\(E_\gamma\)-Resolvability

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Abstract—The conventional channel resolvability refers to the minimum rate needed for an input process to approximate the channel output distribution in total variation distance. In this paper, we study \(E_\gamma\)-resolvability, in which total variation is replaced by the more general \(E_\gamma\) distance. A general one-shot achievability bound for the precision of such an approximation is developed. Let \(Q_{X,U}\) be a random transformation, \(n\) be an integer, and \(E \in (0, +\infty)\). We show that in the asymptotic setting where \(n = \exp(nE)\), a (nonnegative) randomness rate above \(\inf_{Q_{UX}} D_{Q_{UX}}(Q_{UX} \| P_X) + I(Q_U, Q_{X,U} - E)\) is sufficient to approximate the output distribution \(P_X^n\) using the channel \(Q_{X,U}^n\), where \(Q_{UX} \rightarrow Q_X\). The achievable rate is then used to derive: 1) a one-shot Rényi divergences for a fixed-input randomness rate.

Index Terms—Resolvability, source coding, broadcast channel, mutual covering lemma, wiretap channel.

I. INTRODUCTION

Channel resolvability, introduced by Han and Verdú [1], is defined as the minimum randomness rate required to synthesize an input so that its corresponding output distribution approximates a target output distribution. While the resolvability problem itself differs from classical topics in information theory such as data compression and transmission, [1] unveils its potential utility in operational problems through the solution of the strong converse problem of identification coding [2]. Other applications of distribution approximation in information theory include common randomness of two random variables [3], strong converse in identification through channels [1], random process simulation [4], secrecy [5–8], channel synthesis [9, 10], lossless and lossy source coding [1], [4], [11], and the empirical distribution of a capacity-achieving code [12, 13]. The achievability part of resolvability (also known as the soft-covering lemma in [10]) is particularly useful, and coding theorems via resolvability have certain advantages over what is obtained from traditional typicality-based approaches (see e.g. [7]).

If the channel is stationary memoryless and the target output distribution is induced by a stationary memoryless input, then the resolvability is the minimum mutual information over all input distributions inducing the (per-letter) target output distribution, regardless of whether the approximation error is measured in total variation distance [1], [14, Th. 1], normalized relative entropy [1], [3, Th. 6.3], or unnormalized relative entropy [8]. In contrast, relatively few measures for the quality of the approximation of output statistics have been proposed for which the resolvability can be strictly smaller than mutual information. As shown by Steinberg and Verdú [4], one exception is the Wasserstein distance measure, in which case the finite precision resolvability for the identity channel can be related to the rate-distortion function of the source where the distortion is the metric in the definition of the Wasserstein distance [4].

In this paper we generalize the theory of resolvability by considering a distance measure, \(E_\gamma\), defined in Section II, of which the total variation distance is a special case where \(\gamma = 1\). The \(E_\gamma\) metric\(^1\) is more forgiving than total variation distance when \(\gamma > 1\), and the larger \(\gamma\) is, the less randomness is needed at the input for approximation in \(E_\gamma\). Various achievability and converse bounds for resolvability in the \(E_\gamma\) metric are derived in Section III-IV.

If we fix an input randomness rate and consider the minimum exponential growth rate of \(\gamma\) such that the approximation error measured in \(E_\gamma\) is small, we are effectively dealing with resolvability in a large deviations regime. Using general bounds on \(E_\gamma\) and related metrics (not specific to the resolvability problem) developed in Section II-C, we conclude that, in fact, the growth rate of the exponent of \(\gamma\) is the same as (see Section V):

\(^1\)“Metric” or “distance” are used informally since, other than nonnegativity, \(E_\gamma\), in general, does not satisfy any of the other three requirements for a metric.
tribution and a target distribution is close to 1. (That is, the excess relative information defined in Section II is small.)

2) The linear growth rate of the minimum relative entropy between the true output distribution and the target distribution.

3) The linear growth rate of the minimum smooth Rényi $\alpha$-divergence (of any order except for $\alpha = 1$) between the true output distribution and the target distribution.

In the case of a discrete memoryless channel with a given stationary memoryless target output, we provide a single-letter characterization of the minimum exponential growth rate of $\gamma$ to achieve approximation in $E_{\gamma}$ (Theorem 33). The corresponding problem for the worst case target distribution (which is generally not stationary memoryless even for stationary memoryless channels) has a different flavor; a converse bound (Theorem 47) can be derived by drawing connections to the identification coding problem (as in [1]), but which generally does not match the achievability bound unless $\gamma = 1$.

In addition to achievability results in terms of the expectation of the approximation error over a random codebook, we also prove achievability under the high probability criteria\(^2\) for a random codebook (Section IV-B). The implications of the latter problem in secrecy have been noted by several authors [15]–[17]. Here we adopt a simple non-asymptotic approach based on concentration inequalities, dispensing with the finiteness and stationarity assumptions required by the previous proof method [15] based on Chernoff bounds.

The $E_{\gamma}$ metric provides a very convenient tool for change-of-measure purposes: if $E_{\gamma}(P || Q)$ is small, and the probability of some event is large under $P$, then the probability of this event under $Q$ is essentially lower-bounded by $\frac{1}{\gamma}$ (see (18)). In the special case of $\gamma = 1$ (total variation distance), this change-of-measure trick has been widely used, see [10], [11], but the general $\gamma \geq 1$ case is more interesting, since $P$ and $Q$ need not be essentially the same, thereby opening up novel applications of the resolvability theorem (e.g. the one-shot mutual covering lemma in Section VII). In this paper we present three information theoretic applications of $E_{\gamma}$-resolvability (Sections VI-VIII):

- One-shot lower bound on the probability that the distortion lies below a certain threshold in lossy compression (successful decompression). Compared with the proof based on the soft-covering lemma ( achievability part of resolvability in total variation distance) [11], the new bound is capable of recovering the exact success exponent, previously obtained using the method of types (see [18]) for discrete memoryless settings. In contrast, our derivation applies to general sources and dispenses with memoryless and finite alphabet assumptions.

- A one-shot generalization of the mutual covering lemma, with a proof significantly different from the original one based on second moments [19], [20]. In [21] we applied the one-shot mutual covering lemma to derive a one-shot version of Marton’s inner bound for the broadcast channel with a common message, without using time-sharing/common randomness.

- One-shot achievability for wiretap channels, where a novel secrecy measure in terms of the eavesdropper ability to perform list decoding and detect the absence of message is proposed. The previous proofs for wiretap channels using the conventional resolvability (soft-covering lemma) [6], [7] are only suitable when the rate is low enough to achieve perfect secrecy. In contrast, $E_{\gamma}$-resolvability yields lower bounds on the minimum size of the eavesdropper list for an arbitrary rate of communication. This interpretation of security in terms of list size is reminiscent of equivocation [22], and indeed we recover the same formula in the asymptotic setting, even though there is no direct correspondence between both criteria. Moreover, we also consider a more general case where the eavesdropper wishes to reliably detect whether a message is sent, while being able to produce a list including the actual message if it decides it is present. This is a practical setup because “no message” may be valuable information which the eavesdropper wants to ascertain reliably. The idea is reminiscent of the stealth communication problem (see [23], [24] and the references therein) which also involves a hypothesis test on whether a message is sent. However, the setup and the analysis (including the covering lemma) are quite different from [23] and [24]. In comparison, our results are more suitable for the regime with higher communication rates and lower secrecy demands. In the discrete memoryless case, we obtain single-letter expressions of the tradeoff between the transmission rate, eavesdropper list, and the exponent of the false-alarm probability for the eavesdropper (i.e. declaring the presence of a message when there is none).

II. PRELIMINARIES

In this paper, several distance measures between two probability distributions play an important role. In this section, we introduce these distance measures and discuss some of their relations. At the end of this section, the channel resolvability problem is formulated.

A. Excess Relative Information Metric

Given two nonnegative $\sigma$-finite measures $\nu \ll \mu$ on $X$, define the relative information (for each $x \in X$) as the logarithm of the Radon-Nikodym derivative:

$$I_{\nu||\mu}(x) := \log \frac{d\nu}{d\mu}(x).$$

(1)

For $\gamma > 0$ and a probability measure $P$, we define the excess relative information metric with threshold $\gamma$,

$$\tilde{F}_{\gamma}(P||\mu) := \mathbb{P}[I_{P||\mu}(X) > \log \gamma]$$

(2)

where $X \sim P$. As such, it can be expressed in terms of the relative information spectrum (see [25]) as

$$\tilde{F}_{\gamma}(P||\mu) = 1 - F_{P||\mu}(\log \gamma).$$

(3)
Note that (2) is nonnegative and vanishes when \( P = \mu \) provided that \( \gamma > 1 \), and can therefore be considered as a measure of the discrepancy between \( P \) and \( \mu \). In addition to playing an important role in one-shot analysis (see [26]), (2) provides richer information than the relative entropy measure since
\[
D(P \parallel \mu) := \mathbb{E}[\tau_{P \parallel \mu}(X)] = \int_{(0, +\infty)} \tau \mathbb{P}[\tau_{P \parallel \mu}(X) > \tau] d\tau
\]
(4)
Moreover, the excess relative information is also related to total variation distance since for probability measures \( P \) and \( Q \),
\[
\frac{1}{2} |P - Q| = \mathbb{P}[\tau_{P \parallel Q}(X) > 0] = \mathbb{P}[\tau_{P \parallel Q}(Y) > 0],
\]
(6)
where \( X \sim P \) and \( Y \sim Q \) [27]. However, perhaps surprisingly, the excess relative information metric does not satisfy the data processing inequality, in contrast to the relative entropy and total variation distance:

\textbf{Proposition 1:} Suppose \( P_X \rightarrow P_Y | X \rightarrow P_Y, Q_X \rightarrow P_Y | X \rightarrow Q_Y \) and \( P_X \ll Q_X \) and \( Q_X \ll P_X \). Then there exists \( \gamma > 0 \) such that
\[
\bar{F}_\gamma(P_X \parallel Q_X) < \bar{F}_\gamma(P_Y | Q_Y)
\]
(7)
where \( (X, Y) \sim P_{XY} \), unless \( P_{X|Y} = Q_{X|Y} \) almost surely.

Note that in the absence of the condition \( P_X \ll Q_X \) it is indeed possible to find examples where the cdf of the relative information at the input is dominated by that at the output.

\textbf{Proof:} Suppose, for the sake of contradiction,
\[
\mathbb{P}\left[\frac{dQ_X}{dP_X}(X) \geq \lambda \right] \leq \mathbb{P}\left[\frac{dQ_Y}{dP_Y}(Y) \geq \lambda \right]
\]
(8)
for all \( \lambda > 0 \), where \( \frac{dQ_X}{dP_X} \) is well defined because \( P_Y \gg Q_Y \) follows from \( P_X \gg Q_X \). However, since
\[
1 = \mathbb{E}\left[\frac{dQ_X}{dP_X}(X)\right] = \int_0^\infty \mathbb{P}\left[\frac{dQ_X}{dP_X}(X) \geq \lambda \right] d\lambda
\]
(9)
\[
= \mathbb{E}\left[\frac{dQ_Y}{dP_Y}(Y)\right] = \int_0^\infty \mathbb{P}\left[\frac{dQ_Y}{dP_Y}(Y) \geq \lambda \right] d\lambda,
\]
(10)
(11)
(12)
(8) must hold with equality all \( \lambda > 0 \) except for a set of measure zero. But both sides of (8) are decreasing, left continuous functions of \( \lambda \), so in fact equality holds in (8) for all \( \lambda > 0 \). This implies that \( D(P_X \parallel Q_X) = D(P_Y \parallel Q_Y) \), and hence \( P_{X|Y} = Q_{X|Y} \) almost surely. \( \square \)

The failure of the data processing inequality suggests that the excess relative information itself has a limited operational significance; rather, we shall mainly resort to the simplicity of its definition and its connection to other distance measures which have more significant operational meanings. In [27] several bounds on total variation distance using the excess relative information metric are derived. Next we refine those results by providing the tightest possible bounds (i.e. the locus of possible values of \( \bar{F}_\gamma(P \parallel Q) \) as a function of the total variation distance \( |P - Q| \); see Figure 1).

