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On Limiting Characterizations of Memoryless Multiuser Capacity Regions

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Abstract—**The restriction to Gaussian inputs in the limiting expression for the capacity regions of memoryless Gaussian interference and multiple-access channels is shown to fall short of achieving capacity even if the inputs are allowed to be dependent and nonstationary. In addition, the equality between the limiting and the single-letter characterizations of memoryless multiple-access channel capacity is established directly, without recourse to independent coding theorems.**

Index Terms—**Memoryless channels, multiuser information theory, Gaussian channels, multiple-access channels, interference channels.**

I. Introduction

The capacity of information-stable channels is given in terms of limiting expressions involving mutual informations. In the special case of memoryless channels, those limiting expressions usually reduce to single-letter characterizations (not involving limits). Despite many efforts during the last twenty years, no single-letter characterization of the capacity region of the memoryless interference channel has been found, except in particular cases with strong interference (e.g., [1]). In general, only outer bounds and achievable regions have been found. The best achievable region was obtained in [2] and two of its extreme points were shown to be optimal in [3]. Outer bounds have been obtained in [4], [5].

Limiting characterizations of capacity regions for multiuser channels date back to the work of Shannon [6] on two-way channels. It was shown in [7] that the capacity region of the discrete memoryless multiple-access channel (MAC) is given by

$$\mathcal{C}_{\mathrm{MAC}} = \lim_{n\to\infty} \mathrm{co} \cdot\left(\bigcup_{P_{X_1^n X_2^n}=P_{X_1^n}P_{X_2^n}} \left\{(R_1,R_2)\colon \begin{matrix} R_1 \le \frac{1}{n} I(X_1^n;Y^n) \\ R_2 \le \frac{1}{n} I(X_2^n;Y^n)\end{matrix}\right\}\right), \tag{1}$$

where (X_1^n, X_2^n) and Y^n are the length-n input and output of the channel and $\mathrm{co}(\cdot)$ denotes the convex hull operation. Simultaneously, Ahlswede [8] showed that the capacity region of the discrete memoryless interference channel is equal to the limiting expression

$$\mathcal{C}_{\mathrm{IC}} = \lim_{n\to\infty} \mathrm{co} \cdot\left(\bigcup_{P_{X_1^n X_2^n}=P_{X_1^n}P_{X_2^n}} \left\{(R_1,R_2)\colon \begin{matrix} R_1 \le \frac{1}{n} I(X_1^n;Y_1^n) \\ R_2 \le \frac{1}{n} I(X_2^n;Y_2^n)\end{matrix}\right\}\right). \tag{2}$$

In fact, the convex hull operation in the above limiting expressions can be omitted since the capacity region is expressed as the limit of the capacity regions of the n-block channels whose input distributions are allowed to time share several distributions.

The limiting expression in (1) is superseded by the well-known single-letter characterization of [8], [9] discrete memoryless MAC capacity

$$\mathcal{C}_{\mathrm{MAC}} = \mathrm{co} \cdot\left(\bigcup_{P_{X_1 X_2}=P_{X_1}P_{X_2}} \left\{(R_1,R_2)\colon \begin{matrix} R_1 \le I(X_1;Y|X_2) \\ R_2 \le I(X_2;Y|X_1) \\ R_1+R_2 \le I(X_1X_2;Y)\end{matrix}\right\}\right). \tag{3}$$

The general limiting expressions for the capacity of single-user [10] and multiple-access channels [11] with memory have lead to closed-form expressions in several important instances. No such explicit computations have been reported with (1) and (2). In fact, apparently (3) has never been derived from (1); it has always been the result of an independent coding theorem.

