Probability of Error in MMSE Multiuser Detection

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***Abstract*—Performance analysis of the minimum-mean-square-error (MMSE) linear multiuser detector is considered in an environment of nonorthogonal signaling and additive white Gaussian noise. In particular, the behavior of the multiple-access interference (MAI) at the output of the MMSE detector is examined under various asymptotic conditions, including: large signal-to-noise ratio; large near–far ratios; and large numbers of users. These results suggest that the MAI-plus-noise contending with the demodulation of a desired user is approximately Gaussian in many cases of interest. For the particular case of two users, it is shown that the maximum divergence between the output MAI-plus-noise and a Gaussian distribution having the same mean and variance is quite small in most cases of interest. It is further proved in this two-user case that the probability of error of the MMSE detector is better than that of the decorrelating linear detector for all values of normalized crosscorrelations not greater than** $\frac{1}{2}\sqrt{2+\sqrt{3}} \cong 0.9659$.

***Index Terms*—Asymptotic normality, divergence, decorrelating detector. MMSE detection, multiuser detection, probability of error.**

I. Introduction

MULTIUSER detection refers to the process of demodulating one or more user data streams from a nonorthogonal multiplex. A number of approaches to this problem have been studied over the past decade, a survey of which can be found in [14]. Among the first advances in this area was the development of multiuser detectors that are optimum in the sense of minimizing error probability or maximizing likelihood [13]. Such methods have been shown to offer very attractive performance characteristics, although this performance comes at the expense of complexity that is exponential in the number of users. The key algorithmic structure of optimum multiuser detection is that of a bank of matched linear filters, followed by a dynamic programming algorithm. More recently, there has been considerable interest in linear multiuser detection: hard-limiting the output of a linear filter (for the detection of binary data). The conventional matched-filter detector, which neglects the presence of multiaccess interference (MAI), is the simplest linear multiuser detector. By correlating with a signal that takes into account the structure of the multiaccess interference, it is possible to obtain a rather dramatic improvement of the bit-error rate of the conventional detector. Although they do not achieve minimum bit-error rate, linear multiuser detectors have been shown to satisfy alternative optimization criteria based on performance indices such as asymptotic efficiency or near–far resistance. Two key linear multiuser detectors are the *decorrelating detector*, which chooses the linear filter to eliminate the output multiple-access interference (MAI) [5]; and the *minimum-mean-square-error* (MMSE) *detector*, which chooses the linear filter to minimize the average mean-square value of the output MAI-plus-noise [6], [10], [15].

In addition to low complexity, it has been shown in [3] that the MMSE multiuser detector offers the significant practical advantage that it can be adapted blindly, i.e., without the use of training sequences or knowledge of interfering signature waveforms [3]. Thus adaptive forms of this detector are among the most likely candidates for practical application of multiuser detection. However, since the MMSE criterion is not directly related to error probability or to the distribution of the background noise, it is of considerable interest to study the error-probability performance of MMSE detectors in an environment of background Gaussian noise, and this is the subject of the present paper.

In particular, in this paper we analyze the behavior of the output of the MMSE detector under various asymptotic conditions of interest in multiuser applications such as wireless communications. These asymptotes include large signal-to-noise ratios and large numbers of users. Also, our study encompasses situations ranging from perfect power control to large near–far ratios. In each case, the results suggest that the error probability for the MMSE detector can be well-approximated by assuming that the output MAI-plus-noise is Gaussian. This is in marked contrast to the bit-error rate of the conventional detector, where the Gaussian approximation is known to be notoriously unreliable (e.g., [11]). To illustrate this point consider Fig. 1, which compares the error probabilities of the conventional detector and the MMSE detector in the case of eight synchronous equal-power users, where all normalized crosscorrelations are equal to 0.1. While for the conventional detector the Gaussian approximation is quite loose for all but very low signal-to-noise ratios, the exact and Gaussian approximation curves are indistinguishable in the MMSE case. The results obtained in this paper explain the excellent accuracy of this approximation.

The two-user case is analyzed in a nonasymptotic setting. We show that the worst case (Kullback–Leibler) divergence between the actual MAI-plus-noise distribution and the Gaussian distribution having the same mean and variance is quite small in this case. Moreover, the fidelity of the Gaussian approximation suggests that the MMSE detector is superior in terms of error probability to the decorrelating detector for all received signal-to-noise ratios, a result that is also proven here

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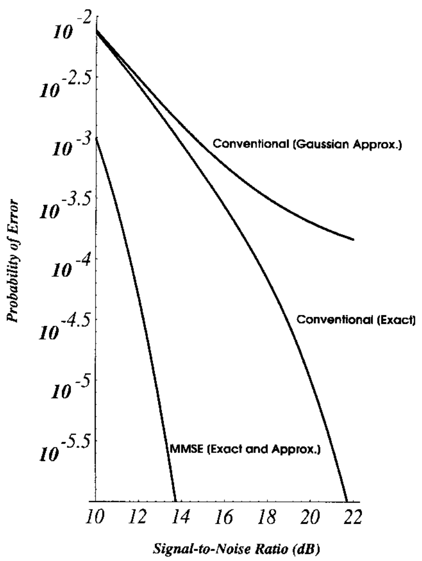

Fig. 1. Comparison of the error probabilities of the conventional detector and the MMSE detector. (Eight synchronous equal-power, equicorrelated users; normalized crosscorrelations are equal to 0.1.)

for the two-user case under the assumption that the magnitude of the normalized signal crosscorrelation is not larger than $\frac{1}{2}\sqrt{2+\sqrt{3}} \cong 0.9659$.

This paper is organized as follows. In Section II, we present the model and detector structure of interest and provide some background needed in the remainder of the paper. Section III contains an analysis of the asymptotic properties of the MMSE detector with a fixed number of users. Section IV analyzes the behavior in the case of equi-correlated signals. Section V examines the two-user case in more detail, as noted above. Finally, Section VI contains some concluding remarks.

II. Background and Signal Model

Consider a K-user binary communication system, employing normalized modulation waveforms $s_1, s_2, \cdots, s_K$, and signaling through an additive white Gaussian noise channel. The received signal in such a channel can be modeled as

$$r(t) = S(t) + \sigma n(t) \tag{1}$$

where $\{n(t)\}$ is white Gaussian noise with unit power spectral density and $\{S(t)\}$ is the superposition of the data signals of the K users, given by

$$S(t) = \sum_{k=1}^{K} A_k \sum_{i=-M}^{M} b_k(i) s_k(t - iT - \tau_k) \tag{2}$$

with the parameters and other quantities being defined as follows:

A_k	received amplitude of the kth user;
$(2M+1)$	frame length;
$b_k(i)$	ith symbol of the kth user (assumed to be binary, ± 1);
τ_k	relative delay of the kth user;
T	inverse of the data rate.

It is assumed that the s_k, the kth user's normalized signaling waveform, is supported only on the interval $[0, T]$; and that $\{b_k(i)\}$ is a collection of independent equiprobable ± 1 random variables.

In this paper, we will restrict attention to the synchronous case of the model (2), in which $\tau_1 = \tau_2 = \cdots = \tau_K \equiv 0$. Since the full model (2) can be viewed as a synchronous model with $(2M+1)K$ users [14], this restriction is not significant for the purposes of bit-error-rate analysis. In this synchronous case, it is easily seen that a sufficient statistic for demodulating the ith data bits of the K users is given by the K-vector $\boldsymbol{y}$ whose kth component is the output of a filter matched to s_k in the ith data interval; i.e.,

$$y_k(i) = \int_{iT}^{(i+1)T} s_k(t - iT) r(t)\, dt, \quad k = 1, 2, \cdots, K. \tag{3}$$

Statistically, this problem is invariant to the choice of the symbol interval i, and so, without loss of generality, we will consider the case $i = 0$, and suppress the index i.

This sufficient vector $\boldsymbol{y}$ can be written as [14]

$$\boldsymbol{y} = \boldsymbol{RAb} + \sigma \boldsymbol{n} \tag{4}$$

where $\boldsymbol{R}$ denotes the normalized crosscorrelation matrix of the signal set $s_1, s_2, \cdots, s_K$[1]

$$\boldsymbol{R}_{k,\ell} = \rho_{k\ell} \triangleq \langle s_k, s_\ell \rangle \tag{5}$$

$\boldsymbol{A} = \mathrm{diag}\,\{A_1, A_2, \cdots, A_K\}$; $\boldsymbol{b}$ is a K-vector whose kth component is b_k; and $\boldsymbol{n}$ is a $\mathcal{N}(0, \boldsymbol{R})$ random vector, independent of $\boldsymbol{b}$.

As noted in Section I, several demodulators have been studied for this channel (see [14]), including the conventional detector

$$\hat{b}_k = \mathrm{sgn}\,(y_k),$$

the optimum multiuser detectors

$$\hat{\boldsymbol{b}} = \arg \max_{\boldsymbol{b} \in \{-1,+1\}^K} p(y|\boldsymbol{b})$$

and

$$\hat{b}_k = \arg \min_{\beta \in \{-1,+1\}} P(\beta \neq b_k),$$

the decorrelating detector

$$\hat{b}_k = \mathrm{sgn}\,((\boldsymbol{R}^{-1}\boldsymbol{y})_k)),$$

and the MMSE detector

$$\hat{b}_k = \mathrm{sgn}\,((\boldsymbol{My})_k) \quad \text{with } \boldsymbol{M} \triangleq (\boldsymbol{R} + \sigma^2 \boldsymbol{A}^{-2})^{-1}, \tag{6}$$

which minimizes the second moment of the difference between the transmitted bit b_k and the output of the linear transformation $\boldsymbol{M}$.