\textbf{Proposition 2:} If \( P \ll Q \) are distributions on \( X \) (not necessarily discrete), and \( \delta := \frac{1}{2}|P - Q| \), then
\[
\bar{F}_\lambda(P \parallel Q) \leq \begin{cases} \frac{\lambda}{1 - \epsilon} & \lambda \in [\frac{1}{1 - \epsilon}, \infty) \\ 1 & \lambda \in (0, \frac{1}{1 - \epsilon}) \end{cases}
\]
(13)
and
\[
\bar{F}_\lambda(P \parallel Q) \geq \begin{cases} 0 & \lambda \in [\frac{1}{1 - \epsilon}, \infty) \\ 1 - (1 - \delta)\lambda & \lambda \in (1, \frac{1}{1 - \epsilon}) \\ 1 - \delta & \lambda \in (0, 1 - \delta) \end{cases}
\]
(14)
where \( X \sim P \). Moreover, the bounds above are tight (that is, the values given on the right sides are the supremum/infimum over \( P \ll Q \) with a total variation distance \( \delta \)).

\textbf{Proof:} See Appendix A. \( \square \)

As shown in [1, Lemma 5], a useful and compact bound on \( \bar{F}_\lambda(P \parallel Q) \) is
\[
\frac{1}{2}|P - Q| \leq \bar{F}_\lambda(P \parallel Q).
\]
(15)
The bound in (15) is not tight, in contrast to (14). In particular, (15) does not show the fact the \( \bar{F}_\lambda(P \parallel Q) \) tends to 1 as \( \lambda \downarrow 0 \).

\textbf{B.} \( E_\gamma \) Metric

\textbf{Definition 3:} Given probability distributions \( P \ll Q \) on the same alphabet and \( \gamma \geq 1 \), define
\[
E_\gamma(P \parallel Q) := \mathbb{P}[\tau_{P \parallel Q}(X) > \log \gamma ] - \gamma \mathbb{P}[\tau_{P \parallel Q}(Y) > \log \gamma ]
\]
(16)
where \( X \sim P \) and \( Y \sim Q \).

We can see that \( E_\gamma \) is an \( f \)-divergence [28] with
\[
f(x) = (x - \gamma)^+.
\]
(17)
From the Neyman-Pearson lemma we have an alternative formula for $E_γ$:

$$E_γ(P\|Q) = \max_{A} \{ P(A) - γ Q(A) \}. \tag{18}$$

$E_γ$ is a basic quantity in binary hypothesis testing: in the Bayesian case, where $P$ and $Q$ have a priori probabilities $π_P$ and $π_Q$, respectively, the probability of making the correct decision is given by

$$π_P \mathbb{P} \left[ t_{P\|Q}(X) > \log \frac{π_Q}{π_P} \right] + π_Q \mathbb{P} \left[ t_{P\|Q}(Y) ≤ \log \frac{π_Q}{π_P} \right]$$

$$= π_Q + π_P E_{π_P}(P\|Q). \tag{19}$$

$E_γ$ has been considered in various fields under different names; for example, in cryptography (more specifically, differential privacy [29], [30]) a computation over a database is said to be $(ε, δ)$-differentially private if the $E_{\text{exp}(ε)}$ distance between the output distributions for any two databases which differ in at most one element [29] is upper-bounded by $δ$. The function (17) is called a hockey-stick function in financial engineering [31], and so $E_γ$ is sometimes called a hockey-stick divergence [32], [33].

It appears that, $E_γ$ was introduced to information theory by [34] to simplify the expression of the DT bound therein. Also, [35] derived a general channel coding converse using any $g$-divergence (a divergence satisfying the data processing inequality, including all $f$-divergences), which recovers the Wolfowitz converse when specialized to $E_γ$, and admits generalization to the quantum setting [32], [33]. The use of $E_γ$ for change-of-measure appeared in [21], [36], and [37].

Below, we prove some basic properties of $E_γ$ useful for later sections. Additional properties of $E_γ$ can be found in [38] and [39, Th. 21].

**Proposition 4:** Assume that $P ≪ S ≪ Q$ are probability distributions on the same alphabet, and $γ, γ_1, γ_2 ≥ 1$.

1) $[1, \infty) → [0, \infty): γ \mapsto E_γ(P\|Q)$ is convex, non-increasing, and continuous.

2) For any event $A$,

$$Q(A) ≥ \frac{1}{γ} (P(A) - E_γ(P\|Q)). \tag{20}$$

3) Triangle inequalities:

$$E_{γ_1 γ_2}(P\|Q) ≤ E_{γ_1}(P\|S) + γ_1 E_{γ_2}(S\|Q), \tag{21}$$

$$E_γ(P\|Q) + E_γ(P\|S) ≥ \frac{γ}{2} |S - Q| + 1 - γ. \tag{22}$$

4) Monotonicity: if $P_{XY} = P_{X} P_{Y|X}$ and $Q_{XY} = Q_{X} Q_{Y|X}$ are joint distributions on $X \times Y$, then

$$E_γ(P_{X}\|Q_{X}) ≤ E_γ(P_{X\|Q}_{X\|Y}) \tag{23}$$

where equality holds for all $ γ \geq 1$ if and only if $P_{Y|X} = Q_{Y|X}$.

5) Given $P_X, P_{Y|X}$ and $Q_{Y|X}$, define

$$E_γ(P_{Y|X}\|Q_{Y|X}|P_X) := \mathbb{E}[E_γ(P_{Y|X}(\cdot|X)\|Q_{Y|X}(\cdot|X))] \tag{24}$$

where the expectation is w.r.t. $X \sim P_X$. Then we have

$$E_γ(P_X P_{Y|X}\|P_{X} Q_{Y|X}|P_X) = E_γ(P_{Y|X}\|Q_{Y|X}|P_X). \tag{25}$$

6) $1 - γ \left( 1 - \frac{1}{2} |P - Q| \right) ≤ E_γ(P\|Q) ≤ \frac{1}{2} |P - Q|. \tag{26}$

**Proof:** The proofs of 1), 2), 4), 5) and the second inequality in (26) are omitted since they follow directly from (18) or are similar to the corresponding properties for total variation distance.

For 3), observe that

$$E_{γ_1 γ_2}(P\|Q) = \max_{A} (P(A) - γ_1 S(A) + γ_1 S(A) - γ_1 γ_2 Q(A)) \tag{27}$$

$$≤ \max_{A} (P(A) - γ_1 S(A)) + \max_{A} (γ_1 S(A) - γ_1 γ_2 Q(A)) \tag{28}$$

$$E_{γ_1}(P\|S) + γ_1 E_{γ_2}(S\|Q), \tag{29}$$

and that

$$E_γ(P\|Q) + E_γ(P\|S)$$

$$= \max_{A} (P(A) - γ Q(A)) + \max_{A} (1 - P(A) - γ + γ S(A)) \tag{30}$$

$$≥ \max_{A} (P(A) - γ Q(A)) + 1 - P(A) - γ + γ S(A)) \tag{31}$$

$$= γ \max_{A} (S(A) - Q(A)) + 1 - γ \tag{32}$$

$$= \frac{γ}{2} |S - Q| + 1 - γ. \tag{33}$$

As far as 6) note that the left inequality in (26) follows by setting $S = P$ in (22).

In view of the right inequality in (26), in the resolvability problem, when the rate of the codebook is not large enough to soft-cover the output distribution in total variation, it may be still possible to do so in the $E_γ(P\|Q)$ metric.
C. Smooth Rényi Divergence and Relationships Between the Distance Measures

To discuss the smooth Rényi divergence, it is convenient to generalize some of our definitions to allow nonnegative finite measures that are not necessarily probability measures:

**Definition 5:** For any \( \gamma \geq 1 \), nonnegative finite measures \( \mu \) and \( \nu \) on \( \mathcal{X} \), \( \mu \ll \nu \), define

\[
|\mu - \nu| := \int |d\mu - d\nu|,
\]
and\(^3\)

\[
E_\gamma (\mu \| \nu) := \sup_{A} (\mu (A) - \gamma \nu (A))
\]

\[
= (\mu - \gamma \nu)(I_{|\mu| > \log \gamma})
\]

\[
= \frac{1}{2}(\mu - \gamma \nu)(\mathcal{X}) - \frac{1}{2}(\mu - \gamma \nu)(I_{|\mu| \leq \log \gamma})
\]

\[
+ \frac{1}{2}(\mu - \gamma \nu)(I_{|\mu| > \log \gamma})
\]

\[
= \frac{1}{2}\mu (\mathcal{X}) - \frac{\gamma}{2} \nu (\mathcal{X}) + \frac{1}{2}(\mu - \gamma \nu).
\]

Note that \( E_1(\mu \| \mu) = \frac{1}{2}|\mu - \mu| \) when \( \mu \) is not a probability measure.

The following result is a generalization of the \( \gamma_1 = 1 \) case of (21) to unnormalized measures, and the proof is immediate from the definition of (35) and the subadditivity of the sup operator.

**Proposition 6:** Triangle inequality: if \( \mu, \nu \) and \( \theta \) are nonnegative finite measures on the same alphabet, \( \mu \ll \nu \ll \theta \ll \nu \), then

\[
E_\gamma (\mu \| \nu) \leq E_1 (\mu \| \theta) + E_\gamma (\theta \| \nu).
\]

The Rényi divergence, defined as follows, is not an \( f \)-divergence, but is a monotonic function of the Hellinger distance [35].

**Definition 7:** [Rényi \( \alpha \)-Divergence] Let \( \mu \) be a nonnegative finite measure and \( \nu \) a probability measure on \( \mathcal{X} \), \( \mu \ll \nu \), \( X \sim \nu \). For \( \alpha \in (0,1) \cup (1, +\infty) \),

\[
D_\alpha (\mu \| Q) := \frac{1}{\alpha - 1} \log \mathbb{E} \left[ \left( \frac{d\mu}{dQ} (X) \right)^\alpha \right],
\]

and

\[
D_0 (\mu \| Q) := \log \frac{1}{Q(I_{\mu \| Q > -\infty})},
\]

\[
D_{\infty} (\mu \| Q) := \mu \ll \nu \text{ ess sup}_{\|\|Q}
\]

which agree with the the limits as \( \alpha \downarrow 0 \) and \( \alpha \uparrow \infty \).

\( D_\alpha (P \| Q) \) is non-negative and monotonically increasing in \( \alpha \). More properties about the Rényi divergence can be found, e.g. in [25], [40], and [41].

**Definition 8:** [Smooth Rényi \( \alpha \)-Divergence] For \( \alpha \in (0,1) \), \( \epsilon \in (0,1) \),

\[
D_\alpha^{+\epsilon} (P \| Q) := \sup_{\mu \in B^\epsilon (P)} D_\alpha (\mu \| Q);
\]

\[
D_\alpha^{-\epsilon} (P \| Q) := \inf_{\mu \in B^\epsilon (P)} D_\alpha (\mu \| Q),
\]

\[
D_\alpha^{-\epsilon} (P \| Q) := \inf_{\mu \in B^\epsilon (P)} D_\alpha (\mu \| Q),
\]

where \( B^\epsilon (P) := \{ \mu \ll \nu : \nu (P) \leq \epsilon \} \) is the \( \epsilon \)-neighborhood of \( P \) in \( E_1 \).

**Remark 9:** Our smooth \( \infty \)-divergence agrees with the smooth max Rényi divergence in [42], although the definitions look different. However, our smooth 0-divergence is different from the smooth min Rényi divergences in [42, Definition 1] and [43] except for non-atomic measures.\(^4\)

**Remark 10:** The smooth Rényi divergence is a natural extension of the smooth Rényi entropy \( H_\alpha^\epsilon \) defined in [44] (which can be viewed as a special case where the reference measure is the counting measure). In [42] the smooth min and max Rényi divergences are introduced, which correspond to the \( \alpha = 0 \) and \( \alpha = +\infty \) cases of Definition 8. Moreover, we have introduced \(+/-\) in the notation to emphasize the difference between the two possible ways of smoothing in (43) and (44).

The following quantity, which characterizes the binary hypothesis testing error, is a \( g \)-divergence but not a monotonic function of any \( f \)-divergence [35]. We will see in Proposition 13 that it is a monotonic function of the 0-smooth Rényi divergence.

