 1 holds, the receiver can distinguish the various signals. Then, the multiuser interference degrades the signal-to-noise ratio by a nonzero constant factor that has a diminishing effect on the total capacity as the signal-to-noise ratios go to infinity.

Theorem 2: For any signal-to-noise ratios, $S'_k = \alpha_k S'$ such that $0 < \alpha_k$ for all k and $\sum_{k=1}^{K} \alpha_k = 1$, we have, for any pulses,

$$\lim_{S' \to 0} \frac{C_A}{S'} = \lim_{S' \to 0} \frac{C_S}{S'} = \frac{1}{2}. \tag{20}$$

Furthermore, if the pulses are linearly independent when synchronized and satisfy Condition 1, then

$$\lim_{S' \to \infty} \frac{C_A}{\log S'} = \lim_{S' \to \infty} \frac{C_S}{\log S'} = \frac{K}{2}. \tag{21}$$

Proof: From Theorem 1, we have

$$\lim_{S' \to 0} \frac{C_A}{S'} \geq \lim_{S' \to 0} \frac{1}{2} \frac{\log\left[1 + S'\right]}{S'} = \frac{1}{2}. \tag{22}$$

From (16), we have

$$\lim_{S' \to 0} \frac{C_S}{S'} \leq \lim_{S' \to 0} \frac{1}{2} \frac{\sum_{k=1}^{K} \log\left[1 + \alpha_k S'\right]}{S'} = \frac{1}{2}. \tag{23}$$

Since C_A is always less than C_S, we have the result in (20) for low signal-to-noise ratios.

Now, we proceed to the high signal-to-noise ratio case. We first note that, by the first part of Lemma 1, $H(w)$ is nonnegative definite; therefore, it follows that [9, p. 482],

$$\left[\det\left(I_K + \Sigma(w) H(w)\right)\right]^{\frac{1}{K}} \geq 1 + \left[\det\left(\Sigma(w) H(w)\right)\right]^{\frac{1}{K}}$$
$$= 1 + \left[\prod_{k=1}^{K} S_k(w)\right]^{\frac{1}{K}} \left[\det H(w)\right]^{\frac{1}{K}}. \tag{24}$$

Then, by the second part of Lemma 1, since the pulses satisfy Condition 1, we have

$$\left[\det\left(I_K + \Sigma(w) H(w)\right)\right]^{\frac{1}{K}} \geq 1 + \left[\sum_{k=1}^{K} S_k(w)\right]^{\frac{1}{K}} \lambda, \quad \text{for some } \lambda > 0. \tag{25}$$

Substituting (25) into (7) and letting $S_k(w) = S'_k$ for all k and w, we have

$$C_A \geq \frac{K}{2} \log\left[1 + \prod_{k=1}^{K} \alpha_k^{\frac{1}{K}} S' \lambda\right]. \tag{26}$$

Taking the limit as $S' \to \infty$, we have

$$\lim_{S' \to \infty} \frac{C_A}{\log S'} \geq \lim_{S' \to \infty} \frac{K}{2} \frac{\log \left[1 + \Pi_{k=1}^{K} \alpha_k^{\frac{1}{K}} S' \lambda \right]}{\log S'} = \frac{K}{2}. \quad (27)$$

From (18) and the fact that C_A is always less than C_S, we have the desired result for high signal-to-noise ratios. □

III. Channels with RMS Bandwidth Constraints

Theorem 1 shows that no matter what pulses are used, symbol-asynchronism can at most decrease the total capacity by a factor equal to the number of users. At first glance, it seems that the next natural step is to find the pulses that are most insensitive to symbol-asynchronism. However, a closer look suggests that those pulses may not be optimal for the synchronous channel in terms of capacity. For instance, while the identical-pulse assignment minimizes the effect of asynchronism, it gives the lowest possible capacity. On the other hand, TDMA signalling maximizes the synchronous capacity, but suffers from the largest asynchronous degradation.

Without bandwidth constraints, pulses with arbitrarily low cross-correlation over all possible delays can be found. However, in some practical systems that use strictly time-limited pulses such as direct-sequence spread-spectrum, bandwidth is neither free nor strictly limited. In such situations, it is of interest [7] to adopt the root mean-square measure of bandwidth (introduced by Gabor [10] and used frequently for various channels [11]–[13]) and consider the maximum total capacity over all pulses under the RMS bandwidth constraint. Then, the difference between the maximum total capacities of the synchronous and the asynchronous channels gives an indication of how important synchronization is in bandlimited multiple-access channels. Notice that, with the bandwidth constraint, we allow that different sets of pulses are used for the synchronous and the asynchronous channels, a reasonable assumption since the users, who know whether the channel is synchronized or not, can design their pulses accordingly.

Let $S_k(f)$ and $\|s_k(t)\|_2$ be the Fourier transform and the L_2 norm of $s_k(t)$, respectively. We define

$$C_S^{\text{rms}} \triangleq \sup_{T} \sup_{\substack{s_k(t)=0\ t \notin [0,T] \\ \|s_k(t)\|_2 = 1 \\ \int_{-\infty}^{\infty} f^2 |S_k(f)|^2 df \leq B^2 \\ k=1,2,\cdots,K}} \frac{C_S}{T}, \quad (28)$$

$$C_A^{\text{rms}} \triangleq \sup_{T} \sup_{\substack{s_k(t)=0\ t \notin [0,T] \\ \|s_k(t)\|_2 = 1 \\ \int_{-\infty}^{\infty} f^2 |S_k(f)|^2 df \leq B^2 \\ k=1,2,\cdots,K}} \frac{C_A}{T} \quad (29)$$

as the total capacities of the RMS bandlimited two-user synchronous and asynchronous channels, respectively.