Note that the conventional, decorrelating, and MMSE detectors are *linear detectors* of the form

$$\hat{b}_k = \mathrm{sgn}\,((\boldsymbol{Ly})_k) \tag{7}$$

[1] Throughout this paper $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $L^2[0, T]$.

where $\boldsymbol{L}$ is a $K \times K$ matrix. A more general linear detector for the kth user can be written canonically in the form of a correlator [3]

$$\hat{b}_k = \mathrm{sgn}\left(\langle s_k + x_k, r\rangle\right) \tag{9}$$

where x_k is an L_2 function orthogonal to s_k; i.e.,

$$\langle s_k, x_k\rangle = 0.$$

From the model (2) we see that

$$\begin{aligned}\langle s_k + x_k, r\rangle = &A_k b_k + \sum_{\ell \neq k} A_\ell b_\ell [\rho_{k\ell} + \langle x_k, s_\ell\rangle] \\ &+ \sigma \langle s_k + x_k, w\rangle,\end{aligned} \tag{10}$$

which has mean-square value

$$\begin{aligned}E\{|\langle s_k + x_k, r\rangle|^2\} = &[A_k^2 + \sigma^2] + \sum_{\ell \neq k} A_\ell^2 [\rho_{k\ell} + \langle x_k, s_\ell\rangle]^2 \\ &+ \sigma^2 ||x_k||^2.\end{aligned} \tag{11}$$

Note that the quantity (11) consists of three terms: an irreducible term, $A_k^2 + \sigma^2$, which would be present even in a single-user channel; a term $\Sigma_{\ell \neq k} A_\ell^2 [\rho_{k\ell} + \langle x_k, s_\ell\rangle]^2$, comprising the multiple-access interference (MAI) at the output of the correlation with $s_k + x_k$; and a term $\sigma^2 ||x_k||^2$, due to the passage of background noise through the correlation with x_k. Of course, only the latter two terms can be affected by the choice of x_k, and the three linear detectors described above correspond to choosing x_k using various optimality criteria. The conventional detector chooses x_k to minimize the final term in (11) by setting $x_k = 0$; the decorrelating detector chooses x_k to eliminate completely the MAI term in (11); and the MMSE detector chooses x_k to minimize the sum of these two terms [3]. Note that it is evident from (10) that any reasonable choice of x_k will be in the span of $\{s_2, s_3, \cdots, s_K\}$, since energy outside of this span will not affect the MAI term but will increase the energy in the background noise term. Thus all interesting detectors of the form (8) will also be of the form (7).

The advantage of the conventional detector in this framework is that its implementation requires knowledge of only the signature waveform of the user being demodulated. However, the conventional detector suffers from very significant performance degradation in the presence of significant MAI, resulting in inefficiency of the overall communication system. The decorrelator, on the other hand, eliminates much of this degradation by forcing the MAI to zero. However, its implementation requires knowledge of all users' signature waveforms to demodulate any given user. The MMSE detector enjoys the favorable performance properties of the decorrelator [6] (in particular, it has optimum near–far resistance), but it too requires knowledge of all users' signature waveforms, as well as requiring knowledge of the signal-to-noise ratios of all users. Thus it would appear at first glance that the MMSE detector is more cumbersome than the decorrelator. However, note that the MMSE detector minimizes the quantity $E\{|\langle s_k + x_k, r\rangle|^2\}$, which is the mean-square value of the output of a linear correlation of the received signal with $s_k + x_k$. Thus the MMSE detector minimizes the energy of a quantity to which the receiver has access, and therefore it can be found adaptively for a given user k with knowledge of only the signature waveform of that user (i.e., s_k) [3]; that is, the MMSE detector admits an adaptive implementation that uses only the knowledge required by the conventional detector.

To analyze bit-error rate we will, without loss of generality, henceforth consider the linear detection of bit b_1. For any linear detector of the form (7), it follows from (4) that this bit is demodulated as the algebraic sign of the quantity

$$z_1 = \frac{y_1}{B_1} = b_1 + \beta_2 b_2 + \cdots + \beta_K b_K + \frac{\sigma}{B_1}\tilde{n}_1 \tag{12}$$

where $\beta_1, \cdots, \beta_K$ are the *leakage coefficients*, defined by

$$\beta_k = B_k / B_1, \quad k = 2, 3, \cdots, K \tag{13}$$

with

$$B_k = A_k (\boldsymbol{LR})_{1,k}, \quad k = 1, 2, \cdots, K \tag{14}$$

and where

$$\tilde{n}_1 \sim \mathcal{N}(0, (\boldsymbol{LRL})_{1,1}). \tag{15}$$

Note that the noise in (12) consists of two parts: a Gaussian part, $(\sigma / B_1)\tilde{n}_1$, due to the background Gaussian noise; and a discrete part

$$m_K = \beta_2 b_2 + \cdots + \beta_K b_K \tag{16}$$

comprising the MAI at the normalized output of the linear transformation.

In the special case of the conventional detector we obtain

$$\begin{aligned}(\boldsymbol{LRL})_{1,1} &= 1 \\ B_1 &= A_1 \\ \beta_k &= \frac{A_k}{A_1}\rho_{1k}, \qquad k = 2, 3, \cdots, K\end{aligned} \tag{17}$$

and in the case of the decorrelating detector we get

$$\begin{aligned}(\boldsymbol{LRL})_{1,1} &= (\boldsymbol{R}^{-1})_{1,1} \\ B_1 &= A_1 \\ \beta_k &= 0, \qquad k = 2, 3, \cdots, K.\end{aligned} \tag{18}$$

In the following sections we will study the behavior of the leakage coefficients, of $(\boldsymbol{LRL})_{1,1}$, and of B_1 for the MMSE detector (6).

By symmetry, it is straightforward to see that the probability of error $(P(\hat{b}_1 \neq b_1))$ of the linear detector in (7) is given by

$$\begin{aligned}P_1 &= P\left(\beta_2 b_2 + \cdots + \beta_K b_K + \frac{\sigma}{B_1}\tilde{n}_1 > 1\right) \\ &= \frac{1}{2^{K-1}} \sum_{b_2, \cdots, b_K \in \{-1,+1\}^{K-1}} \\ &\quad \cdot Q\left(\frac{A_1}{\sigma}\frac{(\boldsymbol{LR})_{1,1}(1 + \beta_2 b_2 + \cdots + \beta_K b_K)}{\sqrt{(\boldsymbol{LRL})_{1,1}}}\right)\end{aligned} \tag{19}$$

where Q denotes the complementary unit cumulative Gaussian distribution.

The Gaussian approximation to P_1 substitutes the binomial distribution of $\beta_1 b_2 + \cdots + \beta_K b_K$ by a Gaussian distribution with identical variance $(\beta_1^2 + \cdots + \beta_K^2)$ to yield

$$\tilde{P}_1 = Q\left(\frac{1}{\sqrt{\dfrac{\sigma^2(\boldsymbol{LRL})_{1,1}}{A_1^2(\boldsymbol{LR})_{1,1}} + \beta_1^2 + \cdots + \beta_K^2}}\right). \tag{20}$$

From (19) and (20) we can expect the Gaussian approximation to be accurate if $\beta_1^2 + \cdots + \beta_K^2$ is negligible when compared with

$$\min\left\{1, \frac{A_1^2(\boldsymbol{LR})_{1,1}}{\sigma^2(\boldsymbol{LRL})_{1,1}}\right\} \tag{21}$$

or if the binomial distribution of the MAI in (16) does not have a vanishing variance but it satisfies the central limit theorem. Interestingly, the latter alternative cannot occur whenever the MMSE detector is used and the crosscorrelation matrix $\boldsymbol{R}$ is nonsingular. To see this, note first that, in this situation, the leakage coefficients satisfy

$$\sum_{j=2}^{K} |\beta_j| < 1 \tag{22}$$

because if $\boldsymbol{R}$ is nonsingular and $\boldsymbol{L} = \boldsymbol{M}$, P_1 in (19) vanishes as $\sigma \to 0$. But, according to (22), the variance of the MAI is upper-bounded by

$$\begin{aligned}\sum_{j=2}^{K} \beta_j^2 &\le (\max_{j=2,\cdots,K} |\beta_j|) \sum_{j=2}^{K} |\beta_j| \\ &< (\max_{j=2,\cdots,K} |\beta_j|) < 1.\end{aligned} \tag{23}$$

Thus if the variance of the MAI is nonvanishing with K, so is the maximum leakage coefficient. Consequently, (16) does not converge to a Gaussian random variable since the variance of at least one of its binary terms is a nonvanishing fraction of its overall variance.