**Definition 11:** For nonnegative finite measures \( \mu \) and \( \nu \) on \( \mathcal{X} \), define

\[
\beta_\alpha (\mu, \nu) := \min_{A : \mu (A) \geq \alpha} \nu (A).
\]

**Remark 12:** In the literature, the definition of \( \beta_\alpha (P, Q) \) is usually restricted to probability measures and allows randomized tests:

\[
\beta_\alpha (P_W, Q_W) := \min \int P_{Z|W} (1|w) dQ_W (w)
\]

where the minimization is over all random transformations \( P_{Z|W} : Z \rightarrow (0,1) \) such that

\[
\int P_{Z|W} (1|w) dP_W (w) \geq \alpha.
\]

In contrast to \( E_\gamma \), allowing randomization in the definition can change the value of \( \beta_\alpha \) except for non-atomic measures. Nevertheless, many important properties of \( \beta_\alpha \) are not affected by this difference.

We conclude this section with some inequalities relating those distance measures we have discussed, which will be used to establish the asymptotic equivalence in the large deviation regime in Section V.

**Proposition 13:** Suppose \( \mu \) is a finite nonnegative measure and \( P \) and \( Q \) are probability measures, all on \( \mathcal{X} \), \( X \sim P \), \( \epsilon \in (0,1) \), and \( \gamma \geq 1 \).

1) For \( \alpha > 1 \),

\[
E_\gamma (P \| Q) \leq \hat{F}_\gamma (P \| Q) \leq \frac{\alpha}{\alpha - 1} E_\gamma^\alpha (P \| Q).\]

\(^4\)With the definition of smooth min Rényi divergence in [42, Definition 1, Proposition 13.7] and [43] would hold with the \( \beta_\alpha \) as defined in (46) rather than (45).
2) \( D_a^\pm (P \| Q) \) is increasing in \( a \in [0, 1] \) and \( D_a^\pm (P \| Q) \) is increasing in \( a \in [1, \infty] \).

3) If \( P \) is a probability measure, then
\[
D(P \| Q) \leq \int_0^\infty P[t_P(X) > \tau] \, dr;
\]
\[
D(P \| Q) \geq E_\gamma(P \| Q) \log \gamma - 2e^{-1} \log e.
\]

4) If \( a \in [0, 1) \) then\(^5\)
\[
D_a(\mu \| Q) \leq \log \gamma - \frac{1}{1-a} \log (\mu(X) - E_\gamma(\mu \| Q)).
\]
If \( a \in (1, \infty) \) then
\[
D_a(\mu \| Q) \geq \log \gamma + \frac{1}{a-1} \log E_\gamma(\mu \| Q).
\]

5) Suppose \( a \in [0, 1) \), \( E_\gamma(P \| Q) < 1 - \epsilon \), then
\[
D_a^\pm (P \| Q) \leq \log \gamma - \frac{1}{1-a} \log(1 - \epsilon - E_\gamma(P \| Q)).
\]

Suppose \( a \in (1, \infty] \), \( E_\gamma(P \| Q) > \epsilon \), then
\[
D_a^\pm (P \| Q) \geq \log \gamma + \frac{1}{a-1} \log (E_\gamma(P \| Q) - \epsilon).
\]

6) \( D_\infty^\pm (P \| Q) \leq \log \gamma \iff E_\gamma(P \| Q) \leq \epsilon \). That is, \( D_\infty^\pm (P \| Q) = \log \inf\{\gamma : E_\gamma(P \| Q) \leq \epsilon\} \).

7) \( D_0^+ (P \| Q) = -\log b_1-\epsilon(P, Q) \).

8) Fix \( \tau \in \mathbb{R} \) and \( \epsilon \geq P[t_P(X) \leq \tau] \), where \( X \sim P \) and \( P \) is non-atomic. Then
\[
D_0^\pm (P \| Q) \geq \tau + \log \frac{1}{1-\epsilon}.
\]

Moreover, \( D_0^\pm (P \| Q) \geq \tau \) holds when \( P \) is not necessarily non-atomic.

Proof: See Appendix B. \( \square \)

D. The Resolvability Problem

The setup in Figure 3 is the same as in the original paper on channel resolvability under total variation distance \(1\). Given a random transformation \( Q_X|U \) and a target distribution \( \pi_X \), we wish to minimize the size \( M \) of a codebook \( \{c_m\}_{m=1}^M \) such that when the codewords are equiprobably selected, the output distribution
\[
Q_X|_{\{c_m\}_{m=1}^M} := \frac{1}{M} \sum_{m=1}^M Q_X|U = c_m
\]
approaches \( \pi_X \). The difference from \(1\) is that we use \( E_\gamma \) (and other metrics) to measure the level of the approximation.

The fundamental one-shot tradeoff is the minimum \( M \) required for a prespecified degree of approximation. As usual, One-shot bounds are general in that no structural assumptions

\(^5\)The special case of \((51)\) for \( a = \frac{1}{2}, \mu(X) = 1 \) and \( \gamma = 1 \) is equivalent to a well-known bound on a quantity called fidelity using total variation distance, see \([45]\).

Fig. 3. Setup for channel resolvability.

on either the random transformation or the target distribution are imposed. The corresponding asymptotic results can usually be recovered quite simply from the one-shot bounds using, say, the law of large numbers in the memoryless case. Asymptotic limits are of interest because of the compactness of the expressions, and because good one-shot bounds are not always known, especially in the converse parts (see for example the converse of resolvability in Section IV-C). Unless otherwise stated, the alphabets considered in this paper are not restricted to be finite or countable. This applies to all the one-shot results in this paper. Finite alphabets are only assumed in the converses in Sections IV-C and IV-D, for which we restrict to finite input alphabets or even to discrete memoryless channels (DMC).\(^6\)

Next, we define the achievable regions in the general asymptotic setting (when sources and channels are arbitrary, see \([1], [6]\)) as well as the case of stationary memoryless channels and memoryless outputs. Boldface letters such as \( X \) denote a general sequence of random variables \((X^n)_{n=1}^\infty\), and sans-serif letters such as X denote the specific distributions in iid settings.

**Definition 14:** Given a channel\(^7\) \((Q_X^n|U^n)_{n=1}^\infty\) and a sequence of target distributions \((\pi_X^n)_{n=1}^\infty\), let \( X^n \sim \pi_X^n \). The triple \((G, R, X)\) is \( \epsilon \)-achievable \((0 < \epsilon < 1) \) if there exists \((c_m^n)_{n=1}^\infty\), where \( c_m^n := (c_1, \ldots, c_m) \) and \( c_m \in U^n \) for each \( m = 1, \ldots, M_n \), and \((\gamma_n)_{n=1}^\infty\), so that
\[
\limsup_{n \to \infty} \frac{1}{n} \log M_n \leq R;
\]
\[
\limsup_{n \to \infty} \frac{1}{n} \log \gamma_n \leq G;
\]
\[
\sup_n E_{\gamma_n}(Q_X^n|_{\{c_m^n\}_{m=1}^M}) = Q_X^n \leq \epsilon.
\]

Moreover, \((G, R, X)\) is said to be \( \epsilon \)-achievable if it is \( \epsilon \)-achievable for all \( 0 < \epsilon < 1 \). Define the asymptotic fundamental limits\(^8\)
\[
G_\epsilon(R, X) := \min\{g : (g, R, X) \text{ is } \epsilon\text{-achievable}\};
\]
\[
G(R, X) := \sup_{\epsilon > 0} G_\epsilon(R, X),
\]
and
\[
S_\epsilon(G, X) := \min\{r : (G, r, X) \text{ is } \epsilon\text{-achievable}\};
\]
\[
S(G, X) := \sup_{\epsilon > 0} S_\epsilon(G, X),
\]
which, in keeping with \([1]\), we refer to as the resolvability function. In the special case of \( Q_X^n|_{U^n} = Q_X^n|U \) and target

\(^6\)A DMC is a stationary memoryless channel whose input and output alphabets are finite, which is denoted by the corresponding per-letter random transformation (such as \( Q_X|U \)) in this paper.

\(^7\)In this setting a “channel” refers to a sequence of random transformations.

\(^8\)We can write \( \min \) instead of \( \inf \) in \((61)\) and \((62)\) since for fixed \( X \) the set of \((G, R)\) such that \((G, R, X)\) is \( \epsilon \)-achievable is necessarily closed; similarly in \((64)\) and \((66)\).
\[ \pi_X^n = \pi_X^{\otimes n} \text{ we may write the quantities in (61) and (62) as } G(R, \pi_X^n) \text{ and } S(G, \pi_X^n). \]

Note that by [1], [14], (0, R, \pi_X^n) is achievable if and only if \( R \geq I(P_U^n, Q_{X^n|U}) \) for some \( P_U \) satisfying \( P_U \rightarrow Q_{X^n|U} \rightarrow \pi_X^n \).

In Section V we show that the same exponent (61) is obtained with other distance measures.

In addition to approximation of a given target distribution, [1] also considered the minimum rate of randomness needed to approximate a worst-case output distribution under total variation distance, which has implications for identification coding. The \( E_7 \) version of this problem amounts to finding the achievable pairs defined as follows:

**Definition 15:** Given a channel \( (Q_{X^n|U^n})_{n=1}^\infty \), the pair \((G, R)\) is \( \epsilon \)-achievable \((0 < \epsilon < 1)\) if for any sequence of input distributions \( P_{U^n} \), the triple \((G, R, X)\) is \( \epsilon \)-achievable, where \( X = (X^n)_{n=1}^\infty \) and \( P_{U^n} \rightarrow P_{X^n} \rightarrow P_X \). Moreover, \((G, R)\) is achievable if it is \( \epsilon \)-achievable for all \( 0 < \epsilon < 1 \).

We define the asymptotic fundamental limits
\[
G(R) := \min \{ g : (g, R) \text{ is achievable} \} \tag{64}
\]
and the resolvability functions
\[
S_\epsilon(G) := \min \{ r : (g, r) \text{ is } \epsilon \text{-achievable} \}; \tag{65}
S(G) := \sup_{\epsilon > 0} S_\epsilon(G). \tag{66}
\]

Note that Definition 15 considers output distributions that are induced by some input distribution, which is a subclass of all the distributions on the output alphabet. In contrast, the memoryless output distribution in Definition 14 need not be induced by any input distribution. The reason for this dichotomy will be explained in Remark 34 and at the beginning of Section IV-D.

A useful property is that the convergence in (57) and (58) can be made uniform in the target distribution. A similar observation was made in the proof of [1, Lemma 6] in the context of resolvability in total variation distance.

**Proposition 16:** If \((G, R)\) is \( \epsilon \)-achievable for \((Q_{X^n|U^n})_{n=1}^\infty\), then there exists \((\gamma_n)_{n=1}^\infty\) and \((M_n)_{n=1}^\infty\) such that
\[
\limsup_{n \to \infty} \frac{1}{n} \log \gamma_n \leq G; \tag{67}
\limsup_{n \to \infty} \frac{1}{n} \log M_n \leq R; \tag{68}
\sup_{\epsilon > 0} \inf_{n, P_{U^n} \in \mathcal{M}_n} E_{\gamma_n}(Q_{X^n}[M_n] \mid Q_X^n) \leq \epsilon, \tag{69}
\]
where \( Q_{U^n} \rightarrow Q_{X^n|U^n} \rightarrow Q_X^n \).

**Proof:** Fix arbitrary \( G' > G \) and \( R' > R \). Observe that the sequence
\[
\sup_{Q_{U^n} \in \mathcal{M}_n} \inf_{n} E_{\exp(nG')} (Q_{X^n}[M_n]) \mid Q_X^n) \tag{70}
\]
where \( M_n := \exp(nR') \), must be upper-bounded by \( \epsilon \) for large enough \( n \). Otherwise, there would be a sequence \((Q_{U^n})_{n=1}^\infty\) such that the infimum in (70) is not upper-bounded by \( \epsilon \) for large enough \( n \), which is a contradiction since we can find \((cM_n)_{n=1}^\infty\) and \((\gamma_n)_{n=1}^\infty\) in Definition 14 such that \( M_n < \exp(nR') \) and \( \gamma_n \leq \exp(nG') \) for \( n \) large enough, and apply the monotonicity of \( E_7 \) in \( \gamma \). Finally, since \((G', R')\) can be arbitrarily close to \((G, R)\) the claim follows by the diagonalization argument [46, p. 56].