This correspondence contains two results. In Section II, it is shown that multivariate Gaussian input distributions do not achieve the limiting characterizations of the capacity regions of Gaussian multiple-access or interference channels, regardless of the memory or nonstationarity allowed. In Section III it is shown that expressions (1) and (3) are equal without recourse to independent derivations via coding theorems. The relevance of those observations is as follows. Lacking a single-letter characterization of the interference channel capacity region (which may not exist), it is tempting to evaluate (2) directly. Recently, it has been reported in [12] that a boundary point in the capacity region of a particular Gaussian interference channel is obtained by evaluating the limiting expression with Gaussian inputs for $n = 2$. Previous experience with well-known linear Gaussian channels subject to power constraints indicates that their capacity is achieved by Gaussian inputs. Thus, it would simplify the computation of (2) enormously if the input random variables therein could be restricted to be Gaussian, (albeit with arbitrary memory structure). The result in Section II shows that such a common shortcut is not possible here. Besides serving as an independent check, the relevance of the result in Section III stems from the fact that in the more general case of the interference channel, only the limiting expression in (2) but not the single-letter characterization, the counterpart to (3), is known.

II. Multivariate Gaussian Distributions are not Optimal

In this Section, we show that Gaussian inputs do not achieve the limiting characterization of the Gaussian MAC capacity region even if they are allowed to be dependent and nonstationary. *A fortiori,* the same conclusion holds for the more general interference channel.

Let us define $\boldsymbol{C}^*$ to be the limiting expression in (1) for the white Gaussian MAC with the union taken over only multivariate Gaussian distributions. Then,

$$\boldsymbol{C}^* = \lim_{n\to\infty} \bigcup_{\substack{\Sigma_1,\Sigma_2\ge 0\\ \mathrm{tr}(\Sigma_k)\le n w_k}} \boldsymbol{C}^*(\Sigma_1,\Sigma_2), \tag{4}$$

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where

$$C_n^*(\Sigma_1, \Sigma_2) = \left\{ (R_1, R_2) : \begin{array}{l} R_1 \le \frac{1}{2n} \log\det\left[I + \Sigma_1(\Sigma_2 + I)^{-1}\right] \\ R_2 \le \frac{1}{2n} \log\det\left[I + \Sigma_2(\Sigma_1 + I)^{-1}\right] \end{array} \right\} \quad (5)$$

and we have assumed, without loss of generality, that the powers of the users are w_1 and w_2 and the variance of the white Gaussian noise is unity, In this section, we find an outer bound to C^* and show that the outer bound is a proper subset of the capacity region of the white Gaussian MAC

$$C_G = \left\{ (R_1, R_2) : \begin{array}{l} R_1 \le \frac{1}{2} \log[1 + w_1] \\ R_2 \le \frac{1}{2} \log[1 + w_2] \\ R_1 + R_2 \le \frac{1}{2} \log[1 + w_1 + w_2] \end{array} \right\}. \quad (6)$$

We need the following lemma whose proof is a straightforward exercise.

Lemma 1: Let A, B, C be any positive semidefinite matrices such that $A \ge B$ (i.e., $A - B$ is also positive semidefinite). Then,

$$\det[I + BC] \le \det[I + AC]. \quad (7)$$

With this lemma, we obtain an outer bound to C^* in the following theorem.

Theorem 1: If $w_1 > 0$ and $w_2 > 0$, then

$$C^* \subseteq \bigcup_{\substack{\lambda_1 \in [0, \lambda_1^*] \\ \lambda_2 \in [0, \lambda_2^*]}} \left\{ (R_1 R_2) : \begin{array}{l} R_1 \le \frac{1}{2} \log[1 + w_1 + \alpha_2 \lambda_2] - \frac{\alpha_2}{2} \log[1 + \lambda_2] \\ R_2 \le \frac{1}{2} \log[1 + w_2 + \alpha_1 \lambda_1] - \frac{\alpha_1}{2} \log[1 + \lambda_1] \\ R_1 + R_2 \le \min\left\{ \frac{\alpha_1}{2} \log\left[1 + \frac{w_1}{\alpha_1}\right] + \frac{1 - \alpha_1}{2} \cdot \log\left[1 + \frac{w_2}{1 - \alpha_1} + \lambda_1\right], \right. \\ \left. \cdot \frac{\alpha_2}{2} \log\left[1 + \frac{w_2}{\alpha_2}\right] + \frac{1 - \alpha_2}{2} \cdot \log\left[1 + \frac{w_1}{1 - \alpha_2} + \lambda_2\right] \right\} \end{array} \right\}, \quad (8)$$