Following [12], it is straightforward to show that if we partition

$$\{2, 3, \cdots, K\} = G \cup \overline{G} \tag{24}$$

such that

$$\sum_{j \in G} |\beta_j| < 1 \tag{25}$$

then the following Chernoff upper bound holds:

$$P_1 < \exp\left\{-\frac{\left(1 - \sum_{j\in G} |\beta_j|\right)^2}{2\left(\dfrac{\sigma^2}{B_1^2}(\boldsymbol{LRL})_{1,1} + \sum_{j\in\overline{G}} \beta_j^2\right)}\right\}. \tag{26}$$

In particular, taking $G = \emptyset$ we have

$$P_1 < \exp\left\{-\frac{1}{2\left(\dfrac{\sigma^2}{B_1^2}(\boldsymbol{LRL})_{1,1} + \beta_1^2 + \cdots + \beta_K^2\right)}\right\} \tag{27}$$

a bound that exceeds the Gaussian approximation (20) by at least a factor of 2. Furthermore, whenever the crosscorrelation matrix $\boldsymbol{R}$ is nonsingular, then (22) yields another upper bound on the MMSE error probabililty by taking the special case of (26) where $\overline{G} = \emptyset$

$$P_1 < \exp\left\{-B_1^2\left(1 - \sum_{j=2}^{K} |\beta_j|\right)^2 \Big/ (2\sigma^2(\boldsymbol{MRM})_{1,1})\right\}. \tag{28}$$

III. Signal-to-Noise Ratio Asymptotics

In general, the probability of error of the MMSE detector is given by (19) with $\boldsymbol{L} = \boldsymbol{M} \equiv (\boldsymbol{R} + \sigma^2 \boldsymbol{A}^{-2})^{-1}$. In order to find approximations to P_1 in this case, we will investigate conditions under which the leakage coefficients vanish; i.e.,

$$\beta_k \triangleq \frac{B_k}{B_1} \equiv \frac{A_k(\boldsymbol{MR})_{1,k}}{A_1(\boldsymbol{MR})_{1,1}} \to 0. \tag{29}$$

If true for all $k = 2, 3, \cdots, K$, (29) will yields asymptotic normality for the total interference term. The following proposition gives such conditions.

Proposition 3.1: Fix an arbitrary normalized crosscorrelation matrix $\boldsymbol{R}$, and fix an arbitrary $k = 2, \cdots, K$. Then $\beta_k \to 0$ if any of the following conditions is satisfied:

1) $\sigma \to 0$
2) $\dfrac{A_1}{\sigma} \to \infty$
3) $\dfrac{A_k}{\sigma} \to 0$
4) $\dfrac{A_k}{\sigma} \to \infty$

where in 1) we assume that the amplitudes remain constant and arbitrary, and in 2)–4), we assume that all other signal-to-noise ratios are constant and arbitrary.

Proof: Recall from (17) that

$$B_k = A_k(\boldsymbol{MR})_{1k}. \tag{30}$$

Since

$$\begin{aligned}\boldsymbol{MR} &= \boldsymbol{M}[\boldsymbol{R} + \sigma^2\boldsymbol{A}^{-2} - \sigma^2\boldsymbol{A}^{-2}] \\ &= \boldsymbol{I} - \sigma^2\boldsymbol{MA}^{-2}\end{aligned} \tag{31}$$

we have

$$(\boldsymbol{MR})_{11} = 1 - \frac{\sigma^2}{A_1^2} M_{11} \tag{32}$$

and

$$(\boldsymbol{MR})_{1k} = -\frac{\sigma^2}{A_k^2} M_{1k}. \tag{33}$$

So the ratio of interest is

$$\beta_k = \frac{B_k}{B_1} = \frac{A_k}{A_1}\frac{(\boldsymbol{MR})_{1k}}{(\boldsymbol{MR})_{11}} = \frac{\sigma}{A_1}\frac{\sigma}{A_k}\frac{-M_{1k}}{1-\dfrac{\sigma^2}{A_1^2}M_{11}}. \tag{34}$$

We will, therefore, concentrate on finding a partitioned form of the inverse of $[\boldsymbol{R}+\sigma^2\boldsymbol{A}^{-2}]$ that allows simplification of (34). The following result is useful in this regard.

Fact 1 (cf., [4]): Suppose $\alpha > 1, \rho$ is a column vector, and $\boldsymbol{H}$ is an invertible matrix. Then

$$\begin{bmatrix} \alpha & \rho^T \\ \rho & \boldsymbol{H} \end{bmatrix}^{-1} = \begin{bmatrix} \mu & -\mu\rho^T\boldsymbol{H}^{-1} \\ -\mu\boldsymbol{H}^{-1}\rho & \boldsymbol{H}^{-1}+\mu\boldsymbol{H}^{-1}\rho\rho^T\boldsymbol{H}^{-1} \end{bmatrix} \tag{35}$$

with $\mu^{-1} = \alpha - \rho^T\boldsymbol{H}^{-1}\rho$.

Let us apply Fact 1 to the quantity

$$\frac{M_{1k}}{1-\dfrac{\sigma^2}{A_1^2}M_{11}} \tag{36}$$

arising in (34). In this case, the relevant quantities from (35) are

$$\alpha = 1 + \frac{\sigma^2}{A_1^2} \tag{37}$$

$$\rho = \rho_1 = \begin{bmatrix} \rho_{12} \\ \vdots \\ \rho_{1K} \end{bmatrix} \tag{38}$$

and

$$\boldsymbol{H} = \boldsymbol{R}_1 + \begin{bmatrix} \sigma^2/A_2^2 & & 0 \\ & \ddots & \\ 0 & & \sigma^2/A_K^2 \end{bmatrix} \tag{39}$$

where $\boldsymbol{R}_1$ is obtained by deleting the first row and the first column from $\boldsymbol{R}$. We thus have[2]

$$\begin{aligned} \frac{-M_{1k}}{1-\dfrac{\sigma^2}{A_1^2}M_{11}} &= \frac{M_{11}(\rho_1^T\boldsymbol{H}^{-1})_k}{1-\dfrac{\sigma^2}{A_1^2}M_{11}} = \frac{(\rho_1^T\boldsymbol{H}^{-1})_k}{\dfrac{1}{M_{11}}-\dfrac{\sigma^2}{A_1^2}} \\ &= \frac{(\rho_1^T\boldsymbol{H}^{-1})_k}{1+\dfrac{\sigma^2}{A_1^2}-\rho_1^T\boldsymbol{H}^{-1}\rho_1-\dfrac{\sigma^2}{A_1^2}} \\ &= \frac{(\rho_1^T\boldsymbol{H}^{-1})_k}{1-\rho_1^T\boldsymbol{H}^{-1}\rho_1} \end{aligned} \tag{40}$$

which is the $1k$ element of the inverse matrix $\boldsymbol{M}^{-1}$ when $\alpha = 1$ (i.e., $\frac{A_1}{\sigma} \to \infty$) (cf. Fact 1).

We now consider each of the cases of the proposition separately.

[2] $(\rho_1^T\boldsymbol{H}^{-1})_k$ refers to the coefficient corresponding to the kth user; i.e., the $(k-1)$th component of $\rho_1^T\boldsymbol{H}^{-1}$.

Case 1 ($\sigma \to 0$): It is clear that $\boldsymbol{H} \to \boldsymbol{R}_1$, and, therefore,

$$\frac{(\rho_1^T\boldsymbol{H}^{-1})_k}{1-\rho_1^T\boldsymbol{H}^{-1}\rho_1} \text{ converges to } \frac{(\rho_1^T\boldsymbol{R}_1^{-1})_k}{1-\rho_1^T\boldsymbol{R}_1^{-1}\rho_1} = \frac{(\rho_1^T\boldsymbol{R}_1^{-1})_k}{\eta_1}. \tag{41}$$

(Here we are assuming that $\boldsymbol{R}_1^{-1}$ exists—a requirement that can be relaxed.) Thus

$$\beta_k = \frac{\sigma}{A_1}\frac{\sigma}{A_k}\frac{(\rho_1^T\boldsymbol{H}^{-1})_k}{1-\rho_1^T\boldsymbol{H}^{-1}\rho_1} \to 0. \tag{42}$$

Case 2 ($A_1/\sigma \to \infty$): Again we see immediately that the quantity

$$\frac{\sigma}{A_k}\frac{(\rho_1^T\boldsymbol{H}^{-1})_k}{1-\rho_1^T\boldsymbol{H}^{-1}\rho_1} \tag{43}$$

depends only on constant parameters, and thus, $\beta_k = (\sigma/A_1)\mathcal{C} \to 0$.

Case 3 ($A_k/\sigma \to 0$): We will now use Fact 1 to compute $\boldsymbol{H}^{-1}$ itself. In the case where $\alpha \to \infty$, the inverse takes the asymptotic form

$$\frac{1}{\alpha}\begin{bmatrix} 1 & -\rho^T\boldsymbol{H}^{-1} \\ -\boldsymbol{H}^{-1}\rho & \boldsymbol{H}^{-1}\rho\rho^T\boldsymbol{H}^{-1} \end{bmatrix}. \tag{44}$$

So $(\rho_1^T\boldsymbol{H}^{-1})_k$ is now given by

$$\frac{1}{1+\dfrac{\sigma^2}{A_k^2}}\mathcal{C} \tag{45}$$

and

$$\rho_1^T\boldsymbol{H}^{-1}\rho_1 = \frac{1}{1+\dfrac{\sigma^2}{A_k^2}}\mathcal{C}'. \tag{46}$$

Thus

$$\beta_k = \frac{\sigma}{A_1}\frac{\sigma}{A_k}\frac{\left(1+\dfrac{\sigma^2}{A_k^2}\right)^{-1}\mathcal{C}}{1-\left(1+\dfrac{\sigma^2}{A_k^2}\right)^{-1}\mathcal{C}'} = \frac{\sigma}{A_1}\frac{\sigma}{A_k}\frac{\mathcal{C}}{1+\dfrac{\sigma^2}{A_k^2}-\mathcal{C}'} \tag{47}$$

where the right side is seen to vanish as $A_k/\sigma \to 0$.