**III. Source Resolvability**

This section is devoted to the source resolvability problem [1], which can be viewed as channel resolvability with identity channels. For source resolvability, we derive simple and tight one-shot bounds, the ideas of which are not easily extendable to the general channel resolvability problem. To get a better grasp of the source resolvability problem, recall the definition of \( M \)-type distributions in [1].

**Definition 17** [1]: A distribution \( P \) on \( X \) is said to be an \( M \)-type \((M \in \mathbb{N})\) if for each \( x \in X' \), the probability \( P(x) \) is a multiple of \( \frac{1}{M} \).

Clearly, \( P \) is an \( M \)-type if and only if there exists an identity random transformation \( Q_{X|U} \) and a codebook \((c_m)_{m=1}^M\) such that \( Q_{X'=M} = P \). Hence the source \( E_7 \)-resolvability can also be viewed as minimizing \( M \) such that there exists an \( M \)-type distribution that approximates a given source distribution in \( E_7 \).

One of the advantages of studying the source resolvability problem is that, in contrast to the general channel resolvability problem, we will be able to give a general one-shot converse. Moreover, as noted in [1], source resolvability is intimately connected to almost-lossless source coding and random number generation. Now in the general case of source approximation under \( E_7 \), suppose a random variable \( X \) can be approximated by an \( M \)-type random variable \( \hat{X} \) in the sense that
\[
E_7(P_{\hat{X}} \parallel PX) \leq \delta \tag{71}
\]
for some \( \gamma, \delta > 0 \). Then we can approximately generate \( X \) from an equiprobable \( U_M \in \{1, \ldots, M\} \) as \( \hat{X} = \phi(U_M) \) with some deterministic function \( \phi \). Again, (71) has the operational meaning that the probability \( P_{\hat{X}}(E) \) of any error \( E \) associated with the random number is guaranteed to be small, say \( \leq 10^{-2} \), provided that the error probability under the target distribution is very small, that is, provided that \( P_X(E) \leq \frac{1}{\gamma}(10^{-2} - E_7(P_{\hat{X}} \parallel PX)) \).

**A. One-Shot Achievability Bound**

We develop a simple one-shot achievability bound without using random coding, which is asymptotically tight for general sequences of sources.\(^9\)

For a discrete random variable \( X \sim P \) on \( X' \), define the information for \( x \in X' \)
\[
i_X(x) := \log \frac{1}{P(x)} \tag{72}
\]

**Theorem 18:** For a discrete \( X \sim P_X \), integer \( M > 0 \), and \( \gamma \geq 1 \), there exists an \( M \)-type distribution \( P_{\hat{X}} \) such that\(^10\)
\[
E_7(P_{\hat{X}} \parallel PX) \leq \left[1 - \frac{\gamma}{2} \right] \left[ i_X(x) \leq \log \gamma M \right]^+ \tag{73}
\]

\(^9\)In contrast, the result obtained by particularizing the bound for channel resolvability to identity channels in Section IV-A is not as general (Remark 22), which is one of the reasons why source resolvability is discussed separately.

\(^10\)For \( a \in \mathbb{R} \), we use the notation \([a]^+ := \max\{a, 0\}\).
Theorem 18 and Theorem 20, we have

\[ \gamma \] are defined on the same discrete alphabet. Then for any

\( \mu \) also be viewed as a distribution on \( X \cup \{e\} \), and we have

\[ E_\gamma (P_X \parallel P_X) = P_X (e) \]

and so the result follows from (75) and (76). \( \square \)

Remark 19: When \( \gamma = 1 \), the bound (73) without the factor 1/2 holds [46].

B. One-shot Converse Bound

The following result generalizes the bound [46, Lemma 2.1.2] to values other than \( \gamma = 1 \).

Theorem 20: Suppose \( P_X \) and an M-type distribution \( P_\gamma \) are defined on the same discrete alphabet. Then for any \( \gamma > 0 \), \( a > 0 \),

\[ E_\gamma (P_X \parallel P_X) \geq 1 - \gamma \mathbb{P}[X < \log \gamma M + a] - \exp(-a). \]

Proof: Define a set

\[ A := \{ x \in X : \log \gamma M + a \} \cup \text{supp}(\hat{X}) \]

Then we have

\[ P_X (A) \leq \mathbb{P}[X < \log \gamma M + a] + \mathbb{P}[X \geq \log \gamma M + a, X \in \text{supp}(\hat{X})] \leq \mathbb{P}[X < \log \gamma M + a] + \frac{1}{\gamma M} \exp(-a) \cdot M \]

where (81) is because \( |\text{supp}(\hat{X})| \leq M \). Therefore the result follows since \( E_\gamma (P_X \parallel P_X) \geq P_X (A) - \gamma P_X (A) \) and \( P_X (A) = \gamma \).

For the asymptotic analysis, we are interested in the regime where \( \gamma \) and \( M \) grow exponentially in \( n \). Combining Theorem 18 and Theorem 20, we have

Corollary 21: For an arbitrary sequence \( X = (X^n)_{n=1}^\infty \), and the identity channel, \( \epsilon \in (0, 1) \) and \( G > 0 \),

\[ S_\epsilon (G, X) = \inf \left\{ R : \limsup_{n \to \infty} \frac{1}{n} \log \frac{1}{\mathbb{P}[X^n < R + G]} \leq G \right\}. \]

Proof: In Theorem 18, note that for any \( (G, R) \), if

\[ \limsup_{n \to \infty} \frac{1}{n} \log \frac{1}{\mathbb{P}[X^n < R + G]} \leq G \]

then by Theorem 18 we see that \( (G + \delta, R) \) is \( \epsilon \)-achievable for any \( \delta > 0 \). Since the set of \( \epsilon \)-achievable pairs is closed, \( (G, R) \) is also \( \epsilon \)-achievable. This shows the \( \leq \) part of (82).

If \( (G, R) \) is such that

\[ \liminf_{n \to \infty} \frac{1}{n} \log \frac{1}{\mathbb{P}[X^n < R + G]} > G, \]

then by Theorem 20, \( (G, R) \) is not \( \epsilon \)-achievable, hence \( S_\epsilon (G, X) > R \). This shows the \( \geq \) part of (82). \( \square \)

Remark 22: In Section IV-A, we develop achievability bounds for channel resolvability (e.g. (127)), which imply the following bound on source resolvability

\[ S(G, X) \leq \sup_{\epsilon > 0} \left\{ R : \liminf_{n \to \infty} \mathbb{P} \left[ \frac{1}{n} \log \frac{1}{\mathbb{P}[X^n < R + G]} \geq 1 - \epsilon \right] \right\} \]

when particularized to the identity channel, where

\[ \bar{H}(X) := \sup_{\epsilon > 0} \left\{ R : \liminf_{n \to \infty} \mathbb{P} \left[ \frac{1}{n} \log \frac{1}{\mathbb{P}[X^n < R]} \geq 1 - \epsilon \right] \right\} \]

is the sup-entropy rate defined in [1]. In view of Corollary 21, the achievability bound (85) is not tight in general. Indeed, the achievability construction in the proof of Theorem 18 is based on quantizing a scaling of the source distribution, which is more efficient than the random coding argument in the channel counterpart.

In the case of discrete memoryless sources, we obtain a more explicit formula by performing large deviation analysis of Corollary 21:

Corollary 23: For any per-letter distribution \( \pi_X \) on a finite alphabet, real number \( G > 0 \), and the identity channel,

\[ S_G (G, \pi_X) = \inf_{Q_X : D(Q_X || \pi_X) \leq G} \left[ D(Q_X || \pi_X) + H(Q_X) - G \right] \]

Proof: By large deviations/method of types,

\[ \limsup_{n \to \infty} \frac{1}{n} \log \frac{1}{\mathbb{P}[X^n < R + G]} = \inf_{Q_X : D(Q_X || \pi_X) + H(Q_X) - G \leq R} D(Q_X || \pi_X), \]

and the result follows. \( \square \)

This result is a special case of channel resolvability for memoryless outputs (Theorem 33).


IV. CHANNEL RESOLVABILITY

In this section, we first derive achievability bounds for channel resolvability using random coding. For discrete memoryless channels with iid target distributions, we prove a converse bound which, combined with the achievability bounds, yields the exact expression for $G_r(R, \pi_X)$. For the worst-case distributions, we prove a converse bound which does not match the achievability bound, but addresses certain properties of $S(G)$.

Since the excess relative information is a trivial upper bound on $E_g$, it suffices to derive achievability (upper-bounds) for the former and converse (lower-bounds) for the latter. In fact, by Proposition 13-1 we know that the two metrics are equivalent in large deviation analysis.

A. One-Shot Achievability Bound

We first present the result in a simple special case where the target distribution matches the input distribution according to which the codewords are generated.

**Theorem 24 (Softer-Covering Lemma):** Fix $\mathcal{Q}_{UX} = \mathcal{Q}_U \mathcal{Q}_{X|U}$. For an arbitrary codebook $[c_1, \ldots, c_M]$, define

$$Q_{X[c^M]} := \frac{1}{M} \sum_{m=1}^{M} Q_{X|U=c_m}.$$  

Then for any $\gamma, \epsilon, \sigma > 0$ satisfying $\gamma > 1 > \epsilon + \sigma$ and $\tau \in \mathbb{R}$,

$$\mathbb{E}[\tilde{F}_\gamma(Q_{X|U^M}, Q_X)] \leq \mathbb{P}\left[I_{U;X}(U; X) > \log M \sigma\right] + \frac{1}{\epsilon} \mathbb{P}[I_{U;X}(U; X) > \log M - \tau] + \frac{\exp(-\tau)}{(\gamma - 1 - \epsilon - \sigma)^2}$$  

where $U^M \sim Q_U \times \cdots \times Q_U$, $(U, X) \sim Q_U Q_{X|U}$, and the information density

$$i_{U;X}(u; x) := \log \frac{dQ_{X|U=u}}{dQ_X}(x).$$

Remark 25: Theorem 24 implies (the general asymptotic version of) the soft-covering lemma based on total variation (see [1], [6], [10]). Indeed if $M_n = \exp(nR)$ at blocklength $n$ where $R > I(U; X)$, we can select $\tau_n \leftarrow \frac{\epsilon}{2}(R - I(U; X))$. Moreover for any $\gamma > 1$ we can pick constant $\epsilon, \sigma > 0$ such that $\gamma - 1 > \epsilon + \sigma$ in the theorem, to show that

$$\lim_{n \to \infty} \mathbb{E}[\tilde{F}_\gamma(Q_{X|U^M_n}, Q_X)] = 0$$

which, by (15) and by taking $\gamma \downarrow 1$, implies that

$$\lim_{n \to \infty} \mathbb{E}[Q_{X|U^M_n} - Q_X] = 0.$$  

We refer to Theorem 24 as the softer-covering lemma since for a larger $\gamma$ it allows us to use a smaller codebook to cover the output distribution more softly (i.e. approximate the target distribution under a weaker metric).

Proof: Define the “atypical” set

$$\mathcal{A}_\epsilon := \{(u, x) : i_{U;X}(u; x) \leq \log M - \tau\}.$$  

Now, let $X^M$ be such that $(U^M, X^M) \sim Q_{UX} \times \cdots \times Q_{UX}$.

The joint distribution of $(U^M, \hat{X})$ is specified by letting $\hat{X} \sim Q_{X|U^M}$ conditioned on $U^M = c^M$. We perform a change-of-measure step using the symmetry of the random codebook:

$$\mathbb{E}[\tilde{F}_\gamma(Q_{X|U^M}, Q_X)] = \mathbb{P}\left[\frac{dQ_{X|U^M}}{dQ_X}(\hat{X}) > \gamma\right]$$  

$$= \frac{1}{M} \sum_{m=1}^{M} \mathbb{P}\left[\frac{dQ_{X|U^M}(X_m)}{dQ_X}(X_m) > \gamma\right]$$  

$$= \mathbb{P}\left[\frac{dQ_{X|U^M}}{dQ_X}(X_1) > \gamma\right]$$

where (99) is because of (92), and (100) is because the summands in (99) are equal. Note that $X_1$ is correlated with only the first codeword $U_1$.