where $\alpha_1 \triangleq w_1/(2w_2 + w_1)$, $\alpha_2 \triangleq w_2/(2w_1 + w_2)$, and λ_1^* and λ_2^* are the unique solutions of

$$\frac{\alpha_1}{2} \log\left[1 + \frac{w_1}{\alpha_1}\right] + \frac{1 - \alpha_1}{2} \log\left[1 + \frac{w_2}{1 - \alpha_1} + \lambda_1^*\right] = \frac{1}{2} \log[1 + w_1 + w_2] \quad (9)$$

and

$$\frac{\alpha_2}{2} \log\left[1 + \frac{w_2}{\alpha_2}\right] + \frac{1 - \alpha_2}{2} \log\left[1 + \frac{w_1}{1 - \alpha_2} + \lambda_2^*\right] = \frac{1}{2} \log[1 + w_1 + w_2], \quad (10)$$

respectively. □

Proof: For any n and $\Sigma_1, \Sigma_2 \in \mathbb{R}^{n \times n}$ such that $\Sigma_1, \Sigma_2 \ge 0$ and $\operatorname{tr}(\Sigma_i) \le n w_i$ for $i = 1, 2$, we shall show that $C_n^*(\Sigma_1, \Sigma_2)$ is a subset of the right hand side of (8). Since Σ_2 is symmetric, we can diagonalize Σ_2 by a unitary matrix P_2 and write $\Lambda_2 = P_2 \Sigma_2 P_2^T$ where Λ_2 is a diagonal matrix with diagonal entries λ_{2i} arranged in descending order (i.e., $\lambda_{21} \ge \lambda_{22} \ge \cdots \ge \lambda_{2n} \ge 0$). Let $\alpha_1, \alpha_2, \lambda_1^*$ and λ_2^* be defined as in the theorem and let $k_2 = \lceil \alpha_2 n \rceil$ (the smallest integer larger than or equal to $\alpha_2 n$) and $\lambda_2 = \lambda_{2k_2}$. Then, it is clear that $\alpha_2 \in (0, 1)$ and $\alpha_2 \lambda_2 \le n^{-1} \sum_{i=1}^{k_2} \lambda_{2i} \le w_2$. Also, let $\Sigma_1^* = P_2 \Sigma_1 P_2^T$ and let its ith diagonal entry be σ_{1i}^*. Furthermore, let

$$\Lambda_2^* = \begin{bmatrix} \lambda_2 I_{k_2} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad (11)$$

$$\Lambda_2^{**} = \begin{bmatrix} \hat{\Lambda}_2 & 0 \\ 0 & \lambda_2 I_{n-k_2} \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad (12)$$

where $\hat{\Lambda}_2$ is the $k_2 \times k_2$ matrix formed by the first k_2 rows and the first k_2 columns of Λ_2 and I_k is the $k \times k$ identity matrix.

Then, for any $(R_1^*, R_2^*) \in C_n^*(\Sigma_1, \Sigma_2)$, we have

$$R_1^* \le \frac{1}{2n} \log\det\left[I + \Sigma_1(\Sigma_2 + I)^{-1}\right] \quad (13)$$

$$= \frac{1}{2n} \log\det\left[I + \Sigma_1^*(\Lambda_2 + I)^{-1}\right] \quad (14)$$

$$\le \frac{1}{2n} \log\det\left[I + \Sigma_1^*(\Lambda_2^* + I)^{-1}\right] \quad (15)$$

$$\le \frac{1}{2n} \sum_{i=1}^{k_2} \log\left[1 + \frac{\sigma_{1i}^*}{1 + \lambda_2}\right] + \frac{1}{2n} \sum_{i=k_2+1}^{n} \log[1 + \sigma_{1i}^*] \quad (16)$$