It turns out that it is convenient to define the time-bandwidth product, $\gamma \triangleq 2BT$, equivalent signal-to-noise ratio, $S_k \triangleq W_k/(N_0 B)$, and $\Sigma \triangleq \text{diag}[S_1 \cdots S_K]$. Since $1 \leq \gamma$ is a necessary condition [10] for the existence of a unit-energy pulse satisfying the strictly time-limited and RMS bandlimited constraints, we can express C_S^{rms} and C_A^{rms} in terms of γ and Σ as

$$C_S^{\text{rms}} = \sup_{1 \leq \gamma} \sup_{\substack{s_k(t)=0\ t \notin \left[0, \frac{\gamma}{2B}\right] \\ \|s_k(t)\|_2 = 1 \\ \int_{-\infty}^{\infty} f^2 |S_k(f)|^2 df \leq B^2 \\ k=1,2,\cdots,K}} C'_S, \quad (30)$$

$$C_A^{\text{rms}} = \sup_{1 \leq \gamma} \sup_{\substack{s_k(t)=0\ t \notin \left[0, \frac{\gamma}{2B}\right] \\ \|s_k(t)\|_2 = 1 \\ \int_{-\infty}^{\infty} f^2 |S_k(f)|^2 df \leq B^2 \\ k=1,2,\cdots,K}} C'_A, \quad (31)$$

where

$$C'_S = \frac{B}{\gamma} \log\left[\det\left(I_K + \gamma \Sigma H\right)\right], \quad (32)$$

$$C'_A = \sup_{\substack{S_k(w) \geq 0\ w \in [-\pi, \pi] \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} S_k(w)\, dw = S_k \\ k=1,2,\cdots,K.}} \inf_{\tau_2, \cdots, \tau_K \in \left[0, \frac{\gamma}{2B}\right]} \frac{B}{2\pi\gamma} \cdot \int_{-\pi}^{\pi} \log\left[\det\left(I_K + \gamma \Sigma(w) H(w)\right)\right] dw \quad (33)$$

in information units per second, where, as before, $\Sigma(w) = \text{diag}[S_1(w) \cdots S_K(w)]$.

The total capacity of the RMS bandlimited two-user synchronous channel can be simplified [7], [14] as

$$C_S^{\text{rms}} = \sup_{1 \leq \gamma \leq \sqrt{5/2}} \frac{B}{\gamma} \log \cdot \left[1 + \gamma S_1 + \gamma S_2 + \gamma^2 S_1 S_2 \left(1 - \frac{4}{9}\left(\frac{5}{2} - \gamma^2\right)^2\right)\right] \quad (34)$$

and the optimal pulse waveforms that achieve the capacity are [7], [14]

$$s_1(t) = \sqrt{\frac{4 - \bar{\gamma}^2}{3}} \sqrt{\frac{4B}{\bar{\gamma}}} \sin \frac{2\pi Bt}{\bar{\gamma}} + \sqrt{\frac{\bar{\gamma}^2 - 1}{3}} \sqrt{\frac{4B}{\bar{\gamma}}} \sin \frac{4\pi Bt}{\bar{\gamma}}, \quad (35)$$

$$s_2(t) = \sqrt{\frac{4 - \bar{\gamma}^2}{3}} \sqrt{\frac{4B}{\bar{\gamma}}} \sin \frac{2\pi Bt}{\bar{\gamma}} - \sqrt{\frac{\bar{\gamma}^2 - 1}{3}} \sqrt{\frac{4B}{\bar{\gamma}}} \sin \frac{4\pi Bt}{\bar{\gamma}}, \quad (36)$$

where $\bar{\gamma}$ is the corresponding optimal γ in (34).

For the RMS bandlimited asynchronous channel, equation (31) is the only known expression and is computationally formidable. In the following theorem, we give a series of lower bounds (with gradual increase in computational complexity) to the asynchronous total capacity. In Corollary 2, we use one of these lower bounds to show that the decrease in the total capacity from the synchronous to the asynchronous two-user channel is at most 12%.

Theorem 3: Let C_A^{rms} be the total capacity of the RMS bandlimited CDMA white Gaussian two-user asynchronous channel, then, for every integer $n \geq 1$,

$$C_A^{\text{rms}} \geq \sup_{1 \leq \gamma \leq 2} \frac{B}{\gamma} \log\left[1 + \gamma S_1 + \gamma S_2 + \gamma^2 S_1 S_2 (1 - \tilde{\rho}_n^2)\right], \tag{37}$$

where

$$\tilde{\rho}_n \triangleq \inf_{\substack{a_k \in \mathbb{R}^n \\ \|a_k\|_2 = 1 \\ a_k^T \Lambda_n a_k \leq \gamma^2 \\ k=1,2.}} \max_{\theta \in [0,1]} \max\left\{ |a_1^T S_n a_2|, |a_1^T D_n a_2| \right\} \tag{38}$$

and S_n and D_n are matrices formed by taking the first k rows and k columns of S and D, respectively. The matrices S and D are defined by

$$S_{ij} \triangleq \begin{cases} \cos(i\pi\theta), & \text{if } i = j \text{ and } i \text{ is even}, \\ (1 - 2\theta)\cos(i\pi\theta) + \dfrac{2}{\pi i}\sin(i\pi\theta), & \text{if } i = j \text{ and } i \text{ is odd}, \\ 0, & \text{if } i \neq j \text{ and } i, j \text{ are both even}, \\ \dfrac{4}{\pi(i^2 - j^2)}\left[i \sin(j\pi\theta) - j\sin(i\pi\theta)\right], & \text{if } i \neq j \text{ and } i, j \text{ are both odd}, \\ -\dfrac{4j}{\pi(i^2 - j^2)}\sin(i\pi\theta), & \text{if } i \neq j \text{ and } i \text{ is even and } j \text{ is odd}, \\ -\dfrac{4i}{\pi(i^2 - j^2)}\sin(j\pi\theta), & \text{if } i \neq j \text{ and } i \text{ is odd and } j \text{ is even}, \end{cases} \tag{39}$$

$$D_{ij} \triangleq \begin{cases} (1 - 2\theta)\cos(i\pi\theta) + \dfrac{2}{\pi i}\sin(i\pi\theta), & \text{if } i = j \text{ and } i \text{ is even}, \\ \cos(i\pi\theta), & \text{if } i = j \text{ and } i \text{ is odd}, \\ \dfrac{4}{\pi(i^2 - j^2)}\left[i \sin(j\pi\theta) - j\sin(i\pi\theta)\right], & \text{if } i \neq j \text{ and } i, j \text{ are both even}, \\ 0, & \text{if } i \neq j \text{ and } i, j \text{ are both odd}, \\ -\dfrac{4i}{\pi(i^2 - j^2)}\sin(j\pi\theta), & \text{if } i \neq j \text{ and } i \text{ is even and } j \text{ is odd}, \\ -\dfrac{4j}{\pi(i^2 - j^2)}\sin(i\pi\theta), & \text{if } i \neq j \text{ and } i \text{ is odd and } j \text{ is even}, \end{cases} \tag{40}$$

and $\Lambda_n = \operatorname{diag}[1^2 2^2 3^2 \cdot n^2]$.