Next, because of the relation

$$\mathbb{E}[\tilde{F}_\gamma(Q_{X|U^M}, Q_X)] = \frac{1}{M} \sum_{m=1}^{M} \exp(i_{U;X}(c_m, x)),$$

we can upper-bound (100) by the union bound as

$$\mathbb{P}[\exp(i_{U;X}(U_1; X_1)) > M \sigma]\left[\frac{1}{M} \sum_{m=2}^{M} \exp(i_{U;X}(U_m; X_1))1_{\mathcal{A}_\epsilon}(U_m, X_1) > \epsilon\right]$$

$$+ \mathbb{P}\left[\frac{1}{M} \sum_{m=2}^{M} \exp(i_{U;X}(U_m; X_1))1_{\mathcal{A}_\epsilon}(U_m, X_1) > \gamma - \epsilon - \sigma\right]$$

where we used the fact that $1_{\mathcal{A}_\epsilon} + 1_{\mathcal{A}_{\mu}} = 1$. Notice that the first term of (102) may be regarded as the probability of “atypical” events and accounts for the first term in (93). The second term of (102) can be upper-bounded with Markov’s inequality:

$$\frac{1}{M \epsilon} \sum_{m=2}^{M} \mathbb{E}[\exp(i_{U;X}(U_m; X_1))1_{\mathcal{A}_\epsilon}(U_m, X_1)]$$

$$\leq \frac{1}{M \epsilon} \sum_{m=2}^{M} \mathbb{E}[1_{\mathcal{A}_\epsilon}(U_m, X)]$$

$$\leq \frac{1}{\epsilon} \mathbb{P}[i_{U;X}(U; X) > \log M - \tau]$$

accounting for the second term in (93) where (103) is a change-of-measure step using the fact that $(U_1, X_1) \sim Q_U \times Q_X$ for $m \geq 2$.

Finally we take care of the last term in (102), again using the independence of $U_m$ and $X_1$ for $m \geq 2$. Observe that for any $x \in \mathcal{X}$,

$$\mu := \mathbb{E}\left[\frac{1}{M} \sum_{m=2}^{M} \exp(i_{U;X}(U_m; x))1_{\mathcal{A}_\epsilon}(U_m, x)\right]$$

$$\leq \frac{1}{M} \sum_{m=2}^{M} \mathbb{E}\left[\exp(i_{U;X}(U_m; x))\right]$$

$$= \frac{M - 1}{M}$$

$$\leq 1,$
whereas
\[
\text{Var}\left(\frac{1}{M} \sum_{m=2}^{M} \exp(i_U X(U_m; x))1_{A_r}(U_m, x)\right)
= \frac{1}{M^2} \sum_{m=2}^{M} \text{Var}\left(\exp(i_U X(U_m; x))1_{A_r}(U_m, x)\right)
\leq \frac{1}{M} \text{Var}\left(\exp(i_U X(U; x))1_{A_r}(U, x)\right)
\leq \frac{1}{M} \mathbb{E}\left[\exp(2\epsilon i U X(U; x))1_{A_r}(U, x)\right]
\leq \exp(-\tau)\mathbb{E}\left[\exp(i_U X(U; x))\right]
= \exp(-\tau),
\]
(113)
where the change of measure step (112) uses (97). It then follows from Chebyshev’s inequality that
\[
\mathbb{P}\left[\frac{1}{M} \sum_{m=2}^{M} \exp(i_U X(U_m; x))1_{A_r}(U_m, x) > \gamma - \epsilon - \sigma - 1\right]
\leq \mathbb{P}\left[\frac{1}{M} \sum_{m=2}^{M} \exp(i_U X(U_m; x))1_{A_r}(U_m, x) - \mu > \gamma - \epsilon - \sigma - 1\right]
\leq \frac{1}{(\gamma - \epsilon - \sigma - 1)^2}.
(115)
\]

For the asymptotic analysis, we are interested in the regime where \(M\) and \(\gamma\) are growing exponentially. In this case, the right side of (93) can be regarded as essentially modulo nuisance parameters. This can be seen from the choice of parameters in Corollary 29. Thus the sum rate of \(M\) and \(\gamma\) has to exceed the sup information rate in order that the approximation error vanishes asymptotically.

Extending Theorem 24 to the more general scenario where the target distribution may not have any relation with the input distribution, we have the following result, where we allow \(\pi X\) to be an arbitrary positive measure.

**Theorem 26 (Softer-Covering Lemma: Unmatched Target Distribution):** Fix \(\pi X\) and \(Q_{UX} = Q_U Q_{X|U}\). For an arbitrary codebook \([c_1, \ldots, c_M]\), define \(Q_{X|U^M}\) as in (92). Then for any \(\gamma, \epsilon, \sigma > 0\) satisfying \(\gamma > \epsilon + \sigma\), \(\tau \in \mathbb{R}\) and \(0 < \delta < 1\),
\[
\mathbb{E}\left[\tilde{F}_\gamma\left(Q_{X|U^M}\parallel \pi X\right)\right]
\leq \mathbb{P}\left[i_{Q_{X|\pi X}}(X) > \log(\gamma - \sigma - \epsilon) \text{ or } i_U X(U; X) + i_{Q_{X|\pi X}}(X) > \log \delta M\sigma\right]
+ \frac{\gamma - \epsilon - \sigma}{\epsilon} \mathbb{P}\left[i_U X(U; X) > \log M - \tau\right]
+ \exp(-\tau)\left(\frac{\gamma - \epsilon - \sigma}{1 - \delta}\right)^2
\]
(117)
where \(U^M \sim Q_U \times \cdots \times Q_U\) and \((U, X) \sim Q_U Q_{X|U}\).

**Proof Sketch:** Similarly to the proof of Theorem 24, we first use a symmetry argument and change-of-measure step so that the random variable of the channel output is correlated only with the first codeword, to obtain
\[
\mathbb{E}\left[\tilde{F}_\gamma\left(Q_{X|U^M}\parallel \pi X\right)\right] \leq \mathbb{P}\left[\frac{dQ_{X|U^M}}{d\pi X}(X) > \gamma\right].
(118)
\]
Then in the next union bound step we have to take care of another “atypical” event that \(\frac{dQ_{X|U^M}}{d\pi X}(X) > \gamma_2\), where
\[
\gamma_2 := \gamma - \epsilon - \sigma.
(119)
\]
More precisely, we have
\[
\mathbb{P}\left[\frac{dQ_{X|U^M}}{d\pi X}(X) > \gamma\right]
\leq \mathbb{P}\left[\tilde{\xi}(X) > \gamma_2\right]
+ \mathbb{P}\left[\frac{1}{M} \sum_{m=2}^{M} \eta(U_m, X_1)1_{A_r}(U_m, X_1) > \epsilon, \tilde{\xi}(X_1) \leq \gamma_2\right]
+ \mathbb{P}\left[\frac{1}{M} \sum_{m=2}^{M} \eta(U_m, X_1)1_{A_r}(U_m, X_1) > \gamma
- \epsilon - \delta\sigma, \tilde{\xi}(X_1) \leq \gamma_2\right]
(120)
\]
where we have defined
\[
\eta(u, x) := \frac{dQ_{X|U=x}}{d\pi X}(x);
(121)
\]
\[
\tilde{\xi}(x) := \frac{dQ_X}{d\pi X}(x).
(122)
\]
As before the first term of (120) may be regarded as the probability of “atypical” events and accounts for the first term in (117). The second and the third terms of (120) can be upper-bounded by
\[
\mathbb{P}\left[\frac{1}{M} \sum_{m=2}^{M} \eta(U_m, X_1)1_{A_r}(U_m, X_1) > \epsilon\right]
\leq \mathbb{P}\left[\frac{1}{M} \sum_{m=2}^{M} \exp(i_U X(U_m; X))1_{A_r}(U_m, X_1) > \epsilon\right]
\]
(123)
and
\[
\mathbb{P}\left[\frac{1}{M} \sum_{m=2}^{M} \eta(U_m, X_1)1_{A_r}(U_m, X_1) > \gamma - \epsilon - \delta\sigma\right]
\leq \mathbb{P}\left[\frac{1}{M} \sum_{m=2}^{M} \exp(i_U X(U_m; X))1_{A_r}(U_m, X_1) > 1 + \frac{(1-\delta)\sigma}{\gamma_2}\right].
(124)
\]
The rest of the proof is similar to that of Theorem 24 and is omitted.

For the purpose of asymptotic analysis in the stationary memoryless setting, the right side of (117) can be regarded as essentially
\[
\mathbb{P}\left[i_{Q_{X|\pi X}}(X) > \log \gamma\right]
+ \mathbb{P}\left[i_U X(U; X) + i_{Q_{X|\pi X}}(X) > \log M\gamma\right]
(125)
\]
modulo nuisance parameters.
Remark 27: By setting \( \tau \leftarrow +\infty \) and letting \( \delta \uparrow 1 \), the bound in Theorem 26 can be weakened in the following slightly simpler form:

\[
\mathbb{E}[\tilde{F}_\gamma(Q_{X[U]} \| \pi_X)] \leq \mathbb{P}
\left[ \frac{dQ_X}{d\pi_X}(X) > \gamma - \sigma \right]
\]

\[
+ \mathbb{P}
\left[ \frac{dQ_X}{d\pi_X}(X|U) \geq M\sigma \right]
\]

\[
+ \frac{\gamma - \sigma - \epsilon}{\epsilon}.
\]

(126)

In fact, assuming \( \tau = +\infty \) we can simplify the proof of the theorem and strengthen (126) to

\[
\mathbb{E}[\tilde{F}_\gamma(Q_{X[U]} \| \pi_X)] \leq \mathbb{P}
\left[ \frac{dQ_X}{d\pi_X}(X) > \gamma_2 \right]
\]

\[
+ \mathbb{P}
\left[ \frac{dQ_X}{d\pi_X}(X|U) > M(\gamma - \epsilon) \right]
\]

\[
+ \frac{\gamma_2}{\epsilon}.
\]

(127)

for any \( \gamma_2 > 0 \) and \( 0 < \epsilon < \gamma \). As we show in Corollary 29, the weakened bounds (126) and (127) are still asymptotically tight provided that the exponent with which the threshold \( \gamma \) grows is strictly positive. However, when the exponent is zero (corresponding to the total variation case), we do need \( \tau \) in the bound for asymptotic tightness.

Remark 28: If \( Q_X = \pi_X \), we can set \( \gamma_2 \leftarrow 1 \) and \( \epsilon \leftarrow \frac{1}{\gamma} \) in (127) to obtain the simplification

\[
\mathbb{E}[\tilde{F}_\gamma(Q_{X[U]} \| \pi_X)] \leq \mathbb{P}
\left[ \exp(U;X;X) > \frac{M\gamma}{2} \right] + \frac{2}{\gamma}.
\]

(128)

Corollary 29: Fix per-letter distributions \( \pi_X \) on \( X \) and \( Q_{UX} = Q_UQ_{X|U} \) on \( U \times X \), and \( E, R \in (0, \infty) \). For each \( n \), define \( \gamma_n := \exp(nE) \) and \( M_n := |\exp(nR)| \); let \( U_{M_n} = (U_1, \ldots, U_{M_n}) \) have independent coordinates distributed according to \( Q_{\pi_X}^{\otimes n} \). Given any \( c^M = (c^m)_{m=1}^{M_n} \), where each \( c^m \in \mathcal{U}^n \), define

\[
Q_{X[c^M]} := \frac{1}{M_n} \sum_{m=1}^{M_n} Q_{\pi_X}^{\otimes n}(c^m).
\]

Then

\[
\lim_{n \to \infty} \mathbb{E}[F_{\gamma_n}(Q_{X[U]} \| \pi_X^{\otimes n})] = \lim_{n \to \infty} \mathbb{E}[F_{\gamma_n}(Q_{X[U]} \| \pi_X^{\otimes n})]
\]

\[
= 0.
\]

(130)

(131)

provided that

\[
E > D(Q_X \| \pi_X) + [I(Q_U, Q_{X|U}) - R]^+.
\]

(132)

Proof: Choose \( E' \) such that

\[
E > E' > D(Q_X \| \pi_X) + [I(Q_U, Q_{X|U}) - R]^+.
\]

(133)

We define \( Q_{\pi_X}^{\otimes n} \) by \( Q_{\pi_X}^{\otimes n}(c^M) := \prod_{i=1}^{n} Q_{X[c^M]}(c^i) \), for any \( c^M \). Also note that we differentiate between per-letter symbols such as \( U \) and one-shot/block symbols such as \( U \) (so that \( U = U^n \) in this corollary).