Case 4 ($A_k/\sigma \to \infty$): Now Fact 1 with $\alpha \to 1$ shows that $\boldsymbol{H}$ will converge to the constant matrix

$$\begin{bmatrix} \beta' & -\beta'\overline{\rho}^T\boldsymbol{G}^{-1} \\ -\beta'\boldsymbol{G}^{-1}\overline{\rho} & \boldsymbol{G}^{-1}+\beta'\boldsymbol{G}^{-1}\overline{\rho}\overline{\rho}^T\boldsymbol{G}^{-1} \end{bmatrix} \tag{48}$$

where $\beta' = 1/1-\overline{\rho}^T\boldsymbol{G}^{-1}/\overline{\rho}$ and $\overline{\rho}$ is the $(K-2)$-dimensional row obtained by striking out columns 1 and k, and $\boldsymbol{G}$ is the $(K-2)\times(K-2)$ matrix obtained from $[\boldsymbol{R}+\sigma^2\boldsymbol{A}^{-2}]$ by striking out columns and rows 1 and k.

Thus

$$\beta_k = \frac{\sigma}{A_k}\frac{\sigma}{A_1}\mathcal{C} \to 0 \quad \text{as} \quad \frac{A_k}{\sigma} \to \infty. \tag{49}$$

□

The following conclusions can be drawn from Proposition 3.1.

- Under Condition 1) ($\sigma \to 0$), the MMSE detector approaches the decorrelator, and so

$$P_1 \sim Q(\sqrt{\mathrm{SNR}_o}) \tag{50}$$

where SNR_o denotes the reciprocal of the variance of $z_1 - b_1$, given in this case by

$$\mathrm{SNR}_o = \frac{A_1^2}{\sigma^2(\boldsymbol{R}^{-1})_{1,1}}. \tag{51}$$

This conclusion also follows trivially from (6) (see also [6]).
- Under Condition 2) $(A_1/\sigma \to \infty)$, $m_K \to 0$. So, the MAI-plus-noise is asymptotically Gaussian, and again we have (50).
- Condition 4) $(A_k/\sigma \to \infty)$ says that, asymptotically, the larger an interfering user's amplitude, the smaller its effect on bit-error rate. This ensures "near–far resistance" of the MMSE detector.

We note that an asymptotic case not treated by Proposition 3.1 is that of low-SNR for the desired user. For this case, the following is straightforward to show:

$$\frac{A_1}{\sigma} \to 0 \Rightarrow P_1 \to \frac{1}{2}. \tag{52}$$

The behavior of the leakage coefficients under the condition that background noise dominates MAI can be characterized by the following result, which shows that the leakage coefficients converge to those of the conventional detector (cf. (17)).

Proposition 3.2: Fix $\boldsymbol{R}$, and $A_1, \cdots, A_K$. Then, for $k > 1$

$$\lim_{\sigma \to \infty} \beta_k = \frac{A_k \rho_{1k}}{A_1}. \tag{53}$$

(Note that we can use this result when the SNR of the desired user is not necessarily small by letting A_1 grow with σ.)

Proof: We recall that the quantity of interest (34) is

$$\beta_k = \frac{\sigma}{A_1}\frac{\sigma}{A_k}\frac{(\rho_1^T \boldsymbol{H}^{-1})_k}{1 - \rho_1^T \boldsymbol{H}^{-1}\rho_1} \tag{54}$$

where

$$\boldsymbol{H} = \boldsymbol{R}_1 + \sigma^2 \begin{bmatrix} 1/A_2^2 & & \\ & \ddots & \\ & & 1/A_K^2 \end{bmatrix} \tag{55}$$

and so

$$\sigma^2 \boldsymbol{H}^{-1} = \left(\frac{1}{\sigma^2}\boldsymbol{R}_1 + \begin{bmatrix} 1/A_2^2 & & \\ & \ddots & \\ & & 1/A_K^2 \end{bmatrix} \right)^{-1} \underset{\sigma \to \infty}{\longrightarrow} \begin{bmatrix} A_2^2 & & \\ & \ddots & \\ & & A_K^2 \end{bmatrix} \tag{56}$$

which implies

$$\sigma^2(\rho_1^T \boldsymbol{H}^{-1})_k \to A_k^2 \rho_{1k} \tag{57}$$

and

$$1 - \rho_1^T \boldsymbol{H}^{-1}\rho_1 \to 1 \tag{58}$$

which when combined with (34) gives the desired result. □

IV. Equicorrelated Signature Waveforms

Recall the expression (12) for the normalized output of the MMSE linear transformation corresponding to user 1

$$z_1 = b_1 + \beta_2 b_2 + \cdots + \beta_K b_K + \frac{\sigma}{B_1}\tilde{n}_1 \tag{59}$$

where $\beta_k = B_k/B_1$ is the kth leakage coefficient. In the preceding section, we considered the behavior of the ratios β_k for fixed K under several asymptotic conditions involving signal-to-noise ratios. These results implied that, under various such asymptotes, the non-Gaussian portion of the MAI-plus-noise (i.e., $\beta_2 b_2 + \cdots + \beta_K b_K$) vanishes, thereby yielding a Gaussian distribution for the overall interference term. Another asymptote of interest is that in which the number of users K increases without bound. In this section, we examine the asymptotic behavior of the relative MAI

$$m_K = \beta_2 b_2 + \cdots + \beta_K b_K \tag{60}$$

in this limit. In this analysis, we will restrict attention to the case of equicorrelated signals. Although this restriction is made here primarily for analytical convenience, it arises in several practical situations—for example, in a synchronous network using shifts of the same m-sequence for the various signature waveforms [7]. Moreover, the intuition gained from this case can be extrapolated somewhat to more general crosscorrelation patterns.

The asymptotic behavior of the MAI (60) in the equicorrelated-signal case is described by the following result, a proof of which can be found in the Appendix.

Proposition 4.1: Suppose that $\rho_{ij} = \rho_K \in (-1/(K-1), 1)$[3] for all $i \neq j$, and the amplitudes satisfy

$$0 < \inf_{k=1,2,\cdots} |A_k| \le \sup_{k=1,2,\cdots} |A_k| < \infty. \tag{61}$$

If

$$\limsup_{K \to \infty} \rho_K < 1 \tag{62}$$

then

$$\lim_{K \to \infty} m_K = 0 \quad \text{a.s. and in m.s.} \tag{63}$$

Proposition 4.1 states that, remarkably, as the number of users grows, the MMSE detector is able to suppress the effect of the multiaccess interference, by making the leakage coefficients vanish sufficiently fast that the relative MAI also vanishes $(m_K \to 0)$. Note that condition (61) guarantees that the channel does indeed have a number of "effective" interferers (i.e., nonnegligible with respect to the others) that grows without bound. Note also that the asymptotic interference suppression exhibited in Proposition 4.1 is enabled

[3] This range of ρ_K is that for which $\boldsymbol{R}$ is positive-definite.

by the fact that the crosscorrelation matrix is nonsingular, and thus bandwidth grows linearly with K.

In the case of perfect power control and equicorrelated signals, the leakage coefficients admit the following explicit nonasymptotic formula.

Proposition 4.2: Suppose that $\rho_{ij} = \rho \in (-1/(K-1), 1)$ for all $i \neq j$, and

$$A_1 = A_2 = \cdots = A_K = A. \tag{64}$$

Then the leakage coefficients are given by (for $k = 2, 3, \cdots, K$)

$$\beta_k = \frac{\rho \frac{\sigma^2}{A^2}}{\frac{\sigma^2}{A^2} + 1 + (K-2)\rho - \rho^2(K-1)}. \tag{65}$$

Proof: Since the diagonal coefficients of $[\boldsymbol{R} + \sigma^2 \boldsymbol{A}^{-2}]$ are all equal (to $1 + \sigma^2/A^2$) and its off-diagonal coefficients are also equal (to ρ), it is straightforward to show that

$$\boldsymbol{M}_{1,1} = \cdots = \boldsymbol{M}_{K,K} = \left(1 + \frac{\sigma^2}{A^2} - \frac{\rho^2(K-1)}{1 + \frac{\sigma^2}{A^2} + (K-2)\rho}\right)^{-1} \tag{66}$$

and

$$\frac{\boldsymbol{M}_{i,j}}{\boldsymbol{M}_{1,1}} = \frac{\rho}{1 + \frac{\sigma^2}{A^2} + (K-2)\rho}. \tag{67}$$

Upon substitution of (66) and (67) into (34), the desired result follows. □

The decrease of the leakage coefficients as a function of K is made evident in (65). For example, in the case analyzed in Fig. 1 where $K = 8$ and $\rho = 0.1$, β_k ranges (with SNR $\in [10\,\text{dB}, 14\,\text{dB}]$) from $\frac{1}{163}$ to $\frac{1}{452}$.