Proof: For the two-user channel,

$$\begin{aligned} \det(I_2 + \gamma\Sigma(w)H(w)) &= 1 + \gamma S_1(w) + \gamma S_2(w) + \gamma^2 S_1(w) S_2(w) \\ &\quad \cdot \left(1 - \left(\rho_{12}^2 + \rho_{21}^2 - 2\rho_{12}\rho_{21}\cos w\right)\right) \\ &\geq 1 + \gamma S_1(w) + \gamma S_2(w) + \gamma^2 S_1(w) S_2(w) \\ &\quad \cdot \left(1 - \rho_{\max}^2\right), \end{aligned} \tag{41}$$

where $\rho_{\max} \triangleq \max\{|\rho_{12} - \rho_{21}|, |\rho_{12} + \rho_{21}|\}$. Then, from the definition of C_A^{rms} in (31) and (41), we have

$$C_A^{\text{rms}} \geq \sup_{1 \leq \gamma} \frac{B}{\gamma} \log\left[1 + \gamma S_1 + \gamma S_2 + \gamma^2 S_1 S_2 (1 - \tilde{\rho}^2)\right], \tag{42}$$

where (to simplify the notation, we drop the subscript of the relative delay and let $\tau = \tau_2$)

$$\tilde{\rho} \triangleq \inf_{\substack{s_k(t) = 0\ t \notin \left[0, \frac{\gamma}{2B}\right] \\ \|s_k(t)\|_2 = 1 \\ \int_{-\infty}^{\infty} f^2 |S_k(f)|^2 df \leq B^2 \\ k = 1,2}} \sup_{\tau \in \left[0, \frac{\gamma}{2B}\right]} \rho_{\max}. \tag{43}$$

When $\gamma > 2$,

$$\begin{aligned} &\frac{B}{\gamma} \log\left[1 + \gamma S_1 + \gamma S_2 + \gamma^2 S_1 S_2 (1 - \tilde{\rho}^2)\right] \\ &\quad \leq \frac{B}{\gamma}\log\left[1 + \gamma S_1\right] + \frac{B}{\gamma}\log\left[1 + \gamma S_2\right] \\ &\quad \leq \frac{B}{2}\log\left[1 + 2S_1\right] + \frac{B}{2}\log\left[1 + 2S_2\right] \\ &\quad \leq B\log\left[1 + S_1 + S_2\right], \end{aligned} \tag{44}$$

which is the case when $\gamma = 1$. Therefore, it is sufficient to take the supermum over $1 \leq \gamma \leq 2$.

Now, in order to complete the proof, it is sufficient to show that $\tilde{\rho}_n \geq \tilde{\rho}$ for all integers $n \geq 1$. Using a complete orthonormal basis on the space of all RMS bandlimited signals over

$$[0, \gamma/(2B)], \{\phi_i(t) \triangleq \sqrt{4B/\gamma}\sin(2\pi i B t/\gamma)\}_{i=1}^{\infty},$$

we can write

$$s_k(t) = a_k^T \Phi(t), \qquad k = 1, 2, \tag{45}$$

where $\Phi(t) = [\phi_1(t)\phi_2(t)\cdots]^T$. Then, we can show that

$$\|s_k(t)\|_2 = 1 \Leftrightarrow \|a_k\|_2 = 1, \tag{46}$$

$$\int_{-\infty}^{\infty} f^2 |S_k(f)|^2 df \leq B^2 \Leftrightarrow a_k^T \Lambda a_k \leq \gamma^2, \tag{47}$$

where the equivalence relationships are the consequences of Parseval's theorem. Defining

$$S \triangleq \int_{\tau}^{T} \Phi(t)\Phi^T(t-\tau)\,dt + \int_{0}^{\tau} \Phi(t)\Phi^T(t+T-\tau)\,dt, \tag{48}$$

$$D \triangleq \int_{\tau}^{T} \Phi(t)\Phi^T(t-\tau)\,dt - \int_{0}^{\tau} \Phi(t)\Phi^T(t+T-\tau)\,dt, \tag{49}$$

we can rewrite (43), with $\theta \triangleq 2B\tau/\gamma$, as

$$\tilde{\rho} = \inf_{\substack{a_k \in l_2 \\ \|a_k\|_2=1 \\ a_k^T \Lambda a_k \le \gamma^2 \\ k=1,2}} \max_{\theta\in[0,1]} \max\{|a_1^T S a_2|, |a_1^T D a_2|\}. \tag{50}$$

It can be readily checked that S and D defined in (48) and (49) have entries specified in (39) and (40). Comparing $\tilde{\rho}_n$ in (38) with $\tilde{\rho}$ in (50), it is easy to see that $\tilde{\rho}_n \ge \tilde{\rho}$ since $\tilde{\rho}_n$ is obtained by minimizing over a smaller subset. Finally, substituting $\tilde{\rho}_n \ge \tilde{\rho}$ in (42), we obtain the desired result. □

Although the lower bounds to $C_A^{\rm rms}$ given in Theorem 3 may be computationally heavy for large n, the following corollary shows that a tight lower bound with low computational complexity can be obtained.

Corollary 2: Let $C_A^{\rm rms}$ be the total capacity of the RMS bandlimited white Gaussian two-user asynchronous channel, then

$$C_A^{\rm rms} \ge \sup_{1\le\gamma\le 2} \frac{B}{\gamma}\log\left[1+\gamma S_1+\gamma S_2+\gamma^2 S_1 S_2(1-\rho_0^2)\right], \tag{51}$$

where

$$\rho_0 \triangleq \min_{\sqrt{\frac{\gamma^2-1}{24}}\ge\alpha\ge 0} \max_{\theta\in[0,1]} \max\{|a_1^T(\alpha)S_5 a_2(\alpha)|, |a_1^T(\alpha)D_5 a_2(\alpha)|\} \tag{52}$$

and

$$a_1(\alpha) = \left[\sqrt{\frac{9-\gamma^2+16\alpha^2}{8}} \quad 0 \quad \sqrt{\frac{\gamma^2-1-24\alpha^2}{8}} \quad 0 \quad \alpha\right]^T, \tag{53}$$

$$a_2(\alpha) = \left[\sqrt{\frac{9-\gamma^2+16\alpha^2}{8}} \quad 0 \quad -\sqrt{\frac{\gamma^2-1-24\alpha^2}{8}} \quad 0 \quad \alpha\right]^T. \tag{54}$$

Proof: Particularizing Theorem 3 with $n = 5$, and restricting $a_k^T\Lambda_5 a_k = \gamma^2$, $a_{k2} = a_{k4} = 0$ for $k = 1, 2$, we define ρ_0 as

$$\rho_0 \triangleq \inf_{\substack{a_k\in\mathbb{R}^5 \\ \|a_k\|_2=1 \\ a_k^T\Lambda_5 a_k=\gamma^2 \\ a_{k2}=a_{k4}=0 \\ k=1,2}} \max_{\theta\in[0,1]} \max\{|a_1^T S_5 a_2|, |a_1^T D_5 a_2|\}. \tag{55}$$