Set \( \delta = \frac{1}{2}, \epsilon_n = \exp(nE) - \exp(nE') \) and \( \sigma_n = \frac{1}{2}(\gamma_n - \epsilon_n) = \frac{1}{2} \exp(nE') \), and apply (126). Notice that

\[
\mathbb{E}\left[ \log \frac{dQ_{X[U]}}{d\pi_X}(X|U) \right] = n[I(Q_U, Q_{X|U}) + D(Q_X \| \pi_X)]
\]

(134)

where \( (X, U) \sim Q_{UX}^{\otimes n} \), \( Q_{UX} := Q_{UX}^{\otimes n} \), and \( \pi_X := \pi_X^{\otimes n} \). By the law of large numbers, the first and second terms in (126) vanish because

\[
D(Q_X \| \pi_X) < E';
\]

\[
I(Q_U, Q_{X|U}) + D(Q_X \| \pi_X) < E' + R
\]

(135)

(136)

are satisfied.

The \( Q_U \) that minimizes the right side of (132) generally does not satisfy \( Q_U \rightarrow Q_{UX|U} \rightarrow Q_X \). This means that in the large deviation analysis, for the best approximation of a target distribution in \( E, \gamma \), we generally should not generate the codewords according to a distribution that corresponds to the target through the channel. This is a remarkable distinction from approximation in total variation distance, in which case an unmatched input distribution would result in the maximal total variation distance asymptotically. However, if we stick to matching input codeword distributions, then a simple and general asymptotic achievability bound can be obtained. Recall that [1] defined the sup-information rate

\[
\tilde{I}(U; X) := \inf R : \lim_{n \to \infty} \mathbb{P}
\left[ \frac{1}{n} I_{U^n; X^n}(U^n; X^n) > R \right] = 0
\]

(137)

and the inf-information rate

\[
\underline{I}(U; X) := \sup R : \lim_{n \to \infty} \mathbb{P}
\left[ \frac{1}{n} I_{U^n; X^n}(U^n; X^n) < R \right] = 0.
\]

(138)

Theorem 30: For any channel \( W = (Q_{X^n[U^n]}^{\otimes n})_{n=1}^{\infty} \), sequence of inputs \( U = (U^n)_{n=1}^{\infty} \), and \( G > 0 \), we have

\[
S(G, X) \leq \tilde{I}(U; X) - G
\]

(139)

where \( X \) is the output of \( U \) through the channel \( W \). As a consequence,

\[
S(G) \leq \sup_U \tilde{I}(U; X) - G
\]

(140)

For channels satisfying the strong converse property, the right of (140) can be related to the channel capacity because of the relations [1], [47]

\[
\sup_U \tilde{I}(U; X) = \tilde{I}(U; X) \equiv C(W).
\]

(141)

We conclude the subsection by remarking that had we used the soft-covering lemma to bound total variation distance and in turn, bounded the excess relative information with total variation distance, we would not have obtained Theorem 30. Indeed, consider \( M_n = \exp(nR) \) and let \( V_1, \ldots, V_{M_n} \) be i.i.d. according to \( Q_{UX}^{\otimes n} \). Regardless of how fast \( \gamma_n \) grows, we cannot conclude from the optimal upper bound (13) that

\[
\mathbb{E}[\tilde{F}_{\gamma_n}(Q_{X^n[U]} \| \pi_X^n)]
\]

(142)
vanishes unless $\mathbb{E}[Q_{X^{|\{U,M\}|}} - \pi_{X^n}]$ vanishes, which happens when $R > I(U;X)$ by the conventional resolvability theorem. This gives an upper bound $I(U;X)$ which is looser than Theorem 30 when $G > 0$.

B. Tail Bound of Approximation Error for Random Codebooks

For applications such as secrecy and channel synthesis, it is sometimes desirable to prove that the approximation error vanishes not only in expectation (e.g. Theorem 26), but also with high probability (see Footnote 2), in the case of a random codebook [15]–[17]. If the probability that the approximation error exceeds an arbitrary positive number vanishes doubly exponentially in the blocklength, then the analyses in these applications carry through because a union bound argument can be applied to exponentially many events. Previous proofs (e.g. [15]) based on carefully applying Chernoff bounds to each difference $Q_{X^{|U,M|}}(x) - Q_X(x)$ and then taking the union bound over $x$ require finiteness of the alphabets.

Here we adopt a different approach. Using concentration inequalities we can directly bound the probability that the error $E_T(Q_{X^{|U,M|}}||Q_X)$ deviates from its expectation, without any restrictions on the alphabet and in fact the bound only depends on the number of codewords. Therefore if the rate is high enough for the approximation error to vanish in expectation (by Theorem 26), we can also conclude that the error vanishes with high probability. The crux of the matter is thus resolved by the following one-shot result:

**Theorem 31:** Fix $\pi_X$ and $Q_{UX} = Q_U Q_{X|U}$. For an arbitrary codebook $[e_1, \ldots, e_M]$, define $Q_{X^{|U,M|}}$ as in (92). Then, for any $r > 0$,

$$\mathbb{P}[E_T(Q_{X^{|U,M|}}||Q_X) - \mathbb{E}[E_T(Q_{X^{|U,M|}}||Q_X)] > r] \leq \exp(-2Mr^2) \quad (143)$$

where the probability and the expectation are with respect to $U^M \sim Q_X \times Q_U$.

**Proof:** Consider $f : c^M \mapsto E_T(Q_{X^{|U,M|}}||Q_X)$. By the definition (92) and the triangle inequality (21), we have the following uniform bound on the discrete derivative:

$$\sup_{c,c',c'' \in \mathcal{X}} \left| f(c_{i+1}^M, c, c'^{i+1}) - f(c_{i+1}^M, c', c'^{i+1}) \right| \leq \frac{1}{M}, \quad \forall i, c^M. \quad (144)$$

The result then follows by McDiarmid’s inequality (see e.g. [48, Th. 2.2.3]). \qed

**Remark 32:** If we are interested in bounding both the upper and the lower tails then the right side of (143) gains a factor of 2. Other concentration inequalities may also be applied here; the transportation method gives the same bound in this example.

C. Converse for Stationary Memoryless Outputs

In this section we establish a converse of resolvability for stationary memoryless outputs and discrete memoryless channels which matches the achievability bound of Corollary 29 asymptotically.

**Theorem 33 (Resolvability for Stationary Memoryless Outputs):** For a DMC $Q_{X|U}$ and a nonnegative finite measure $\pi_X$, $G_e(R, \pi_X) = \min_{Q_U} [D(\pi_X||Q_X) + I(Q_U, Q_{X|U}) - R^+]$ where $Q_U \rightarrow Q_{X|U} \rightarrow Q_X$, for any $0 < \epsilon < 1$.

**Remark 34:** When resolvability was introduced in [1], the resolvability rate (under total variation distance) was formulated for outputs of stationary memoryless inputs, rather than all the tensor power distributions on the output alphabet, because otherwise there is no guarantee that the output process can be approximated under total variation distance even with an arbitrarily large codebook. Here we can extend the scope because all stationary memoryless distributions on the output alphabet (satisfying the mild condition of being absolutely continuous with respect to some output) can be approximated under $E_T$ as long as $\gamma$ is sufficiently large.

The achievability part of Theorem 33 is already shown in Corollary 29. For the converse, we need a notion of conditional typicality specially tailored for our problem which differs from the definitions of conditional typicality in [18] or [49] (see also [20]). This can be viewed as an intermediate of the above two definitions.

**Definition 35:** [Moderate Conditional Typicality] The $\delta$-typical set of $u^n \in U^n$ with respect to the discrete memoryless channel with per-letter conditional distribution $Q_{X|U}$ is defined as

$$T^n_{(Q_{X|U})}(u^n) := \{x^n : \forall a, b, \quad |\hat{P}_{u^n,x^n}(a, b) - Q_{X|U}(b|a)| \leq \delta Q_{X|U}(b|a)\} \quad (146)$$

where $\hat{P}_{u^n,x^n}$ denotes the empirical distribution of $(u^n, x^n)$.

**Remark 36:** In addition to its broad interest, Definition 35 plays an important role in obtaining the uniform bound in Lemma 35, as well as in Lemma 36. This definition of conditional typicality is of broad interest because of Lemma 37 and Lemma 38 ahead, and in particular the uniform bound in Lemma 37. Note that the definition in [18] corresponds to replacing the term $\delta Q_{X|U}(b|a)$ in (146) with $\delta$, in which case we cannot bound the probability of a sequence in the typical set as in Lemma 38. The “robust typicality” definition of [49] (see also [20]) corresponds to replacing this term with $\delta Q_{X|U}(b|a)\hat{P}_{u^n}(a)$, which does not give the uniform lower bound on the probability of conditional typical set as in Lemma 37.

**Lemma 37:** For fixed $\delta > 0$ and $Q_{X|U}$, there exists a sequence $(\gamma_n)$ such that $\lim_{n \to \infty} \gamma_n = 0$ and

$$Q_{X|U}^n(T^n_{(Q_{X|U})}(u^n)|u^n) \geq 1 - \gamma_n \quad (147)$$

for all $u^n \in U^n$.

**Proof:** We show that the statement holds with

$$\gamma_n = \frac{\mathbb{E}[\|X^n||\mathcal{X}]}{n^2} \left( \frac{1}{q} - 1 \right), \quad (148)$$

where

$$q := \min_{(a,b):Q_{X|U}(b|a)\neq 0} Q_{X|U}(b|a). \quad (149)$$
The number of occurrences \( N(a, b|u^n, X^n) \) of \((a, b) \in \mathcal{U} \times \mathcal{X} \) in \((u^n, X^n)\), where \( X^n \sim \prod_{i=1}^n \mathcal{Q}_X|u_i \), is binomial with mean \( N(a|u^n)Q_X|u(b|a) \) and variance \( N(a|u^n)Q_X|u(b|a)(1 - Q_X|u(b|a)) \). If \( Q_X|u(b|a) = 0 \) the condition that defines the set in (146) is automatically true. Otherwise, by Chebyshev's inequality we have for each \((a, b)\),

\[
\mathbb{P}[N(a, b|u^n, X^n) - N(a|u^n)Q_X|u(b|a)] > n\delta Q_X|u(b|a)] \\
\leq \frac{N(a|u^n)Q_X|u(b|a)(1 - Q_X|u(b|a))}{n^2\delta^2 Q_X|u(b|a)} \\
\leq \frac{1}{n\delta^2} \left( \frac{1}{q} - 1 \right) 
\]

(150)

and the claim follows by taking the union bound.

**Lemma 38:** For each \( u^n \), and \( x^n \in T_{[Q_X|u]}^{Q_X|u}(u^n) \), we have the bound

\[
Q_X|u^n(x^n|u^n) \geq \exp(-n[H(Q_X|u|\hat{P}_{x^n}) + \delta|\mathcal{U}|\log|\mathcal{X}|]).
\]

(152)

**Proof:** Since

\[
Q_X|u^n(x^n|u^n) = \prod_{a \in \mathcal{U}, b \in \mathcal{X}} Q_X|u(b|a)^{N(a|u^n, x^n)},
\]

we have

\[
\frac{1}{n} \log \frac{1}{Q_X|u^n(x^n|u^n)} \\
= \sum_{a, b} \hat{P}_{x^n}(a, b) \log \frac{1}{Q_X|u(b|a)} \\
\leq \sum_{a, b} \{Q_X|u(b|a)\hat{P}_{x^n}(a) + \delta Q_X|u(b|a)\} \log \frac{1}{Q_X|u(b|a)} \\
= H(Q_X|u|\hat{P}_{x^n}) + \delta \sum_{a} H(Q_X|u=a) \\
\leq H(Q_X|u|\hat{P}_{x^n}) + \delta |\mathcal{U}| \log |\mathcal{X}|.
\]

(153)

(154)

(155)

(156)

(157)

**Lemma 39:** For any type \( P_X \) and sequence \( u^n \),

\[
|T_{[Q_X|u]}^{n}(u^n) \cap T_{P_X}| \leq \exp(n[H(P_X) + D(P_X||\pi_X)]) \cdot |\mathcal{A}_n \cap T_{P_X}|
\]

where we have defined

\[
H(P_X) := \max_{\mathcal{Q}_U : |\mathcal{Q}_X - P_X| \leq \delta|\mathcal{U}|} H(Q_X|u|Q_U)
\]

(159)

where \( Q_U \to Q_X|u \to Q_X \), and the maximum in (159) is understood as \(-\infty\) if the set \( \{Q_U : |Q_X - P_X| \leq \delta|\mathcal{U}|\} \) is empty.