Particularizing (19) to the present case, we obtain the following expression for the error probability of the MMSE detector (with which Fig. 1 was generated) (see (68) at the bottom of this page) which for parameters of interest is indistinguishable from the Gaussian approximation

$$\tilde{P}_1 = Q\left(\frac{A}{\sigma}\sqrt{1 - \frac{\rho^2(K-1)}{1 + \frac{\sigma^2}{A^2} + (K-2)\rho}}\right). \tag{69}$$

It is interesting to observe from (67) that, if ρ is held at any value in $[0, 1)$, then both the MMSE detector and the decorrelating detector converge to the conventional detector as $K \to \infty$. However, it would be wrong to conclude that this means that there is little to be gained in such a case by taking the presence of multiaccess interference into account. As $K \to \infty$, the bit-error rate of the conventional detector approaches $1/2$, whereas the bit-error rate of the decorrelating detector and of the MMSE detector approaches

$$Q\left(\frac{A}{\sigma}\sqrt{1-\rho}\right). \tag{70}$$

Although we would expect the results of Proposition 4.1 to hold in many scenarios other than the equicorrelated case, it should be pointed out that there are indeed crosscorrelation matrices for which the MAI at the MMSE output does not vanish as $K \to \infty$ (and as we saw in Section II, it is not asymptotically Gaussian either). Consider the case

$$A_1 = A_2 = \cdots = A_K = A \tag{71}$$

$$\rho_{ij} = 0, \qquad i \neq j,\ i = 2, \cdots, K,\ j = 2, \cdots, K$$

$$\rho_{ij}^2 = \gamma \frac{1 - \rho_{12}^2}{K-2}, \qquad j = 3, \cdots, K \tag{72}$$

where $\gamma \in [0, 1]$ for nonnegativity of $\boldsymbol{R}$. It can be shown that, regardless of the value of K

$$\frac{\beta_2^2}{\beta_3^2 + \cdots + \beta_K^2} = \frac{\rho_{12}^2}{\gamma(1 - \rho_{12}^2)} \tag{73}$$

and

$$\beta_2 = \frac{\rho_{12}}{\frac{A^2}{\sigma^2}\left[\left(1 + \frac{\sigma^2}{A^2}\right)^2 - \rho_{12}^2 - \gamma^2(1 - \rho_{12}^2)\right] - \left(1 + \frac{\sigma^2}{A^2}\right)}. \tag{74}$$

V. A Closer Examination of the Two-User Case

The results in Sections III and IV bring up the question of how far the distribution of the MAI-plus-noise at the MMSE output can diverge from a Gaussian distribution under nonasymptotic conditions. In the following subsection, we show that the answer to this question is "not very much" for the two-user case unless the normalized crosscorrelation between the two signals is very close to 1 in magnitude. Note that this two-user case should, in the context of the Gaussian approximation, be an unfavorable case since the binomially distributed MAI is binary in the two-user case.

Recall from Section II that the output of the linear transformation in the MMSE detector has minimum variance among all possible linear detectors. For this reason, the closeness of the MMSE output distribution to a Gaussian distribution found in Section V-A would suggest that its error probability might be smaller than that of any other linear detector whose output is also almost Gaussian. A notable example of such a linear detector is the decorrelator, whose output error is

$$P_1 = 2^{1-K} \sum_{n=0}^{K-1} \binom{K-1}{n} Q\left(\frac{A}{\sigma} \frac{1 + \frac{\sigma^2}{A^2} + (K-2)\rho - \rho^2(K-1) + \frac{\sigma^2}{A^2}\rho(K-1-2n)}{\sqrt{\left(1 + \frac{\sigma^2}{A^2} + (K-2)\rho\right)^2 - \rho^2(K-1)\left[1 + 2\frac{\sigma^2}{A^2} + (K-2)\rho\right]}}\right) \tag{68}$$

exactly Gaussian. In Section V-B we prove, again for the two-user case, that the error probability of the MMSE detector is smaller than that of the decorrelator provided the normalized crosscorrelation is less in magnitude than about 0.97.

We now turn to the first of these two results, namely, the divergence analysis of the MMSE output.

A. Divergence Analysis of the MMSE Output

We are interested in quantifying the non-Gaussianness of the random variable

$$X = B_2 b_2 + \cdots + B_K b_K + \tilde{n}_1.$$

A useful way to quantify that non-Gaussianness is to analyze the (Kullback–Leibler) divergence of the distribution of the random variable X and the Gaussian random variable $\overline{X}$ with the same variance [9][4]

$$D(f_X \| f_{\overline{X}}) = \int_{-\infty}^{\infty} f_X(x) \log \frac{f_X(x)}{f_{\overline{X}}(x)}\, dx \tag{75}$$

where $f_{\overline{X}}(x) = (1/\sqrt{2\pi}\sigma_1)\exp(-x^2/2\sigma_1^2)$ and

$$\begin{aligned}\sigma_1^2 &= \sum_{k=2}^{K} B_k^2 + \sigma^2(\boldsymbol{MRM})_{11}\\ &= \sum_{k=2}^{K} A_k^2 (\boldsymbol{MR})_{1k}^2 + \sigma^2(\boldsymbol{MRM})_{11}\\ &= \sum_{k=2}^{K} A_k^2 \frac{\sigma^4}{A_k^4} M_{1k}^2 + \sigma^2(M_{11} - \sigma^2(\boldsymbol{MA}^{-2}\boldsymbol{M})_{11})\\ &= \sigma^4 \sum_{k=2}^{K} \frac{M_{1k}^2}{A_k^2} + \sigma^2 M_{11} - \sigma^4 \sum_{j=1}^{K} \frac{M_{1j}^2}{A_j^2}\\ &= \sigma^2 M_{11}\left(1 - \frac{\sigma^2}{A_1^2} M_{11}\right) = \sigma^2 M_{11} \frac{B_1}{A_1}.\end{aligned} \tag{76}$$

Ideally, we would like to compute the maximum non-Gaussianness of f_X over all signal-to-noise ratios $(A_1/\sigma, \cdots, A_K/\sigma)$ for a given normalized $\boldsymbol{R}$. This problem is open. However, we conjecture that the maximum non-Gaussianness is very close to that obtained with only one active interferer. This is based on the fact that $\tilde{n}_1 + \max_{j=2,\cdots,K} |B_j| b_j$ is more non-Gaussian than is X.

The minimum non-Gaussianness of the MAI at the output of the conventional detector given the first moments of the MAI interference has been investigated in [11]. The following result characterizes the non-Gaussianness of the output of the MMSE linear transformation in the synchronous two-user case.

Proposition 5.1: Suppose $K = 2$, and denote the normalized crosscorrelation by ρ and the signal-to-noise ratio of the interferer by $r_2 = A_2^2/\sigma^2$. Define

$$\zeta(\alpha^2) = D(\tfrac{1}{2}\mathcal{N}(-\alpha, 1) + \tfrac{1}{2}\mathcal{N}(\alpha, 1) \| \mathcal{N}(0, 1+\alpha^2)). \tag{77}$$

[4] Note that the non-Gaussianness of X is invariant to scaling of X.

Then we have the following.

a) The non-Gaussianness of the MAI at the output of the linear MMSE transformation is given by

$$\zeta\left(\frac{\rho^2 r_2}{1 + (1-\rho^2)(2r_2 + r_2^2)}\right). \tag{78}$$

b) The worst case non-Gaussianness is given by

$$\max_{r_2>0} \zeta\left(\frac{\rho^2 r_2}{1 + (1-\rho^2)(2r_2 + r_2^2)}\right) = \zeta\left(\frac{1}{2}\frac{\rho^2}{\sqrt{1-\rho^2} + 1 - \rho^2}\right). \tag{79}$$

Proof: We characterize the non-Gaussianness of an equimixture of two identical shifted Gaussian densities.

Fact 2: For all a and $v > 0$ we have

$$\zeta\left(\frac{a^2}{v^2}\right) = \xi\left(\frac{a^2}{v^2}\right) + \frac{1}{2}\log\left(1 + \frac{a^2}{v^2}\right) - \frac{a^2}{v^2}\log e \tag{80}$$

with

$$\xi(\eta^2) = e^{-\eta^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\eta} e^{-u^2/2\eta^2} \cosh(u) \cdot \log\cosh(u)\, du \tag{81}$$

Proof of Fact 2: The following Pythagorean identity can be verified straightforwardly from the definition of divergence

$$\begin{aligned}D(\tfrac{1}{2}\mathcal{N}(a, v^2) + \tfrac{1}{2}\mathcal{N}(-a, v^2) \| \mathcal{N}(0, a^2+v^2))\\ = D(\tfrac{1}{2}\mathcal{N}(a, v^2) + \tfrac{1}{2}\mathcal{N}(-a, v^2) \| \mathcal{N}(0, v^2))\\ - D(\mathcal{N}(0, a^2+v^2) \| \mathcal{N}(0, v^2))\end{aligned} \tag{82}$$

where (see [1])

$$D(\mathcal{N}(0, a^2+v^2) \| \mathcal{N}(0, v^2)) = \frac{1}{2}\frac{a^2}{v^2}\log e - \frac{1}{2}\log\left(1 + \frac{a^2}{v^2}\right) \tag{83}$$

and (with ϕ the standard Gaussian density function)

$$\begin{aligned}D\left(\frac{1}{2}\mathcal{N}(a, v^2) + \frac{1}{2}\mathcal{N}(-a, v^2) \| \mathcal{N}(0, v^2)\right)\\ = E\left\{\log\left(\frac{1}{2}\frac{\phi(Z-a)}{\phi(Z)} + \frac{1}{2}\frac{\phi(Z+a)}{\phi(Z)}\right)\right\}\\ = E\left\{\log\exp\left(-\frac{a^2}{2v^2}\right) + \log\cosh\left(\frac{Za}{v^2}\right)\right\}\end{aligned} \tag{84}$$

$$= -\frac{1}{2}\frac{a^2}{v^2}\log e + E\left\{\log\cosh\left(\frac{Za}{v^2}\right)\right\} \tag{85}$$

where Z has distribution $\frac{1}{2}\mathcal{N}(a, v^2) + \frac{1}{2}\mathcal{N}(-a, v^2)$.