$$\le \sup_{\substack{0 \le S(\zeta) \\ \int_0^1 S(\zeta)\, d\zeta \le w_1}} \frac{1}{2} \int_0^{\alpha_2} \log\left[1 + \frac{S(\zeta)}{1 + \lambda_2}\right] d\zeta + \frac{1}{2} \int_{\alpha_2}^{1} \log[1 + S(\zeta)]\, d\zeta \quad (17)$$

$$= \begin{cases} \frac{1}{2} \log[1 + w_1 + \alpha_2 \lambda_2] - \frac{\alpha_2}{2} \log[1 + \lambda_2], & \text{if } \lambda_2 \le \frac{w_1}{1 - \alpha_2} = w_1 + \frac{w_2}{2} \\ \frac{1 - \alpha_2}{2} \log\left[1 + \frac{w_1}{1 - \alpha_2}\right], & \text{if } \lambda_2 > \frac{w_1}{1 - \alpha_2} = w_1 + \frac{w_2}{2} \end{cases} \quad (18)$$

Equation (14) follows from the fact [13, p. 651] that $\det(I + AB) = \det(I + BA)$, and inequality (15) follows from Lemma 1 since $(\Lambda_2 + I)^{-1} \le (\Lambda_2^* + I)^{-1}$. Inequality (16) follows from the Hadamard inequality and (17) is a consequence of the concavity of the log function. Finally, (18) follows from the water-filling argument.

Similarly, the rate sum must satisfy

$$R_1^* + R_2^* \le \frac{1}{2n} \log\det\left[I + \Sigma_1(\Sigma_2 + I)^{-1}\right] + \frac{1}{2n} \log\det\left[I + \Sigma_2(\Sigma_1 + I)^{-1}\right] \quad (19)$$

$$\le \frac{1}{2n} \log\det\left[I + \Sigma_1(\Sigma_2 + I)^{-1}\right] + \frac{1}{2n} \log\det[I + \Sigma_2] \quad (20)$$

$$= \frac{1}{2n} \log\det[\boldsymbol{I} + \Sigma_1 + \Sigma_2] \quad (21)$$

$$= \frac{1}{2n} \log\det[\boldsymbol{I} + \Sigma_1^* + \Lambda_2] \quad (22)$$

$$\leq \frac{1}{2n} \log\det[\boldsymbol{I} + \Sigma_1^* + \Lambda_2^{**}] \quad (23)$$

$$\leq \frac{1}{2n} \sum_{i=1}^{k_2} \log[1 + \sigma_{1i}^* + \lambda_{2i}] + \frac{1}{2n} \sum_{i=k_2+1}^{n} \log[1 + \sigma_{1i}^* + \lambda_2] \quad (24)$$

$$\leq \sup_{\substack{0 \leq S_1(\zeta), S_2(\zeta) \\ \int_0^1 S_i(\zeta)\, d\zeta \leq w_i \\ i=1,2.}} \frac{1}{2} \int_0^{\alpha_2} \log[1 + S_1(\zeta) + S_2(\zeta)]\, d\zeta + \frac{1}{2} \int_{\alpha_2}^{1} \log[1 + S_1(\zeta) + \lambda_2]\, d\zeta \quad (25)$$

$$= \begin{cases} \frac{\alpha_2}{2} \log\left[1 + \frac{w_2}{\alpha_2}\right] + \frac{1-\alpha_2}{2} \log\left[1 + \frac{w_1}{1-\alpha_2} + \lambda_2\right], \\ \qquad \text{if } \lambda_2 \leq \frac{w_2}{\alpha_2} - \frac{w_1}{1-\alpha_2} = w_1 + \frac{w_2}{2} \\ \frac{1}{2} \log[1 + w_1 + w_2 + (1-\alpha_2)\lambda_2], \\ \qquad \text{if } \lambda_2 > \frac{w_2}{\alpha_2} - \frac{w_1}{1-\alpha_2} = w_1 + \frac{w_2}{2} \end{cases} \quad (26)$$