**Proof:** For any \( u^n \), we have the upper bound

\[
|T_{[Q_X|u]}^{n}(u^n) \cap T_{P_X}| \leq |T_{[Q_X|u]}^{n}(u^n)|
\]

(160)

\[
\leq \left( \sum_{x^n \in T_{[Q_X|u]}^{n}(u^n)} Q_X|u^n(x^n|u^n) \right)^{-1}
\]

(161)

\[
\leq \exp(n[H(Q_X|u|\hat{P}_{x^n}) + \delta|\mathcal{U}| \log |\mathcal{X}|])
\]

(162)

where we used Lemma 38 in (162). Moreover, if \( u^n \) satisfies \( |P_X - Q_X|u \circ \hat{P}_{u^n}| > \delta|\mathcal{U}| \) where \( Q_X|u \circ \hat{P}_{u^n} := \int Q_X|u=a d\hat{P}_{u^n}(a) \), then \( T_{[Q_X|u]}^{n}(u^n) \cap T_{P_X} \) is empty, because

\[
|\hat{P}_{u^n} - Q_X|u \circ \hat{P}_{u^n}| = \sum_{a, b} |\hat{P}_{u^n}(a, b) - Q_X|u(b|a)\hat{P}_{u^n}(a)|
\]

\[
\leq \sum_{a, b} \delta Q_X|u(b|a) \\
= \delta|\mathcal{U}|
\]

(163)

(164)

(165)

implies that any \( x^n \) in \( T_{[Q_X|u]}^{n}(u^n) \) does not have the type \( P_X \). Therefore the desired result follows by taking the maximum of (162) over type \( Q_U \) satisfying \( |Q_X - P_X| \leq \delta|\mathcal{U}| \).

**Proof of Converse of Theorem 33:** Fix a codebook \( (c_1, \ldots, c_M) \) and type \( P_X \). Define

\[
\mathcal{A}_n := \bigcup_{m=1}^{M} T_{[Q_X|u]}^{m}(c_m).
\]

(166)

Then

\[
\pi_X^{n}(\mathcal{A}_n \cap T_{P_X}) = \sum_{x^n \in \mathcal{A}_n \cap T_{P_X}} \pi_X^{n}(x^n)
\]

= \exp(-n[H(P_X) + D(P_X||\pi_X)]) \cdot |\mathcal{A}_n \cap T_{P_X}|

\leq \exp(-n[H(P_X) + D(P_X||\pi_X)]) \cdot \sum_{m=1}^{M} |T_{[Q_X|u]}^{m}(c_m) \cap T_{P_X}|

\leq \exp(-n[H(P_X) + D(P_X||\pi_X)])

\cdot M \exp(n[H(P_X) + \delta|\mathcal{U}| \log |\mathcal{X}|])

= \exp(-n[D(P_X||\pi_X) + H(P_X) - H(P_X) - R - \delta|\mathcal{U}| \log |\mathcal{X}|]).

(167)

(168)

(169)

(170)

(171)

where (170) is from Lemma 39. Whence (171) and the trivial bound

\[
\pi_X^{n}(\mathcal{A}_n \cap T_{P_X}) \leq \pi_X^{n}(T_{P_X})
\]

\leq \exp(-nD(P_X||\pi_X))

\leq \exp(-n[D(P_X||\pi_X) - \delta|\mathcal{U}| \log |\mathcal{X}|])

(172)

(173)

(174)

yield the bound

\[
\pi_X^{n}(\mathcal{A}_n \cap T_{P_X}) \leq \exp(-n[f(\delta, P_X) - \delta|\mathcal{U}| \log |\mathcal{X}|]),
\]

(175)

where we have defined the function

\[
f(\delta, P_X) := D(P_X||\pi_X) + [H(P_X) - H(P_X) - R]^{+}
\]

(176)

for \( \delta > 0 \) and \( P_X \ll \pi_X \). Define\(^{13}\)

\[
g(\delta) := \min_{P_X} f(\delta, P_X),
\]

(177)

\(^{13}\)The reason why we can write minimum in (177) is explained in Remark 41.
Then
\[ \pi_X^\otimes n(A_n) = \sum_{P_X} \pi_X^\otimes n (A_n \cap T_{P_X}) \]  
(178)
\[ \leq \sum_{P_X} \exp(-n[g(\delta) - \delta|U| \log |X|]) \]  
(179)
\[ \leq (n+1)^{|X|} \exp(-n[g(\delta) - \delta|U| \log |X'|]) \]  
(180)
where the summation is over all type \( P_X \) absolutely continuous with respect to \( \pi_X \), and (179) is from (175). Then for any real number \( G < g(\delta) - \delta|U| \log |X'| \) we have
\[ E_{\exp(nG)}(P_X^{|U=U|}||\pi_X^\otimes n) \]  
\[ \geq P_X^{\otimes n} (A_n) - \exp(nG)\pi_X^\otimes n(A_n) \]  
(181)
\[ \geq \frac{1}{M} \sum_{m=1}^M Q_{X|U}^\otimes n(\|T_{U(m)}\|c_m) - \exp(nG)\pi_X^\otimes n(A_n) \]  
(182)
\[ \geq 1 - \gamma_n - \exp(nG)\pi_X^\otimes n(A_n) \]  
(183)
\[ \to 1, \ n \to \infty. \]  
(184)
where
- (182) uses \( T_{n}^\otimes n(\|c_m\|) \subseteq A_n \), and we used the notation of the tensor power for the conditional law \( Q_{X|U}^\otimes n(\|U^n\|) := \prod_{i=1}^n Q_{X|U=U_i} \).
- (183) is from Lemma 37,
- (184) uses (180).

Since \( \delta > 0 \) was arbitrary, we thus conclude
\[ G_e(R, \pi_X) \geq \sup_{\delta > 0} \left\{ g(\delta) - \delta|U| \log |X'| \right\} \]  
(185)
\[ \geq \lim_{\delta \to 0} \inf g(\delta) \]  
(186)
\[ \geq g(0) \]  
(187)
where (187) is from Lemma 40. Since \( g(0) \) is the right side of (145), the converse bound is established. \( \Box \)

**Remark 41:** We can write minimum instead of infimum in (177) and hence (145) because of the lower semicontinuity of \( f \).

**Proof:** Consider a lower semicontinuous function \( \chi \) where \( \chi(\delta, P_X, Q_X) \) equals
\[ D(P_X||\pi_X) + [H(P_X) - H(Q_X||P_X||Q_X) - R]^+ \]  
(188)
if \( |Q_X - P_X| \leq \delta|U| \) and \( +\infty \) otherwise. Then \( f(\delta, P_X) = \min_{Q_X} \chi(\delta, P_X, Q_X) \) is lower semicontinuous, as it is the pointwise infimum of a lower semicontinuous functions over a compact set (see for example the proof in [50, Lemma 9]). The lower semicontinuity of \( g \) follows for the same reason.

The function \( G(R, \pi_X) \) in (145) satisfies some nice properties. Below we write it as \( G(R, \pi_X, Q_X) \) to emphasize its dependence on \( Q_X \), and assume that \( X \) and \( U \) can be arbitrary.

**Proposition 42:** 1) The function being minimized in (145), denoted as \( F(Q_U, R, \pi_X, Q_X) \), is convex in \( Q_U \).

2) Additivity: for any \( R > 0 \), \( \pi_X \), and \( Q_X \) and \( Q_X \{i\} \) (\( i = 1, 2 \)),
\[ G(R, \pi_X, Q_X \{i\} || Q_X \{i\} \{j\}) = \min_{R_1, R_2: R_1+R_2 \leq R} \left\{ G(R_1, \pi_X, Q_X \{i\}) + G(R_2, \pi_X, Q_X \{j\}) \right\} \]  
(189)
where we have abbreviated \( Q_X \times Q_X \) and \( Q_X \{i\} \times Q_X \{i\} \) as \( \pi_X \times \pi_X \) and \( Q_X \{i\} \| Q_X \{j\} \).

3) \( G(R, \pi_X, Q_X) \) is continuous in \( R \).

4) \( G(R, \pi_X, Q_X) \) is convex in \( R \).

**Proof:**
1) The function of interest is the maximum of the two functions \( D(Q_X||\pi_X) \) and
\[ D(Q_X||\pi_X) + I(Q_U, Q_X||U) - R \]  
(190)
where \( (U, X) \sim Q_{UX}, Q_U \rightarrow Q_X \rightarrow Q_X \), and the conditional relative information
\[ I(Q_X||\pi_X) := \log \frac{dQ_X||\pi_X}{d\pi_X}(X), \quad \forall u, x. \]  
(191)
The former is convex and the latter is linear in \( Q_U \).

2) The \( \leq \) direction is immediate from the single-letter formula (145) and the inequality
\[ |a|^+ + |b|^+ \geq |a + b|^+ \]  
(192)
for any \( a, b \in \mathbb{R} \). For the \( \geq \) direction, suppose \( Q_{U_1} \) achieves the minimum in the single-letter formula of \( G(R, \pi_X, \pi_X, Q_X \{i\} || Q_X \{j\}) \). Observe that
\[ F(Q_U \{i\}, R, \pi_X, Q_X \{i\} || Q_X \{j\} || U, \{i\} \{j\}) \]  
\[ - F(Q_U \{i\}, R, \pi_X, Q_X \{i\} || Q_X \{j\}) \]  
\[ = I(X_1; X_2) + I(Q_U \{i\}, Q_X \{i\} || Q_X \{j\} || U, \{i\} \{j\}) - R]^+ \]  
(193)
\[ \geq \sum_{i=1}^2 I(Q_U \{i\}, Q_X \{i\} || U, \{i\} \{j\}) - R]^+ \]  
(194)
\[ = 0 \]  
(195)
where (193) uses
\[ D(Q_X \{i\} ||\pi_X \{i\}) - D(Q_X \{i\} ||\pi_X \{j\}) - D(Q_X \{j\} ||\pi_X \{i\}) \]  
(196)
and (194) follows from (192). Therefore
\[ G(R, \pi_X, Q_X \{i\} || Q_X \{j\}) = F(Q_U \{i\}, R, \pi_X, Q_X \{i\} || Q_X \{j\}) \]  
(197)
But clearly there exists \( R_1 \) and \( R_2 \) summing to \( R \) such that
\[ R.H.S. \ of \ (197) \]  
\[ = F(Q_U \{i\}, R_1, \pi_X, Q_X \{i\} || U_1) + F(Q_U \{i\}, R_2, \pi_X, Q_X \{i\} || U_2) \]  
(198)
\[ \geq F(R_1, \pi_X, Q_X \{i\} || U_1) + F(R_2, \pi_X, Q_X \{i\} || U_2) \]  
(199)
and the result follows.
3) Fix any two numbers $0 \geq R' < R$. Choose $Q_U$ such that

$$G(R, \pi_X, Q_{X|U}) = F(Q_U, R, \pi_X, Q_{X|U}). \quad (200)$$

Then

$$0 \leq G(R', \pi_X, Q_{X|U}) - G(R, \pi_X, Q_{X|U}) \leq F(Q_U, R', \pi_X, Q_{X|U}) - F(Q_U, R, \pi_X, Q_{X|U}) \leq \lbrack I(Q_U, Q_{X|U}) - R' \rbrack^+ - \lbrack I(Q_U, Q_{X|U}) - R \rbrack^+ \leq [R - R']^+$$

where (201) follows because $G(\cdot, \pi_X, Q_{X|U})$ is non-increasing, and (204) uses (192) again. Thus $G(R, \pi_X, Q_{X|U})$ is actually 1-Lipschitz continuous in $R$.