We will now check that

$$E\left\{\log\cosh\left(\frac{Za}{v^2}\right)\right\} = \xi\left(\frac{a}{v}\right). \tag{86}$$

Note that

$$f_Z(x) = \frac{1}{\sqrt{2\pi}v} e^{-(x^2/2v^2)} e^{-(a^2/2v^2)} \cosh\left(\frac{xa}{v^2}\right). \tag{87}$$

Changing the variable of integration $u = xa/v^2$ and letting $\eta = a/v$, we get

$$\begin{aligned} E&\left\{\log\cosh\left(\frac{Za}{v^2}\right)\right\} \\ &= \int_{-\infty}^{\infty} \log\cosh(u) f_Z\left(\frac{uv^2}{a}\right)\frac{v}{\eta}\,du \\ &= \frac{1}{\sqrt{2\pi}} e^{-\eta^2/2} \int_{-\infty}^{\infty} \frac{1}{\eta} e^{-u^2/2\eta^2} \cosh(u)\log\cosh(u)\,du \\ &= \xi(\eta^2). \end{aligned} \tag{88}$$

This completes the proof of Fact 2. □

The function ζ is plotted in Fig. 2 in nats.

We will now find a and v in the case of interest; i.e., for the MMSE detector. From the previous analysis we have

$$a = B_2 \tag{89}$$

and

$$a^2 + v^2 = \sigma_1^2. \tag{90}$$

In the $K = 2$ case these quantities can be computed as follows:

$$\begin{aligned} \boldsymbol{M} &= \begin{bmatrix} 1 + \frac{\sigma^2}{A_1^2} & \rho \\ \rho & 1 + \frac{\sigma^2}{A_2^2} \end{bmatrix}^{-1} \\ &= \left[\left(1 + \frac{\sigma^2}{A_1^2}\right)\left(1 + \frac{\sigma^2}{A_2^2}\right) - \rho^2\right]^{-1} \\ &\quad \cdot \begin{bmatrix} \left(1 + \frac{\sigma^2}{A_2^2}\right) & -\rho \\ -\rho & \left(1 + \frac{\sigma^2}{A_1^2}\right) \end{bmatrix}. \end{aligned} \tag{91}$$

Thus

$$B_2 = A_2 \frac{\sigma^2}{A_2^2} \frac{\rho}{\left(1 + \frac{\sigma^2}{A_1^2}\right)\left(1 + \frac{\sigma^2}{A_2^2}\right) - \rho^2} \tag{92}$$

and

$$\begin{aligned} \sigma_1^2 &= a^2 + v^2 = \sigma^2 M_{11}\left(1 - \frac{\sigma^2}{A_1^2} M_{11}\right) \\ &= \sigma^2 \frac{1 + \frac{\sigma^2}{A_2^2}}{\left(1 + \frac{\sigma^2}{A_2^2}\right)\left(1 + \frac{\sigma^2}{A_1^2}\right) - \rho^2} \\ &\quad \cdot \left[1 - \frac{\sigma^2}{A_1^2} \frac{1 + \frac{\sigma^2}{A_2^2}}{\left(1 + \frac{\sigma^2}{A_2^2}\right)\left(1 + \frac{\sigma^2}{A_1^2}\right) - \rho^2}\right]. \end{aligned} \tag{93}$$

From (89), (90), (92), and (93), it is straightforward to obtain

$$\frac{a^2}{v^2} = \frac{\rho^2 r_2}{1 + (1 - \rho^2)(2r_2 + r_2^2)}. \tag{94}$$

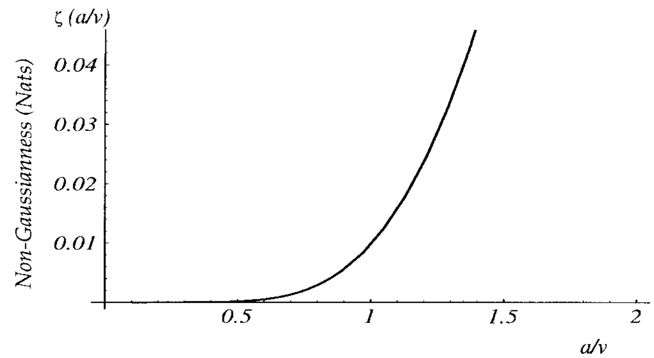

Fig. 2. Non-Gaussianness (nats) of the mixture of two normal distributions with standard deviation v centered at $\pm a$.

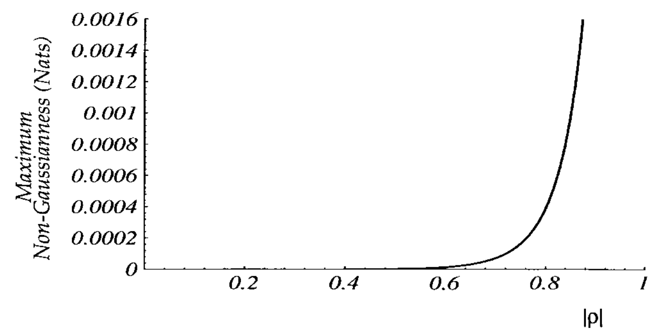

Fig. 3. Maximum non-Gaussianness (nats) of the MMSE decision statistic for worst case received amplitudes. (Two-user channel with normalized crosscorrelation ρ.)

Since ζ is a monotone increasing function, we need to find r_2 that maximizes (94). A quick calculation shows that the maximum is achieved for

$$r_2^2 = \frac{1}{1 - \rho^2} \tag{95}$$

and (79) follows by substitution.

This completes the proof of the proposition. □

Fig. 3 shows the maximum non-Gaussianness in the $K = 2$ case as a function of $|\rho|$. Note its extremely small value for all values of $|\rho|$ except those close to 1. Therefore, the Gaussian approximation to the error probability of the MMSE detector in all but those pathological cases where $|\rho| \to 1$ should be excellent. These results on maximum non-Gaussianness can be used in conjuction with the worst case results of [8] to give an upper bound on bit-error rate.

The reader can easily check that the non-Gaussianness of the MAI at the output of the conventional single-user detector is equal to

$$\zeta(\rho^2 r_2). \tag{96}$$

Naturally, the non-Gaussianness of the conventional detector grows without bound as the signal-to-noise ratio of the interferer goes to infinity. As a point of comparison, if $\rho = 0.8$ and $r_2 = 7$ dB, the non-Gaussianness of the conventional receiver output is 0.12 nats, whereas the non-Gaussianness of the MMSE receiver output is 1.2×10^{-4} nats.

B. MMSE versus Decorrelator Bit-Error Rate

Based on the analytical and numerical evidence available to date, we put forward the following conjecture.

Conjecture: For any choice of $K, \boldsymbol{R}, \sigma^2$ and $A_1, A_2, \cdots, A_K$ the bit-error rate of the MMSE detector is smaller than the bit-error rate of the decorrelating detector.

The following proposition proves the conjecture in the special case of two users whose signature waveforms are not almost identical.

Proposition 5.2: Suppose $K = 2$ and

$$|\langle s_1, s_2\rangle| \le \tfrac{1}{2}\sqrt{2+\sqrt{3}}\ (\cong 0.9659).$$

Then for all A_1, A_2, and σ the error probability of the MMSE detector is less than or equal to the error probability of the decorrelator.

Proof: The probability of error for the MMSE detector in the $K=2$ case is given by (without loss of generality, we consider user 1)

$$P_1 = \frac{1}{2}Q\left(\frac{x+y}{z}\right) + \frac{1}{2}Q\left(\frac{x-y}{z}\right) \tag{97}$$

where

$$x = \frac{1-\rho^2+\phi_2}{\sqrt{\phi_1}} \tag{98}$$

$$y = \rho\sqrt{\phi_2} \tag{99}$$

$$z = \sqrt{(1+2\phi_2)(1-\rho^2)+\phi_2^2} \tag{100}$$

$$\phi_1 = \sigma^2/A_1^2 \tag{101}$$

$$\phi_2 = \sigma^2/A_2^2. \tag{102}$$

Note that the probability of error of the decorrelator P_1^d satisfies

$$P_1^d = P_1|_{\phi_2=0} = Q\left(\frac{A_1}{\sigma}\sqrt{1-\rho^2}\right). \tag{103}$$

We will show that for the given crosscorrelation range

$$\frac{\partial}{\partial\phi_2}P_1 \le 0, \quad \forall \phi_2 \ge 0. \tag{104}$$

The proposition will then follow since P_1^d is independent of ϕ_2. We have

$$\begin{aligned}\frac{\partial}{\partial\phi_2}P_1 &= \frac{1}{2}Q'\left(\frac{x+y}{z}\right)\left[\frac{1/\sqrt{\phi_1}+\rho/(2\sqrt{\phi_2})}{z} - \frac{(x+y)}{z^3}(1-\rho^2+\phi_2)\right]\\ &\quad + \frac{1}{2}Q'\left(\frac{x-y}{z}\right)\left[\frac{1/\sqrt{\phi_1}-\rho/(2\sqrt{\phi_2})}{z} - \frac{(x-y)}{z^3}(1-\rho^2+\phi_2)\right]\\ &= \frac{1}{2}\left[Q'\left(\frac{x+y}{z}\right)+Q'\left(\frac{x-y}{z}\right)\right]\\ &\quad \cdot\left[\frac{1}{z^3\sqrt{\phi_1}}(z^2 - x\sqrt{\phi_1}(1-\rho^2+\phi_2))\right]\\ &\quad + \frac{1}{2}\left[Q'\left(\frac{x+y}{z}\right)-Q'\left(\frac{x-y}{z}\right)\right]\\ &\quad \cdot\left[\frac{1}{2z^3\sqrt{\phi_2}}(\rho z^2 - 2y\sqrt{\phi_2}(1-\rho^2+\phi_2))\right].\end{aligned}$$