Since $\|a_k\|_2 = 1$, $a_k^T\Lambda_5 a_k = \gamma^2$, and $a_{k2} = a_{k4} = 0$, we can parametrize the entires of a_k. Restricting $a_{15} = a_{25} = \alpha$, we have $a_{k3}^2 = (\gamma^2 - 1 - 24\alpha^2)/8$ and $a_{k1}^2 = (9 - \gamma^2 + 16\alpha^2)/8$. Then, further restricting $a_{13} = -a_{23}$ and $a_{11} = a_{21}$, and finding the range of α, we obtain the desired result. □

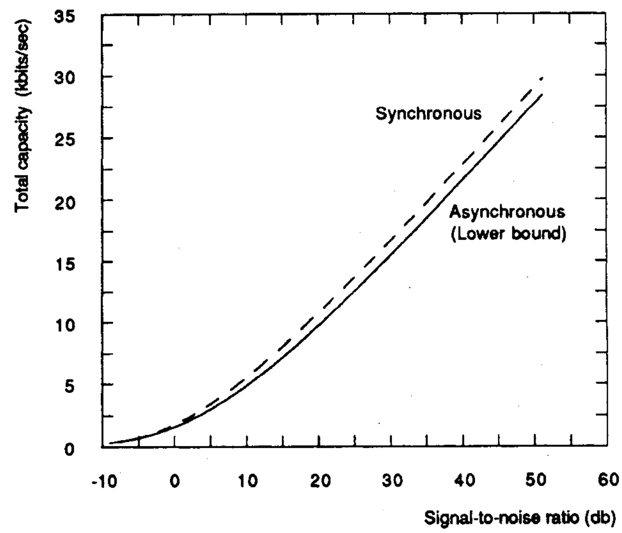

Fig. 1. Total capacity and its lower bound. RMS bandwidth = 1 kHz.

Fig. 1 shows the total capacity of the RMS bandlimited synchronous channel and the lower bound to the total capacity of the RMS bandlimited asynchronous channel in Corollary 2, for different signal-to-noise ratios. To assess the percentage decrease, the corresponding upper bound to $1 - (C_A^{\rm rms}/C_S^{\rm rms})$ is plotted in Fig. 2 for various signal-to-noise ratios. Just to show the importance of the shapes of the pulses, we also plot, in Fig. 2, a lower bound to $1 - (C'_A/C_S^{\rm rms})$ when the optimal pulses for the synchronous channel are used in the asynchronous channel. We can see that while the upper bound to the degradation using optimal pulses is at most 12% for all signal-to-noise ratios, the lower bound to the degradation using the pulses for the synchronous channel is at least 36% for some signal-to-noise ratios.

Figs. 3 and 4 show the optimal pulses for the synchronous channel and the pulses that achieve the lower bound in Corollary 2 for the asynchronous channel, respectively, for a nominal signal-to-noise ratio.

Since the lower bound to the asynchronous total capacity is so close to the synchronous total capacity, it is natural to ask if they are actually equal. The following lemma is used to obtain Theorem 4 that gives the necessary and sufficient condition for the asynchronous and the synchronous total capacities to coincide.

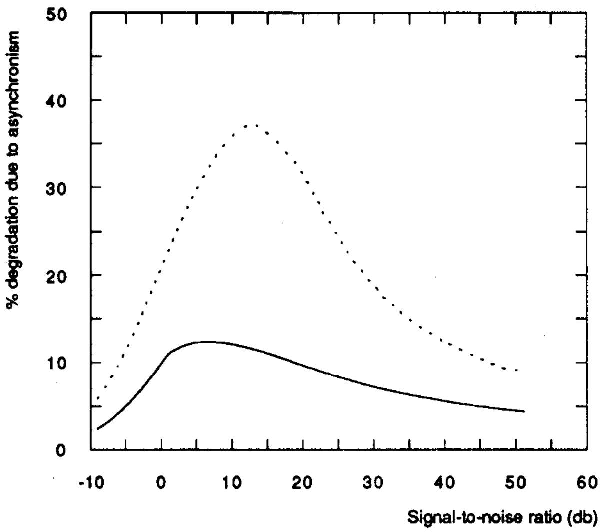

Fig. 2. Bounds to the degradation due to asynchronism versus signal-to-noise ratio. Solid line is an upper bound to $1 - C_A^{\text{rms}}/C_S^{\text{rms}}$. Dotted line is a lower bound to the degradation when the optimal pulses for the synchronous channel (at the same signal-to-noise ratio) is used in the asynchronous channel.

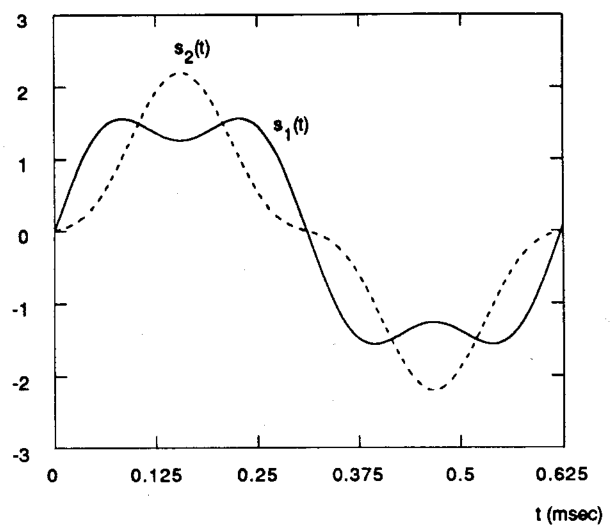

Fig. 3. Pulse waveforms that achieve the lower bound in Corollary 2 for the asynchronous channel.

Lemma 2: Let $f(\tau, s_1(t), s_2(t)) = \min\{\rho_{12} + \rho_{21})^2, (\rho_{12} - \rho_{21})^2\}$, then

$$\inf_{\substack{s_k(t)=0\ t\notin\left[0,\frac{\gamma}{2B}\right] \\ \|s_k(t)\|_2=1 \\ \int_{-\infty}^{\infty} f^2|S_k(f)|^2\,df \le B^2 \\ k=1,2}} f(0, s_1(t), s_2(t)) = \frac{4}{9}\left(\frac{5}{2} - \gamma^2\right)^2. \tag{56}$$

and is achieved only when $(s_1(t), s_2(t))$ belongs to the set $D = \{(\pm p_1(t), \pm p_2(t)), (\pm p_1(t), \mp p_2(t)), (\pm p_2(t), \pm p_1(t)), (\pm p_2(t), \mp p_1(t))\}$, where