On the other hand, we known from (21) that

$$R_1^* + R_2^* \leq \frac{1}{2n} \log\det[\boldsymbol{I} + \Sigma_1 + \Sigma_2] \leq \frac{1}{2} \log[1 + w_1 + w_2]. \quad (27)$$

Since $(\alpha_2/2)\log[1 + w_2/\alpha_2] + ((1-\alpha_2)/2)\log[1 + w_1/(1-\alpha_2) + \lambda_2]$ is increasing in λ_2 and is equal to $(1/2)\log[1 + w_1 + w_2 + (1-\alpha_2)\lambda_2]$ when $\lambda_2 = w_1 + w_2/2$, we have, using λ_2^* defined in (10) and (27),

$$R_1^* + R_2^* \leq \begin{cases} \frac{\alpha_2}{2} \log\left[1 + \frac{w_2}{\alpha_2}\right] + \frac{1-\alpha_2}{2} \log\left[1 + \frac{w_1}{1-\alpha_2} + \lambda_2\right], \\ \qquad \text{if } \lambda_2 \leq \lambda_2^* \\ \frac{1}{2} \log[1 + w_1 + w_2], \\ \qquad \text{if } \lambda_2 > \lambda_2^* \end{cases} \quad (28)$$

and $\lambda_2^* \in (0, w_1 + w_2/2)$. Combining (18) and (28) and noting that $(1/2)\log[1 + w_1 + \alpha_2\lambda_2] - (\alpha_2/2)\log[1 + \lambda_2]$ is strictly decreasing in λ_2, we have

$$(R_1^*, R_2^*) \in \bigcup_{\lambda_2 \in [0, \lambda_2^*]} \cdot \left\{ (R_1, R_2) : \begin{array}{l} R_1 \leq \frac{1}{2} \log[1 + w_1 + \alpha_2\lambda_2] \\ \quad -\frac{\alpha_2}{2} \log[1 + \lambda_2] \\ R_1 + R_2 \leq \frac{\alpha_2}{2} \log\left[1 + \frac{w_2}{\alpha_2}\right] \\ \quad + \frac{1-\alpha_2}{2} \log\left[1 + \frac{w_1}{1-\alpha_2} + \lambda_2\right] \end{array} \right\}. \quad (29)$$

We can repeat the similar argument on R_2^* and $R_1^* + R_2^*$ using α_1 and $\lambda_1 + \lambda_{1k_1}$ where $k_1 = \lceil \alpha_1 n \rceil$ and λ_{1i} is the ith diagonal entry of $\Lambda_1 = \boldsymbol{P}_1 \Sigma_1 \boldsymbol{P}_1^T$. Then, we have

$$(R_1^*, R_2^*) \in \bigcup_{\lambda_1 \in [0, \lambda_1^*]} \cdot \left\{ (R_1, R_2) : \begin{array}{l} R_2 \leq \frac{1}{2} \log[1 + w_2 + \alpha_1\lambda_1] \\ \quad -\frac{\alpha_1}{2} \log[1 + \lambda_1] \\ R_1 + R_2 \leq \frac{\alpha_1}{2} \log\left[1 + \frac{w_1}{\alpha_1}\right] \\ \quad + \frac{1-\alpha_1}{2} \log\left[1 + \frac{w_2}{1-\alpha_2} + \lambda_1\right] \end{array} \right\}. \quad (30)$$

Finally, combining (29) and (30), we have the desired result. □

Corollary 1: The region $\boldsymbol{C}^*$ is a proper subset of $\boldsymbol{C}_G$, if and only if both w_1 and w_2 are strictly greater than zero.