We now make use of the following:

$$\frac{1}{2}\left[Q'\left(\frac{x+y}{z}\right)+Q'\left(\frac{x-y}{z}\right)\right] = -\frac{1}{\sqrt{2\pi}}e^{-(x^2+y^2/2z^2)}\cosh\left(\frac{xy}{z^2}\right) \tag{105}$$

$$\frac{1}{2}\left[Q'\left(\frac{x+y}{z}\right)-Q'\left(\frac{x-y}{z}\right)\right] = +\frac{1}{\sqrt{2\pi}}e^{-(x^2+y^2/2z^2)}\sinh\left(\frac{xy}{z^2}\right) \tag{106}$$

$$z^2 - x\sqrt{\phi_1}(1-\rho^2+\phi_2) = \rho^2(1-\rho^2) \tag{107}$$

and

$$\rho z^2 - 2y\sqrt{\phi_2}(1-\rho^2+\phi_2) = \rho(1-\rho^2-\phi_2^2) \tag{108}$$

from which we have

$$\frac{\partial}{\partial\phi_2}P_1 = -\frac{|\rho|(1-\rho^2)}{2z^3\sqrt{2\pi\phi_2}}e^{-(x^2+y^2/2z^2)}\cosh\left(\frac{xy}{z^2}\right)[2\xi - f(\phi_2)] \tag{109}$$

with $\xi = |\rho|\sqrt{\phi_2/\phi_1}$, and

$$f(\phi_2) = \left(1-\frac{\phi_2^2}{1-\rho^2}\right)\tanh\left(\xi\frac{1-\rho^2+\phi_2}{z^2}\right). \tag{110}$$

Now, note that $\frac{\partial}{\partial\phi_2}P_1 \le 0$ if and only if the term $2\xi - f(\phi_2)$ in square brackets in (109) is nonnegative. This is clearly true if $\phi_2 \ge \sqrt{1-\rho^2}$ or if $\xi = 0$. Let us consider the case $0 \le \phi_2 \le \sqrt{1-\rho^2}$, and examine the maximum value of the second term of this expression with $\rho \ne 0$ and $\xi > 0$ fixed

$$\max_{0\le\phi_2\le\sqrt{1-\rho^2}} f(\phi_2). \tag{111}$$

Consider the term

$$g(\phi_2) \triangleq \frac{1-\rho^2+\phi_2}{z^2} = \frac{1-\rho^2+\phi_2}{(1-\rho^2+\phi_2)^2+\rho^2(1-\rho^2)} \tag{112}$$

appearing in the argument of $\tanh$. Analysis of this term shows that

$$\max_{0\le\phi_2\le\sqrt{1-\rho^2}} g(\phi_2) = \begin{cases} g(0) \equiv 1, & \text{if } |\rho| \le \frac{1}{\sqrt{2}}\\ g(\sqrt{\rho^2-\rho^4}-(1-\rho^2)) \equiv \frac{1}{2\sqrt{\rho^2-\rho^4}}, & \text{if } \frac{1}{\sqrt{2}} < |\rho| \le 1.\end{cases} \tag{113}$$

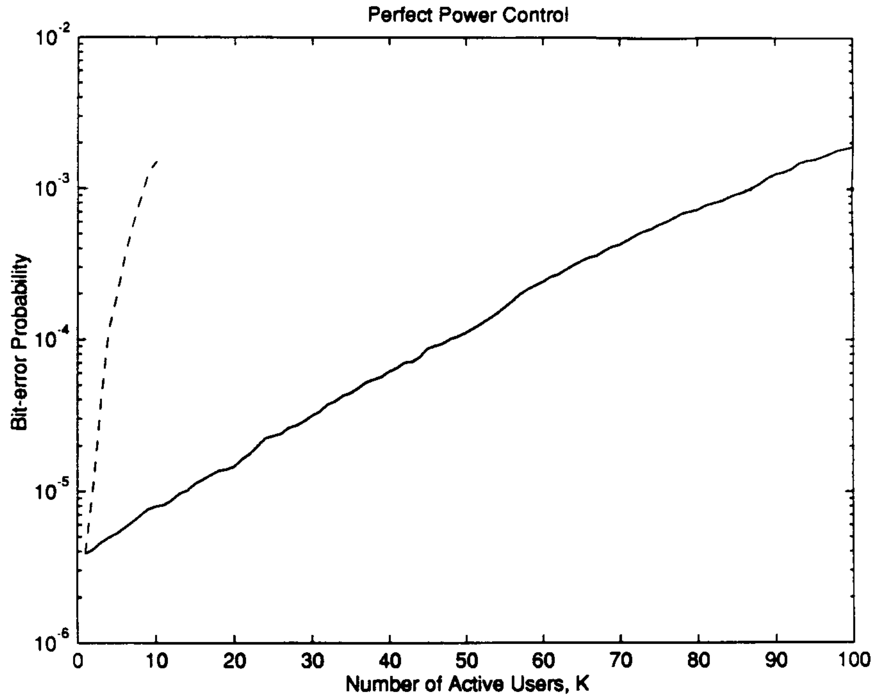

Fig. 4. Bit-error probabilities of the MMSE detector (solid line) and the conventional detector (dashed line). (Perfect power control; SNR $= 10$ dB; 127-length signature sequences.)

It is easily shown that

$$\frac{1}{2\sqrt{\rho^2-\rho^4}} \le 2 \quad \text{when} \quad \frac{1}{\sqrt{2}} < |\rho| \le \frac{1}{\sqrt{2}}\sqrt{\left(1+\frac{\sqrt{3}}{2}\right)} \tag{114}$$

with equality at the right endpoint; and thus from (113) we have

$$\max_{0\le\phi_2\le\sqrt{1-\rho^2}} g(\phi_2) = 2. \tag{115}$$

We conclude that, for $|\rho| \le (1/\sqrt{2})\sqrt{(1+\sqrt{3}/2)}$, the maximum in (111) is bounded by

$$\max_{0\le\phi_2\le\sqrt{1-\rho^2}} f(\phi_2) \le \tanh(2\xi) \tag{116}$$

and the term in square brackets in (109) is thus bounded below by

$$2\xi - \tanh(2\xi) \ge 0. \tag{117}$$

This implies that $(\partial/\partial\phi_2)P_1 \le 0$, for all $\phi_2 \ge 0$, and the proposition follows. □

Remark: It is not generally true that $(\partial/\partial\phi_2)P_1 \le 0$ for ρ outside the range given in the proposition. For example, with $\rho = 0.99$ and $A_2 = 99A_1$ computations show that $(\partial/\partial\phi_2)P_1 > 0$ with ϕ_2 in the approximate range 0.02 to 0.1. However, even in this case it is true that $P_1 \le P_1^d$, but the case $|\rho| > (1/\sqrt{2})\sqrt{(1+\sqrt{3}/2)}$ requires another method of proof. However, the range of ρ in Proposition 5.1 certainly covers any case of interest in practice.

Further evidence for our conjecture is that whereas the limit as $|\rho| \to 1$ of

$$P_1^d = Q\left(\frac{A_1\sqrt{1-\rho^2}}{\sigma}\right) \tag{118}$$

is equal to $\frac{1}{2}$, the limit as $|\rho| \to 1$ of P_1 is equal to

$$P_1 = \frac{1}{2}Q\left(\frac{A_1+A_2}{\sigma}\right) + \frac{1}{2}Q\left(\frac{A_1-A_2}{\sigma}\right) \tag{119}$$

which is less than $\frac{1}{2}$.

VI. Conclusion

In this paper, we have considered the behavior of the output of the linear correlator used in the MMSE linear multiuser detector. We have seen that, under many conditions, the error (MAI-plus-noise) in this output is approximately Gaussian. This property is particularly useful in the performance analysis of multiuser networks involving the MMSE detector, since (unlike the other detectors discussed in Section II) no approximation had been previously investigated analytically for the bit-error rate of the MMSE detector.

For example, Figs. 4 and 5 show a comparison of the performance of the MMSE detector and the conventional detector for a coherent direct-sequence spread-spectrum network employing length-127 signature sequences with various numbers of active users. Two cases are shown: perfect power control, in which all users have the same amplitude; and no

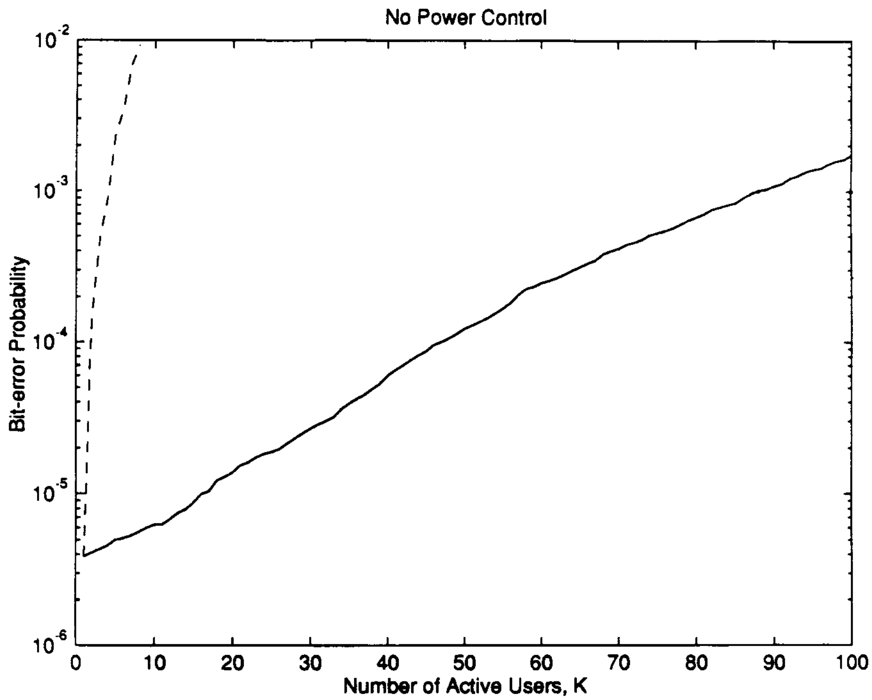

Fig. 5. Bit-error probabilities of the MMSE detector (solid line) and the conventional detector (dashed line). (No power control; SNR $=$ 10 dB; 127-length signature sequences.)

power control, in which the interfering users' amplitudes vary according to a lognormal distribution with a standard deviation of 6 dB. The results shown are averaged over 100 random signature sequences and, in the case of no power control, 100 random choices for the amplitudes. The results for the MMSE detector were obtained with exact calculations for smaller network sizes, and using the Gaussian approximation for larger network sizes $(K > 10)$. Even for $K \leq 10$, the Gaussian approximation and the exact error probability were quite close and would be indistinguishable in the scale shown in these figures. Note that the MMSE detector appears from these calculations to be exceptionally robust, in contrast to the conventional detector, which can reliably support only a few users. Although these curves are far from exhaustive, they would be impossible to generate without use of the Gaussian approximation.