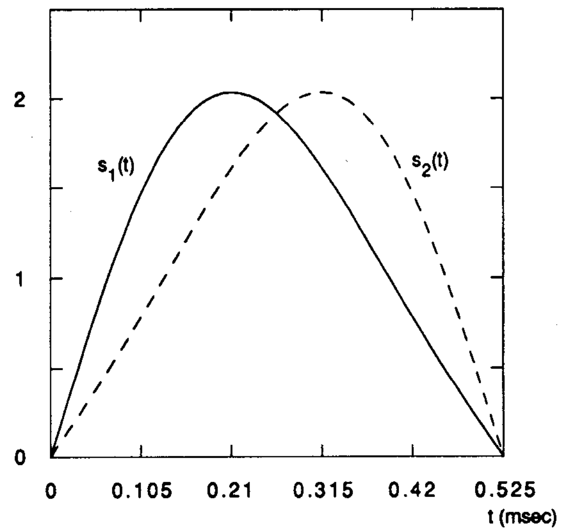

Fig. 4. Optimal pulse waveforms for the synchronous channel.

$$p_1(t) = \sqrt{\frac{4-\gamma^2}{3}}\sqrt{\frac{4B}{\gamma}}\sin\frac{2\pi Bt}{\gamma} + \sqrt{\frac{\gamma^2-1}{3}}\sqrt{\frac{4B}{\gamma}}\sin\frac{4\pi Bt}{\gamma}, \tag{57}$$

$$p_2(t) = \sqrt{\frac{4-\gamma^2}{3}}\sqrt{\frac{4B}{\gamma}}\sin\frac{2\pi Bt}{\gamma} - \sqrt{\frac{\gamma^2-1}{3}}\sqrt{\frac{4B}{\gamma}}\sin\frac{4\pi Bt}{\gamma}. \tag{58}$$

Moreover, for any sequence $(s_{1k}(t), s_{2k}(t))$, satisfying the constraints, such that $f(0, s_{1k}(t), s_{2k}(t)) \to (4/9)(5/2 - \gamma^2)^2$, we have $(s_{1k}(t), s_{2k}(t)) \to D$.

Proof: When $\tau = 0$, $\rho_{21} = 0$, and $f(0, s_1(t), s_2(t)) = \rho_{12}^2$ and the first part of Lemma 2 has been proved in [14], using the standard Lagrange multiplier method.

Now, we show that for any sequence $\{(s_{1k}(t), s_{2k}(t))\}_{k=1}^{\infty}$, satisfying the constraints, such that $f(0, s_{1k}(t), s_{2k}(t)) \to (4/9)(5/2 - \gamma^2)^2$, we have $(s_{1k}(t), s_{2k}(t)) \to D$. For any such sequence $\{(s_{1k}(t), s_{2k}(t))\}_{k=1}^{\infty}$, using the complete orthonormal basis defined in the proof of Theorem 3, $\{\phi_i(t)\}_{i=1}^{\infty}$, we obtain the orthonormal projections, $\{a_{ki}\}_{i=1}^{\infty}, \{b_{ki}\}_{i=1}^{\infty}, \{p_{1i}\}_{i=1}^{\infty}$, for $s_{1k}(t)$, $s_{2k}(t)$ and $p_1(t)$, respectively. Then, the constraints on $s_1(t)$ and $s_2(t)$ become

$$\sum_{i=1}^{\infty} a_{ki}^2 = 1, \qquad \sum_{i=1}^{\infty} b_{ki}^2 = 1, \tag{59}$$

$$\sum_{i=1}^{\infty} a_{ki}^2 i^2 \le \gamma^2, \qquad \sum_{i=1}^{\infty} b_{ki}^2 i^2 \le \gamma^2. \tag{60}$$

It is easy to check that $p_{11}^2 \le a_{k1}^2, b_{k1}^2$. Therefore, we can let $\epsilon_{k1} = a_{k1}^2 - p_{11}^2 \ge 0$ and $\epsilon_{k2} = p_{12}^2 - a_{k2}^2 \ge 0$, $\delta_{k1} = b_{k1}^2 - p_{11}^2 \ge 0$, and $\delta_{k2} = p_{12}^2 - b_{k2}^2 \ge 0$. Then, using the constraints in (59), we have

$$\begin{aligned}\epsilon_{k1} - \epsilon_{k2} &= a_{k1}^2 + a_{k2}^2 - \left(p_{11}^2 + p_{12}^2\right) \\ &= a_{k1}^2 + a_{k2}^2 - 1 \le 0.\end{aligned} \tag{61}$$

Also, from (59) and (60), we have $3a_{k2}^2 \le \sum_{i=1}^{\infty} a_{ki}^2(i^2-1) \le \gamma^2 - 1$, and, therefore,

$$\begin{aligned} 5\epsilon_{k2} - 8\epsilon_{k1} &= \frac{5}{3}(\gamma^2-1) + \frac{8}{3}(4-\gamma^2) - 5a_{k2}^2 - 8a_{k1}^2 \\ &\le \frac{5}{3}(\gamma^2-1) + \frac{8}{3}(4-\gamma^2) - 5a_{k2}^2 - 8a_{k1}^2 \\ &\quad + \left[(\gamma^2-1) - 3a_{k2}^2\right] \le 0. \end{aligned} \tag{62}$$

Similarly, we can apply the same procedures to δ_{k1} and δ_{k2}. So, we have

$$\frac{5}{8}\epsilon_{k1} \le \frac{5}{8}\epsilon_{k2} \le \epsilon_{k1}, \tag{63}$$

$$\frac{5}{8}\delta_{k1} \le \frac{5}{8}\delta_{k2} \le \delta_{k1}. \tag{64}$$

Considering the minimum possible value (i.e,. optimally assigning the signs for a_{ki} and b_{ki}) of $f(0, s_{1k}(t), s_{2k}(t))$, we have

$$\begin{aligned} &f(0, s_{1k}(t), s_{2k}(t)) \\ &\quad \ge \left[\sqrt{a_{k1}^2 b_{k1}^2} - \sqrt{(1-a_{k1}^2)(1-b_{k1}^2)}\right]^2 \\ &\quad = \Big[\sqrt{(p_{11}^2+\epsilon_{k1})(p_{11}^2+\delta_{k1})} \\ &\qquad - \sqrt{(p_{12}^2-\epsilon_{k1})(p_{12}^2-\delta_{k1})}\Big]^2, \end{aligned} \tag{65}$$

which is strictly increasing in ϵ_{k1} and δ_{k1} for $\epsilon_{k1}, \delta_{k1} \in [0, p_{12}^2]$, and is equal to $\frac{4}{9}(\frac{5}{2}-\gamma^2)^2$ when $\epsilon_{k1} = \delta_{k1} = 0$. Therefore, by the assumption, as $k \to \infty$, $\epsilon_{k1}, \delta_{k1} \to 0$. Then, by (63) and (64), $\epsilon_{k2}, \delta_{k2} \to 0$ and we have the desired result. □

Theorem 4 gives a necessary and sufficient condition for the RMS bandlimited synchronous and asynchronous total capacity to coincide. It turns out that it depends only on the signal-to-noise ratios.