Proof: It is clear that if either w_1 and w_2 or both is equal to zero, $\boldsymbol{C}^*$ and $\boldsymbol{C}_G$ are the same. If both w_1 and w_2 are strictly greater than zero, Theorem I gives an outer bound to $\boldsymbol{C}^*$. Let us denote

$$f(\lambda_2) = \frac{1}{2} \log[1 + w_1 + \alpha_2\lambda_2] - \frac{\alpha_2}{2} \log[1 + \lambda_2] \quad (31)$$

$$g(\lambda_2) = \frac{\alpha_2}{2} \log\left[1 + \frac{w_2}{\alpha_2}\right] + \frac{1-\alpha_2}{2} \left[1 + \frac{w_1}{1-\alpha_2} + \lambda_2\right]. \quad (32)$$

Then, it is easy to check that $f(\lambda_2)$ and $g(\lambda_2)$ are strictly decreasing and strictly increasing, respectively, in the interval $[0, \lambda_2^*]$. Therefore, we have

$$\max_{\boldsymbol{C}^*} 2R_1 + R_2 \leq \max_{\lambda_2 \in [0, \lambda_2^*]} f(\lambda_2) + g(\lambda_2), \quad (33)$$

which is strictly less than $f(0) + g(\lambda_2^*)$. Since the point

$$(R_1, R_2) = \left(\frac{1}{2} \log[1 + w_1], \frac{1}{2} \log[1 + w_2/(1 + w_1)] \right),$$

satisfying $2R_1 + R_2 = f(0) + g(\lambda_2^*)$, is in the region $\boldsymbol{C}_G$, we have shown that $\boldsymbol{C}^*$ is a proper subset of $\boldsymbol{C}_G$. □

In this section, we have shown that multivariate Gaussian distributions do not suffice in order to achieve the capacity region in the limiting expression. The significance of this result lies in its implications in the Gaussian interference channel in which only the limiting expression, but no single-letter characterization, is known. However, it is important to note that this result applies only to the limiting expression (1). Gaussian distributions do achieve the Gaussian MAC capacity region in the single-letter characterization and the limiting expression in [11]. Furthermore, every point in the capacity region can be achieved by codewords drawn from i.i.d. Gaussian random variables.

III. Equality of the Two Characterizations

In this section, we show that the limiting characterization in (1) and the single-letter characterization in (3) for the memoryless multiple-access channel are the same directly without recourse to independent coding theorems.

Let us denote the limiting expression in (1) by $\boldsymbol{C}_{\text{lim}}$ and the single-letter characterization in (3) by $\boldsymbol{C}_{\text{sl}}$.

Theorem 2:

$$\boldsymbol{C}_{\text{lim}} = \boldsymbol{C}_{\text{sl}}. \quad (34)$$

Proof: First of all, it is easy to show that $\boldsymbol{C}_{\text{lim}} \subseteq \boldsymbol{C}_{\text{sl}}$. Indeed, for all n and independent X_1^n and X_2^n, we have,

$$\frac{1}{n} \boldsymbol{I}(X_1^n; Y^n) \leq \frac{1}{n} \boldsymbol{I}(X_1^n; Y^n \mid X_2^n) \quad (35)$$

$$\leq \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{I}(X_{1i}; Y_i \mid X_2^n) \quad (36)$$

$$\leq \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{I}(X_{1i}; Y_i \mid X_{2i}), \quad (37)$$

where (35) follows from the independence of X_1^n and X_2^n, and (36) and (37) follow because the channel is memoryless. Similarly, we have

$$\frac{1}{n} I(X_2^n; Y^n) \leq \frac{1}{n} \sum_{i=1}^{n} I(X_{2i}; Y_i \mid X_{1i}) \tag{38}$$

and

$$\frac{1}{n}[I(X_1^n; Y^n) + I(X_2^n; Y^n)] \leq \frac{1}{n} I(X_1^n X_2^n; Y^n) \leq \frac{1}{n} \sum_{i=1}^{n} I(X_{1i}, X_{2i}; Y_i). \tag{39}$$

Therefore, $C_{\text{lim}} \subseteq C_{\text{sl}}$.