This work suggests a number of interesting avenues for further research, including the extension of the results of Section IV to more general crosscorrelation patterns, and the generalization of the results of Section V to $K > 2$.

Appendix
A Proof of Proposition 4.1

Note that, when K varies, the leakage coefficients also vary. In particular, recall that β_k is given from (42) as

$$\beta_k = \frac{\sigma}{A_1}\frac{\sigma}{A_k}\frac{(\rho_1^T \boldsymbol{H}^{-1})_k}{1 - \rho_1^T \boldsymbol{H}^{-1}\rho_1} \tag{120}$$

where ρ_1 and $\boldsymbol{H}$ are given in (38) and (39), respectively. Note that both of these latter quantities will depend on K. To emphasize this variation, we will denote the leakage coefficients by $\beta_2^{(K)}, \beta_3^{(K)}, \cdots, \beta_K^{(K)}$, so that

$$m_K = \beta_2^{(K)} b_2 + \beta_3^{(K)} b_3 + \cdots + \beta_K^{(K)} b_K. \tag{121}$$

We first consider mean-square convergence of m_K to 0. Since $E\{m_K\} = 0$, we need only examine the variance of m_K. From (42), we can write

$$\operatorname{var}(m_K) = \sum_{k=2}^{K} [\beta_k^{(K)}]^2 \leq \bar{\xi}^4 \frac{\rho_1^T \boldsymbol{H}^{-2}\rho_1}{[1 - \rho_1^T \boldsymbol{H}^{-1}\rho_1]^2} \tag{122}$$

where

$$\bar{\xi} = \frac{\sigma}{\inf\limits_{k=1,2,\cdots} |A_k|}. \tag{123}$$

Note that $\boldsymbol{H}$ can be written as

$$\boldsymbol{H} = \boldsymbol{\Delta} + \rho_K \underline{1}\underline{1}^T \tag{124}$$

with

$$\boldsymbol{\Delta} = \operatorname{diag}\left\{1 + \frac{\sigma^2}{A_2^2} - \rho_K, 1 + \frac{\sigma^2}{A_3^2} - \rho_K, \cdots, 1 + \frac{\sigma^2}{A_K^2} - \rho_K\right\}. \tag{125}$$

From (124) we see that $\boldsymbol{H}$ is a rank-1 modification of the positive-definite diagonal matrix $\boldsymbol{\Delta}$. Because of this, we can write $\boldsymbol{H}^{-1}$ in closed form (e.g., [4])

$$\boldsymbol{H}^{-1} = \boldsymbol{\Delta}^{-1} - \rho_K \frac{\boldsymbol{\Delta}^{-1}\underline{1}\underline{1}^T\boldsymbol{\Delta}^{-1}}{1 + \rho_K \underline{1}^T \boldsymbol{\Delta}^{-1}\underline{1}}. \tag{126}$$

Equation (126) yields

$$\rho_1^T \boldsymbol{H}^{-1} = \frac{\rho_K}{1+\rho_K \underline{1}^T \boldsymbol{\Delta}^{-1}\underline{1}} \underline{1}^T \boldsymbol{\Delta}^{-1} \tag{127}$$

from which

$$\rho_1^T \boldsymbol{H}^{-1}\rho_1 = \frac{(\rho_K)^2 \underline{1}^T \boldsymbol{\Delta}^{-1}\underline{1}}{1+\rho_K \underline{1}^T \boldsymbol{\Delta}^{-1}\underline{1}}$$

and

$$\rho_1^T \boldsymbol{H}^{-2}\rho_1 = \frac{(\rho_K)^2 \underline{1}^T \boldsymbol{\Delta}^{-2}\underline{1}}{(1+\rho_K \underline{1}^T \boldsymbol{\Delta}^{-1}\underline{1})^2}. \tag{128}$$

Using (128) in (122), we have

$$\begin{aligned}\mathrm{var}\,(m_K) &\le \overline{\xi}^4 \frac{(\rho_K)^2 \underline{1}^T \boldsymbol{\Delta}^{-2}\underline{1}}{[1+\rho_K(1-\rho_K)\underline{1}^T \boldsymbol{\Delta}^{-1}\underline{1}]^2}\\ &= \overline{\xi}^4 \frac{\underline{1}^T \boldsymbol{\Delta}^{-2}\underline{1}}{(K-1)^2}\left[\frac{x_K}{1+\gamma_K x_K}\right]^2\end{aligned} \tag{129}$$

where $x_K \triangleq (K-1)\rho_K$ and where

$$\gamma_K = (1-\rho_K)\frac{\underline{1}^T \boldsymbol{\Delta}^{-1}\underline{1}}{K-1}. \tag{130}$$

We now note that

$$\begin{aligned}0 < \frac{K-1}{1+\overline{\xi}^2-\rho_K} &\le \underline{1}^T \boldsymbol{\Delta}^{-1}\underline{1}\\ &= \sum_{k=2}^{K} \frac{1}{1+\dfrac{\sigma^2}{A_k^2}-\rho_K} \le \frac{K-1}{1+\underline{\xi}^2-\rho_K}\end{aligned} \tag{131}$$

and

$$\underline{1}^T \boldsymbol{\Delta}^{-2}\underline{1} = \sum_{k=2}^{K} \frac{1}{\left(1+\dfrac{\sigma^2}{A_k^2}-\rho_K\right)^2} \le \frac{K-1}{\left(1+\underline{\xi}^2-\rho_K\right)^2} \tag{132}$$

where

$$\underline{\xi} = \frac{\sigma}{\sup\limits_{k=1,2,\cdots} |A_k|}. \tag{133}$$

Using (131) we have

$$0 < \frac{1-\rho_K}{1+\overline{\xi}^2-\rho_K} \le \gamma_K \le \frac{1-\rho_K}{1+\underline{\xi}^2-\rho_K} < \frac{2}{2+\underline{\xi}^2} < 1 \tag{134}$$

and (since $0 < x_K < (K-1)$)

$$1+\gamma_K x_K > \frac{\underline{\xi}^2}{2+\underline{\xi}^2} > 0. \tag{135}$$

Applying (132), (134), and the fact that the function f defined by

$$f(x) = \frac{x^2}{(1+\alpha x)^2} \tag{136}$$

is strictly increasing on $1+\alpha x > 0$, (129) becomes

$$\begin{aligned}\mathrm{var}\,(m_K) &\le \frac{\overline{\xi}^4}{(K-1)(1+\underline{\xi}^2-\rho_K)^2}\left[\frac{x_K}{1+\gamma_K x_K}\right]^2\\ &\le \frac{\overline{\xi}^4}{(K-1)(1+\underline{\xi}^2-\rho_K)^2}\left[\frac{(K-1)}{1+\gamma_K (K-1)}\right]^2\\ &< \frac{\overline{\xi}^4}{(1+\underline{\xi}^2-\rho_K)^2}\frac{1}{(K-1)\gamma_K} < \frac{\overline{\xi}^4}{\underline{\xi}^4}\frac{1}{(K-1)\gamma_K}.\end{aligned} \tag{137}$$

From (134), we have

$$\begin{aligned}\liminf_{K\to\infty}\gamma_K &\ge \liminf_{K\to\infty}\frac{1-\rho_K}{1+\overline{\xi}^2-\rho_K}\\ &\ge \frac{1-\limsup\limits_{K\to\infty}\rho_K}{1+\overline{\xi}^2-\limsup\limits_{K\to\infty}\rho_K} > 0.\end{aligned} \tag{138}$$

Thus

$$\frac{1}{(K-1)\gamma_K} \sim \mathcal{O}\left(\frac{1}{K}\right) \tag{139}$$

and (137) yields

$$m_K \overset{\mathrm{m.s.}}{\to} 0.$$

To prove almost-sure convergence of m_K to zero, we note that

$$\begin{aligned}|m_K| &\le \sum_{k=2}^{K} |\beta_k^K| \le \sqrt{(K-1)\sum_{k=2}^{K}[\beta_k^K]^2}\\ &= \sqrt{(K-1)\mathrm{var}\,(m_K)}.\end{aligned} \tag{140}$$

Equation (137) then implies

$$|m_K| \le \frac{\overline{\xi}^2}{\underline{\xi}^2\sqrt{\gamma_K}} \tag{141}$$

which, when combined with (139), implies that, for sufficiently large κ, $\{|m_K|\}_{K=\kappa}^{\infty}$ is bounded by a constant. Thus the mean-square convergence of m_K to zero implies the almost-sure convergence of m_K to zero via the Bounded Converence Theorem. □

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