Theorem 4: Let C_S^{rms} and C_A^{rms} be the total capacity of the RMS bandlimited white Gaussian two-user synchronous and asynchronous channels, respectively, with bandwidth B, signal-to-noise ratios S_1 and S_2. Then,

$$C_A^{\mathrm{rms}} \le C_S^{\mathrm{rms}}, \tag{66}$$

with equality, if and only if

$$C_S^{\mathrm{rms}} = B \log\left[1 + S_1 + S_2\right] \tag{67}$$

or, equivalently,

$$\arg \max_{1 \le \gamma \le \sqrt{\frac{5}{2}}} \frac{1}{\gamma} \log\left[1 + \gamma S_1 + \gamma S_2 + \gamma^2 S_1 S_2\left(1 - \frac{4}{9}\left(\frac{5}{2} - \gamma^2\right)^2\right)\right] = 1. \tag{68}$$

Proof: It is easy to obtain (66) since the synchronous total capacity is a special case of the asynchronous total capacity when $\tau = 0$. Hence, the proof involves showing that the condition in (67) or (68) is necessary and sufficient for equality to hold in (66). Note that, from (34), (67), and (68) are equivalent.

We first show the necessary part by showing that if γ^*, the optimum γ in (68), is greater than 1, then

$$C_A^{\mathrm{rms}} < C_S^{\mathrm{rms}}. \tag{69}$$

Let us define

$$\rho_{\min} \triangleq \min\left\{|\rho_{12} - \rho_{21}|, |\rho_{12} + \rho_{21}|\right\}; \tag{70}$$

then, specializing (33) to the two-user case, we have

$$\begin{aligned} C_A' &\le \sup_{\substack{S_k(w)\ge 0\ w\in[-\pi,\pi] \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} S_k(w)\,dw \le S_k \\ k=1,2}} \inf_{\tau\in\left[0,\frac{\gamma}{2B}\right]} \frac{B}{2\pi\gamma} \\ &\quad \cdot \int_{-\pi}^{\pi} \log\left[1 + \gamma S_1(w) + \gamma S_2(w) + \gamma^2 S_1(w) S_2(w)\left(1 - \rho_{\min}^2\right)\right] dw \\ &\le \inf_{\tau\in\left[0,\frac{\gamma}{2B}\right]} \sup_{\substack{S_k(w)\ge 0\ w\in[-\pi,\pi] \\ \frac{1}{2\pi}\int_{-\pi}^{\pi} S_k(w)\,dw \le S_k \\ k=1,2}} \frac{B}{2\pi\gamma} \\ &\quad \cdot \int_{-\pi}^{\pi} \log\left[1 + \gamma S_1(w) + \gamma S_2(w) + \gamma^2 S_1(w) S_2(w)\left(1 - \rho_{\min}^2\right)\right] dw \\ &= \inf_{\tau\in\left[0,\frac{\gamma}{2B}\right]} \frac{B}{\gamma} \log \\ &\quad \cdot \left[1 + \gamma S_1 + \gamma S_2 + \gamma^2 S_1 S_2\left(1 - \left(\rho_{\min}^2\right)\right)\right] \\ &= \frac{B}{\gamma} \log\left[1 + \gamma S_1 + \gamma S_2 + \gamma^2 S_1 S_2 \cdot \left(1 - \left(\sup_{\tau\in\left[0,\frac{\gamma}{2B}\right]} \rho_{\min}^2\right)\right)\right]. \end{aligned} \tag{71}$$

Substituting into the definition of C_A^{rms}, (31), we have

$$C_A^{\mathrm{rms}} \le \sup_{1\le\gamma} \frac{B}{\gamma} \log\left[1 + \gamma S_1 + \gamma S_2 + \gamma^2 S_1 S_2\left(1 - \left(\hat{\rho}^2\right)\right)\right], \tag{72}$$

where

$$\hat{\rho}^2 = \inf_{\substack{s_k(t)=0\ t\notin\left[0,\frac{\gamma}{2B}\right] \\ \|s_k(t)\|_2 = 1 \\ \int_{-\infty}^{\infty} f^2 |S_k(f)|^2\,df \le B^2 \\ k=1,2}} \sup_{\tau\in\left[0,\frac{\gamma}{2B}\right]} f(\tau, s_{1k}(t), s_{2k}(t)), \tag{73}$$

where $f(\tau, s_{1k}(t), s_{2k}(t))$ is defined in Lemma 2.

Considering the synchronous total capacity in (34), it is clear that

$$\begin{aligned} C_S^{\text{rms}} &\geq \frac{B}{\gamma}\log\left[1+\gamma S_1+\gamma S_2+\gamma^2 S_1 S_2\right]\Big|_{\gamma=\sqrt{5/2}} \\ &> \frac{B}{\gamma}\log\left[1+\gamma S_1+\gamma S_2+\gamma^2 S_1 S_2\right] \quad \forall\gamma>\sqrt{5/2} \\ &\geq \frac{B}{\gamma}\log\left[1+\gamma S_1+\gamma S_2+\gamma^2 S_1 S_2(1-\hat{\rho}^2)\right] \\ &\quad \forall\gamma>\sqrt{5/2}, \end{aligned} \tag{74}$$

where the second inequality follows from the fact that the right side is strictly decreasing in γ for $\forall\gamma>0$. Assuming that the optimum γ^* that maximizes C_S^{rms} is strictly greater than 1, we have

$$C_S^{\text{rms}} > \frac{B}{\gamma}\log\left[1+\gamma S_1+\gamma S_2+\gamma^2 S_1 S_2(1-(\hat{\rho}^2))\right]\big|_{\gamma=1}. \tag{75}$$