In order to show that $C_{\text{sl}} \subseteq C_{\text{lim}}$, it suffices to show that for any P_{X_1} and P_{X_2}, the corners of the pentagon, $(I(X_1; Y), I(X_2; Y \mid X_1))$ and $(I(X_1; Y \mid X_2), I(X_2; Y))$, are in C_{lim}. By symmetry, it is sufficient to show that $(I(X_1; Y \mid X_2), I(X_2; Y)) \in C_{\text{lim}}$.

This basically involves finding, for any inputs X_1, X_2 and their output Y, a sequence of X_1^n, X_2^n such that $n^{-1} I(X_1^n,; Y^n)$ and $n^{-1} I(X_2^n; Y^n)$ tend to $I(X_1; Y \mid X_2)$ and $I(X_2; Y)$, respectively. The sequences X_1^n and X_2^n will be chosen as follows: X_1^n is the nth product of X_1 and X_2^n is uniform on an (n, M, ϵ) code for the memoryless channel with transition probability

$$P_c(y \mid b) = \sum_{a \in \mathcal{X}_1} P_{Y|X_1X_2}(y|a, b) P_{X_1}(a) \tag{40}$$

and

$$I(X_2; Y) - 2\delta < \frac{\log M}{n} < I(X_2; Y) - \delta. \tag{41}$$

Let $\bar{X}_2$ be a mixture of the n marginals of X_2^n, i.e.,

$$P_{\bar{X}_2} = \frac{1}{n} \sum_{i=1}^{n} P_{X_{2i}}. \tag{42}$$

It follows from the constant-composition direct coding theorem for discrete memoryless channels [14, p. 117] that for asymptotically large n, it is possible to find a code with arbitrarily small $\delta > 0$ and $\epsilon > 0$ such that $P_{\bar{X}_2}(b)$ is arbitrarily close to $P_{X_2}(b)$ for all $b \in \mathcal{X}_2$. Using the independence of X_1^n, X_2^n, and the nonnegativity of entropy, we can write

$$\frac{1}{n} I(X_1^n; Y^n) = \frac{1}{n} I(X_1^n; Y^n|X_2^n) - \frac{1}{n} I(X_1^n; X_2^n|Y^n) \tag{43}$$

$$\geq \frac{1}{n} \sum_{i=1}^{n} I(X_{1i}, Y_i|X_{2i}) - \frac{1}{n} H(X_2^n|Y^n) \tag{44}$$

$$= I(X_1; Y|\bar{X}_2) - \frac{1}{n} H(X_2^n|Y^n). \tag{45}$$

The first term in the right side of (45) approaches $I(X_1; Y|X_2)$ because of the linearity on the conditioning random variable. Moreover, the Fano inequality and the data processing theorem imply that

$$\frac{1}{n} H(X_2^n|Y^n) \leq \epsilon \frac{\log M}{n} + \frac{1}{n} h(\epsilon) \leq \epsilon I(X_2; Y) + \delta' \tag{46}$$

and

$$\frac{1}{n} I(X_2^n; Y^n) \geq (1 - \epsilon) I(X_2; Y) - \delta'' \tag{47}$$

for sufficiently large n. Since ϵ, δ', and δ'' are arbitrarily small, (45)–(46) imply that $n^{-1} I(X_1^n; Y^n)$ and $n^{-1} I(X_2^n; Y^n)$ are at least arbitrarily close to $I(X_1; Y \mid X_2)$ and $I(X_2; Y)$, respectively, as we wanted to show. □

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The Cutoff Rate of Time Correlated Fading Channels

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***Abstract*—The cutoff rate of Ricean and Rayleigh channels characterized by time correlated fading is derived for M-ary phase shift keying (MPSK) modulation. It is useful for situations where perfect interleaving cannot be achieved. It indicates the practical achievable information rate of the channel when coding is employed. It is used to determine the potential coding gains of coded modulation over the correlated fading channel.**

***Index Terms*—Cutoff rate, fading autocorrelation function.**

I. Introduction

On radio and satellite channels, system performance is degraded by Ricean or Rayleigh fading resulting from multipath propagation. Performance can be improved by channel coding combined with

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