Therefore, it is sufficient to show that for all $\gamma\in(1,\sqrt{5/2}]$,

$$\frac{4}{9}\left(\frac{5}{2}-\gamma^2\right)^2 < \hat{\rho}^2. \tag{76}$$

By Lemma 2, it is clear that $(4/9)(5/2-\gamma^2)^2 \leq \hat{\rho}^2$. Now, we assume that $(4/9)(5/2-\gamma^2)^2 = \hat{\rho}^2$ for some $\gamma\in(1,\sqrt{5/2}]$. Then, there exists a sequence $\{(s_{1k}(t), s_{2k}(t))\}$ such that $\sup_\tau f(\tau, s_{1k}(t), s_{2k}(t))$ decreases to $(4/9)(5/2-\gamma^2)^2$. Since

$$\begin{aligned} \frac{4}{9}\left(\frac{5}{2}-\gamma^2\right)^2 &\leq f(0, s_{1k}(t), s_{2k}(t)) \\ &\leq \sup_\tau f(\tau, s_{1k}(t), s_{2k}(t)), \end{aligned} \tag{77}$$

it follows from the assumption that $f(0, s_{1k}(t), s_{2k}(t)) \to (4/9)(5/2-\gamma^2)^2$. By Lemma 2 and the fact that $\sup_\tau f(\tau, \pm s_1(t), \pm s_2(t)) = \sup_\tau f(\tau, \pm s_2(t), \pm s_1(t))$, we can assume, without loss of generality, that $s_{1k}(t) \to p_1(t)$ and $s_{2k}(t) \to p_2(t)$. Since $f(\tau, s_1(t), s_2(t))$ is continuous in both $s_1(t)$ and $s_2(t)$, we have

$$\lim_{k\to\infty} f(\tau, s_{1k}(t), s_{2k}(t)) = f(\tau, p_1(t), p_2(t)), \tag{78}$$

for each $\tau\in[0,\gamma/(2B)]$.

If $\gamma\in(1,\sqrt{5/2})$, it is straightforward to check that, using (39), (40), (57), and (58), when $s_k(t) = p_k(t)$, $d(\rho_{12}+\rho_{21})^2/d\tau\,|_{\tau=T} < 0$ and $d(\rho_{12}-\rho_{21})^2/d\tau\,|_{\tau=T} < 0$. Therefore, there exists a $\tau\in[0,\gamma/(2B))$ such that

$$\begin{aligned} \frac{4}{9}\left(\frac{5}{2}-\gamma^2\right)^2 &= f(0, p_1(t), p_2(t)) \\ &= f(T, p_1(t), p_2(t)) < f(\tau, p_1(t), p_2(t)). \end{aligned} \tag{79}$$

If $\gamma=\sqrt{5/2}$, then it is easy to check that

$$0 = \frac{4}{9}\left(\frac{5}{2}-\gamma^2\right)^2 < f\left(\frac{\gamma}{4B}, p_1(t), p_2(t)\right). \tag{80}$$

Hence, we have, from (78)–(80), that for any $\gamma\in(1,\sqrt{5/2}]$, there exist $\tau\in[0,\gamma/(2B))$ such that

$$\begin{aligned} \frac{4}{9}\left(\frac{5}{2}-\gamma^2\right)^2 &< \lim_{k\to\infty} f(\tau, s_{1k}(t), s_{2k}(t)) \\ &\leq \lim_{k\to\infty} \sup_\tau f(\tau, s_{1k}(t), s_{2k}(t)) \\ &= \frac{4}{9}\left(\frac{5}{2}-\gamma^2\right)^2, \end{aligned} \tag{81}$$

which leads to a contradiction.

The sufficient part is straightforward from Corollary 2 since

$$\begin{aligned} C_A^{\text{rms}} &\geq B\log\left[1+S_1+S_2\right] \\ &= C_S^{\text{rms}}. \end{aligned} \tag{82}$$

Combining with (66), we have completed the proof. □

Fig. 5 shows the region where the synchronous and the asynchronous total capacities coincide. The behavior we observe along the diagonal of equipower users is due to the following result.

Corollary 3: Let C_S^{rms} and C_A^{rms} be the total capacity of the RMS bandlimited CDMA white Gaussian two-user synchronous and asynchronous channels, respectively, with bandwidth B, signal-to-noise ratios S_1 and S_2. If $S_1 = S_2 \neq 0$, then

$$C_A^{\text{rms}} < C_S^{\text{rms}}. \tag{83}$$

Proof: Let $S \triangleq S_1 = S_2 \neq 0$, we have

$$\begin{aligned} C_S^{\text{rms}} = \max_{1\leq\gamma\leq\sqrt{5/2}} &\frac{B}{\gamma}\log \\ &\cdot\left[1+2\gamma S+\gamma^2 S^2\left(1-\frac{4}{9}\left(\frac{5}{2}-\gamma^2\right)^2\right)\right]. \end{aligned} \tag{84}$$

Then, by Theorem 4, it is sufficient to show that

$$\begin{aligned} \frac{d}{d\gamma}\frac{B}{\gamma}\log&\left[1+2\gamma S+\gamma^2 S^2\left(1-\frac{4}{9}\left(\frac{5}{2}-\gamma^2\right)^2\right)\right]\Bigg|_{\gamma=1} \\ &= B\left[\frac{2S+\frac{8}{3}S^2}{1+2S} - \log\left[1+2S\right]\right] \end{aligned} \tag{85}$$

is strictly positive over all $S>0$. Since the derivative in (85) is zero when $S=0$, and

$$\frac{d}{dS}B\left[\frac{2S+\frac{8}{3}S^2}{1+2S} - \log\left[1+2S\right]\right] = B\frac{4S(1+4S)}{3(1+2S)^2} > 0, \tag{86}$$

for all $S>0$, we have the desired result. □

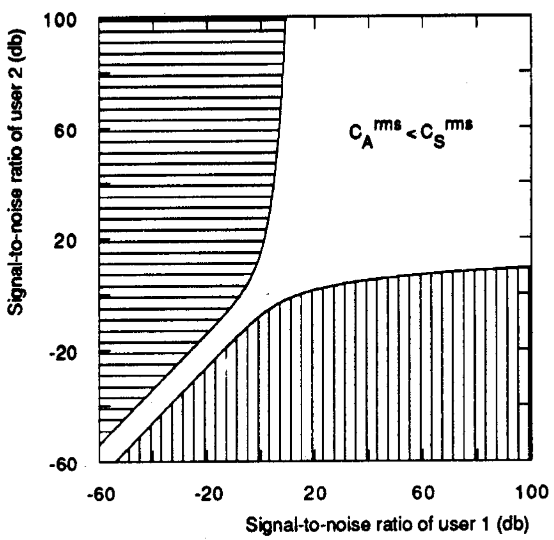

Fig. 5. Signal-to-noise ratio region where the synchronous and the asynchronous capacities coincide.

To conclude we give some asymptotic results comparing the total capacities of the synchronous and asynchronous K-user RMS bandlimited Gaussian channels.

In Fig. 2, we notice that the relative degradation due to symbol asynchronism is very small in both high and low signal-to-noise ratio cases. The low signal-to-noise ratio behavior is a consequence of Theorem 2. However, the high signal-to-noise ratio result does not follow directly since we need to find pulses that satisfy both Condition 1 in Lemma 1 and the RMS bandwidth constraint. In the following theorem, we demonstrate the existence of such a set of pulses and show the insensitivity of the total capacity to asynchronism in high signal-to-noise ratios.

Theorem 5: Let C_S^{rms} and C_A^{rms} be the total capacity of the RMS bandlimited white Gaussian synchronous and asynchronous channels, respectively, with RMS bandwidth B and signal-to-noise ratio $\alpha_k S$, $\alpha_k > 0$ for all k and $\sum_{k=1}^{K} \alpha_k = 1$, then

$$\lim_{S\to 0} \frac{C_S^{\text{rms}}}{S} = \lim_{S\to 0} \frac{C_A^{\text{rms}}}{S} = B \tag{87}$$

and

$$\lim_{S\to \infty} \frac{C_S^{\text{rms}}}{\log S} = \lim_{S\to \infty} \frac{C_A^{\text{rms}}}{\log S} = BK. \tag{88}$$

Proof: The result for low signal-to-noise ratio is a consequence of Theorem 2 in Section II by taking $s_k(t) = \phi_1(t)$ (recall that $\phi_k(t) \triangleq \sqrt{4B/\gamma}\,\sin(2\pi kBt/\gamma)$ is defined for the interval $[0, \gamma/(2B)]$ in the proof of Theorem 3) and $\gamma = 1$.

Now, we show the result for the high signal-to-noise ratio case. For any $\epsilon > 0$, let $\gamma = \sqrt{1+\epsilon}$, and $\delta_k = \epsilon/(k^2 - 1) > 0$ for $k = 2,\cdots,K$. Then, let $s_1(t) = \phi_1(t)$ and $s_k(t) = \sqrt{1-\delta_k}\,\phi_1(t) + \sqrt{\delta_k}\,\phi_k(t)$ for $k = 2,\cdots,K$. Then, it is easy to check that every pulse is RMS bandlimited by B. Moreover, since any length-$\gamma/(2BK)$-truncation of the kth pulse has the form

$$\sqrt{\frac{4B}{\gamma}}\left(a\sin\left(\frac{2\pi Bt}{\gamma}\right) + b\cos\left(\frac{2\pi Bt}{\gamma}\right)\right) + \sqrt{\frac{4B}{\gamma}}\left(c\sin\left(\frac{2\pi kBt}{\gamma}\right) + d\cos\left(\frac{2\pi kBt}{\gamma}\right)\right), \tag{89}$$

for some $a, b, c, d \in \mathbb{R}$ such that $c^2 + d^2 \neq 0$, any collections of K of these truncations must be linearly independent, and Condition 1 in Lemma 1 is satisfied. By Theorem 2, we have

$$\lim_{S\to\infty} \frac{C_A^{\text{rms}}}{\log S} \geq \frac{BK}{\gamma} = \frac{BK}{\sqrt{1+\epsilon}}. \tag{90}$$

Since ϵ can be arbitrarily close to zero, we have

$$\lim_{S\to\infty} \frac{C_A^{\text{rms}}}{\log S} \geq BK. \tag{91}$$

On the other hand, since $\gamma \geq 1$ is a necessary condition for a RMS bandlimited and strictly time-limited pulse to exist, we have

$$\lim_{S\to\infty} \frac{C_S^{\text{rms}}}{\log S} = \frac{BK}{\gamma} \leq BK. \tag{92}$$

Finally, since C_A^{rms} is always less than or equal to C_S^{rms}, we have the desired result. □

Appendix

Proof of Lemma 1: The main step in this proof is to observe that

$$H(w) = \int_{-\infty}^{\infty} s(t) s_w^*(t)\,dt, \tag{A.1}$$

where $s(t) = [s_1(t) s_2(t-\tau_2) \cdots s_K(t-\tau_K)]^T$, $s_w(t) = e^{jw}s(t+T) + s(t) + e^{-jw}s(t-T)$ and the asterisk stands for complex conjugate transpose.

Now, for any complex nonzero K-vector $\boldsymbol{a}$,

$$\begin{aligned} \boldsymbol{a}^* H(w)\boldsymbol{a} &= \int_0^{2T} \boldsymbol{a}^* s(t) s^*(t)\boldsymbol{a}\,dt \\ &\quad + e^{-jw}\int_0^{T} \boldsymbol{a}^* s(t) s^*(t+T)\boldsymbol{a}\,dt \\ &\quad + e^{jw}\int_T^{2T} \boldsymbol{a}^* s(t) s^*(t-T)\boldsymbol{a}\,dt \\ &= \|\boldsymbol{a}^* s(t)\|_2^2 + \langle (\boldsymbol{a}^* s(t)), \hat{s}(t)\rangle_2, \end{aligned} \tag{A.2}$$

where $\langle\cdot,\cdot\rangle_2$ and $\|\cdot\|_2$ are the inner product and the induced norm in L_2, and

$$\hat{s}(t) \triangleq \begin{cases} e^{jw}\boldsymbol{a}^* s(t+T), & \text{if } t\in[0,T], \\ e^{-jw}\boldsymbol{a}^* s(t-T), & \text{if } t\in[T,2T], \\ 0, & \text{elsewhere}. \end{cases} \tag{A.3}$$

Notice that

$$|\langle (a^*s(t)), \hat{s}(t)\rangle_2| \le \|a^*s(t)\|_2 \|\hat{s}(t)\|_2 \le \|a^*s(t)\|_2^2, \tag{A.4}$$

where the first inequality follows from Schwarz inequality and the second inequality follows from (A.3). Therefore, $0 \le a^*H(w)a$ and $H(w)$ is nonnegative definite.

Moreover, if one arranges the delays in increasing order such that $\tau_{k_1} \le \tau_{k_2} \le \cdots \tau_{k_K}$, then there must exist an interval $[\tau_i, \tau_{i+1}]$ with length at least T/K. If we take a length T/K subinterval of $[\tau_i, \tau_{i+1}]$, then by Condition 1, it is impossible to have $a^*s(t) = \beta\hat{s}(t)$ for all t in that interval for some complex number β. Therefore, the first inequality in (A.4) will be a strict inequality. If we let

$$\lambda \triangleq \frac{1}{2} \min_{\|a\|_2=1} \|a^*s(t)\|_2^2 + \langle (a^*s(t)), \hat{s}(t)\rangle_2 > 0, \tag{A.5}$$

then it follows from (A.2) that $H(w) - \lambda I$ is positive definite for all $\tau_2, \cdots, \tau_K \in [0, T]$ and all $w \in [0, 2\pi]$. □

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