4) Fix $R_1, R_2 \geq 0, \alpha \in [0, 1]$, and let $Q_{U_i}$ maximize $F(\cdot, R_i, \pi_X, Q_{X|U})$ for $i = 1, 2$. Define

$$R_a := (1 - \alpha)R_0 + \alpha R_1; \quad (205)$$

$$Q_{U_a} := (1 - \alpha)Q_{U_0} + \alpha Q_{U_1}. \quad (206)$$

In both $I(Q_{U_i}, Q_{X|U}) > R_a$ and $I(Q_{U_i}, Q_{X|U}) \leq R_a$ cases one can explicitly calculate that

$$F(Q_{U_i}, R_i, \pi_X, Q_{X|U}) \leq (1 - \alpha)F(Q_{U_0}, R_0, \pi_X, Q_{X|U}) + \alpha F(Q_{U_1}, R_1, \pi_X, Q_{X|U}) \quad (207)$$

and the convexity follows.

$\square$

D. Converse for Worst-Case Resolvability

The resolvability for the worst-case input distribution and for stationary memoryless outputs have very different flavors. First, let us remark that the resolvability for the worst distribution on the output alphabet (i.e. not necessarily induced by an input distribution) is usually a degenerate unexciting problem. For any DMC $Q_{X|U}$ having an output symbol $x \in X$ such that the one point distribution on $x$ is not induced by any input (it may still be true that the one point distribution is absolutely continuous with respect to the output distribution corresponding to some input), the probability $Q_{X^n}(x^n)$ vanishes for any sequence of input distributions $\{Q_{U^n}\}_{n=1}^\infty$, where $Q_{U^n} \to Q_{X^n} \to Q_{X^n}$. Thus if we pick the output distribution $\pi_X$ as the one point distribution on $x^n$, then the approximation error $E_{\exp(nG)}(Q_{X^n}|\pi_X^n) \uparrow 1$ as $n \to \infty$, no matter how large $G$ and $M_n$ are and what $c_{M_n}$ we pick.

Returning to the resolvability for the worst-case output distribution as formulated in Definition 15, we have shown the achievability bound $S(G) \leq \sup_U I(U; X) - G$ in Theorem 30. Is this bound tight in general? Before delving into the converse, it is instructive to consider the following example.

Example 43: Consider a DMC with $\mathcal{U} = \{0, e, 1\}$, $X = \{0, 1\}$ and

$$Q_{X|U}(x|u) = \begin{cases} 1 - \delta & u \in \{0, 1\}, x = u \\ \frac{\delta}{2} & u \in \{0, 1\}, x \neq u \end{cases} \quad (208)$$

where $0 < \delta < \frac{1}{2}$. Then from the formula of resolvability under total variation distance [1], we have

$$S(0) = C(Q_{X|U}) = 1 - h(\delta) \quad (209)$$

where $h(\cdot)$ is the binary entropy function, while Theorem 30 yields,

$$S(G) \leq [1 - h(\delta) - G]^+ \quad (210)$$

for $G > 0$.

Example 43 with $\delta = 0$ reduces to [1, Example 1]. It is argued in [1] that the worst-case input distribution requiring the maximal asymptotic randomness (209) is the equiprobable distribution on the set of all sequences having the type $\frac{1}{2}, 0, 1/2$ (which is the capacity-achieving single-letter distribution). Naively and from symmetry considerations, one might expect that this is also the worst-case distribution for approximation in $E_{\exp(nG)}$ metric when $G > 0$ and the channel parameter $\delta > 0$. However, this cannot be farther from the truth. Consider a deterministic input sequence $(e, \ldots, e)$, denote by $X^n$ the corresponding output sequence, and denote by $T$ the number of 1’s in the output. Moreover, when the input to the channel is equiprobable on the set of all sequences having the type $\frac{1}{2}, 0, 1/2$, denote by $X^n$ the corresponding output sequence, and denote by $T$ the number of 1’s. Then by the central limit theorem, $\frac{T - n/2}{\sqrt{n}}$ and $\frac{T - n/2}{\sqrt{n}}$ both converge weakly to some Gaussian random variables $N$ and $N$. By carefully bounding the probability of binomial distributions using Stirling’s formula, one can show that for any $\gamma > 0$, $\lim_{n \to \infty} E_{\gamma}(Q_{\hat{U}}\|Q_{\hat{T}}) = E_{\gamma}(Q_{\hat{N}}\|Q_{\hat{N}})$. Thus

$$E_{\exp(nG)}(Q_{\hat{U}}\|Q_{\hat{T}}) \to 0, \quad n \to \infty \quad (211)$$

for any $G > 0$ because $\exp(nG) \to \infty$. For both inputs, the output conditioned on the number of 1’s is the equiprobable distribution on a set of sequences with the same type, so by Proposition 4-5, $E_{\exp(nG)}(Q_{\hat{U}}\|Q_{\hat{X}}) \to 0$ as well. This seems to suggest that the value of $S(G)$ has a jump as $G$ changes from 0 to a positive number (see (209)). So is $S(G)$ discontinuous at $G = 0$ in Example 43?

In the remainder of this section, we answer this question in the negative by developing a general converse bound, implying that the worst-case distribution is not induced by the set of all input sequences of the same type $(1/2, 0, 1/2)$. The basic idea is to use achievability results for the error exponent of identification (ID) channels. The converse bound will not match the achievability bound in Theorem 30, and the exact formula for $S(G)$ seems to be out of reach at this point even for a DMC.

As the first step of the converse, we observe how the achievability of ID coding implies a packing lemma, which is a sharpening of an argument in [1] to the large deviation
analysis. Recall that an \((N,v,\lambda)-\text{ID code}\) [2] consists of distributions \((Q_{U^n_j})_{i=1}^N\) and decoding regions \((D_i)_{i=1}^N\) such that the two types of errors satisfy
\[
\max_{1 \leq i \leq N} Q_{X^n_i}(D^n_j) \leq \mu \\
\max_{i \neq j} Q_{X^n_i}(D^n_j) \leq \lambda
\] (212)
(213)
In the asymptotic setting, the performance of ID code is quantified as follows.

**Definition 44:** The triple \((R, G_1, G_2)\) is said to be achievable if there exists \((N_n, v_n, \lambda_n)-\text{ID code}\) such that
\[
\lim inf_{n \to \infty} \frac{1}{n} \log \log N_n \geq R,
\]
\[
\lim inf_{n \to \infty} \frac{1}{n} \log \frac{1}{1 - \frac{1}{2} \min_{i \neq j} |Q_{X_i^n} - Q_{X_j^n}|} \geq \min(G_1, G_2).
\] (214)
(215)
\[
\lim inf_{n \to \infty} \frac{1}{n} \log \frac{1}{1 - \frac{1}{2} \min_{i \neq j} |Q_{X_i^n} - Q_{X_j^n}|} \geq G_2.
\] (216)

**Lemma 45 (Packing Lemma):** If \((R, G_1, G_2)\) is achievable for ID code, then for each \(n\) there exists \(N_n\) distributions \((Q_{U^n_i})_{i=1}^{N_n}\) such that
\[
\lim inf_{n \to \infty} \frac{1}{n} \log \log N_n \geq R,
\]
\[
\lim inf_{n \to \infty} \frac{1}{n} \log \frac{1}{1 - \frac{1}{2} \min_{i \neq j} |Q_{X_i^n} - Q_{X_j^n}|} \geq \min(G_1, G_2).
\] (217)
(218)

**Proof:** Pick the \((N_n, v_n, \lambda_n)-\text{ID code}\) as in Definition 44. Then for any \(i \neq j\),
\[
\frac{1}{2} |Q_{X_i^n} - Q_{X_j^n}| \geq Q_{X_i^n}(D^n_j) - Q_{X_j^n}(D^n_j) \geq 1 - \mu_n - \lambda_n
\] (219)
\[
\geq 1 - \mu_n - \lambda_n
\] (220)
and the result follows.

For a fixed DMC \(Q_{X|U}\), define the function
\[
e(R) := \max_{Q_{U^n}: I(Q_{U^n}, Q_{X|U}) \geq R} E_{sp}(R, Q_{U^n}, Q_{X|U})
\] (221)
where
\[
E_{sp}(R, Q_{U^n}, Q_{X|U}) := \min_{p_{X|U^n}: I(Q_{U^n}, Q_{X|U}) \leq R} D(p_{X^n|Q_{X|U}} \| Q_{X^n|U} Q_{U^n})
\] (222)
is the well known sphere packing exponent function [18]. Then \(e(R)\) is a nonnegative, non-increasing function of \(R\) with \(e(C(Q_{X|U})) = 0\), so that there is a unique solution \(G^*(R)\) to the equation
\[
e(R + 2G) = G
\] (223)
provided that \(R < C(Q_{X|U})\).

We also need an achievability result of ID coding error exponents. The exponent is known for the first type of error [2], but not for the second type of error.

**Lemma 46** [2]: If \(R, G \geq 0\), and a stationary memoryless channel \(Q_{X|U}\) and a probability measure \(Q_{U}\) satisfy \(I(Q_{U^n}, Q_{X|U}) \geq R + 2G\), the triple
\[
(R, p_{X|U}: I(Q_{U^n}, Q_{X|U}) \leq R + 2G)
\] (224)
is achievable for identification coding.

**Theorem 47:** Let \(Q_{X|U}\) be a stationary memoryless channel whose input alphabet \(U\) is finite. For any \(R < C(Q_{X|U})\) and \(0 < \epsilon < \frac{1}{2}\),
\[
S_\epsilon(G^*(R)) \geq R.
\] (225)

**Remark 48:** Our proof based on identification coding only yields a "strong converse" in the range \(\epsilon \in (0, \frac{1}{2})\), rather than a full strong converse for \(\epsilon \in (0, 1)\).

**Proof:** Let \(G_s := G^*(R)\) and \(Q_{U^n}\) be a distribution achieving the maximum in the definition of \(e(R + 2G_s)\) (see (221)). Then \(I(Q_{U^n}, Q_{X|U}) \geq R + 2G_s\) and
\[
G_s = e(R + 2G_s)
\]
\[
= \min_{p_{X|U^n}: I(Q_{U^n}, Q_{X|U}) \leq R + 2G_s} D(p_{X^n|Q_{X|U}} \| Q_{X^n|U} Q_{U^n})(226)
\]
hold, so that \((R, G_s, G_s)\) is achievable for identification coding by Lemma 46. Therefore by Lemma 45, there exist \((Q_{U^n_i})_{i=1}^{N_n}\) such that (217) holds and
\[
\lim inf_{n \to \infty} \frac{1}{n} \log \frac{1}{1 - \frac{1}{2} \min_{i \neq j} |Q_{X_i^n} - Q_{X_j^n}|} \geq G_s.
\] (217)
\[
\lim sup_{n \to \infty} \frac{1}{n} \log \gamma_n \leq G''
\] (228)
\[
\lim sup_{n \to \infty} \frac{1}{n} \log M_n \leq S(G'');
\] (229)
and for each \(n\) and each \(1 \leq i \leq N_n\),
\[
\frac{1}{2} |\hat{Q}_{X_i^n} - Q_{X_i^n}| \leq \gamma_n
\] (228)
\[
\leq E_{\gamma_n} (\hat{Q}_{X_i^n} \| Q_{X_i^n}) + E_{\gamma_n} (\hat{Q}_{X_i^n} \| Q_{X_i^n}) \leq E_{\gamma_n} (\hat{Q}_{X_i^n} \| Q_{X_i^n}) + \frac{1}{2} |\hat{Q}_{X_i^n} - Q_{X_i^n}|
\] (230)
where (232) and (233) are from (22) and (21), respectively. However the sum of the first two terms in (233) is bounded by \(2\epsilon\) because of (231), and
\[
1 - \frac{1}{2} \min_{i \neq j} |\hat{Q}_{X_i^n} - \hat{Q}_{X_j^n}| \leq \exp(-nG')
\] (234)
for \(n\) large enough. Therefore (233) implies that for large \(n\)
\[
\frac{1}{2} \min_{i \neq j} |\hat{Q}_{X_i^n} - \hat{Q}_{X_j^n}| \geq 1 - \gamma_n \exp(-nG') - 2\epsilon > 0
\] (235)
and so $\tilde{Q}_{X_i} \neq \tilde{Q}_{X_j}$ unless $i = j$. But the number of distinctive $\tilde{M}_n$-type distributions is at most $(|U|^n)^{\tilde{M}_n}$, which should upper-bound $N_n$ for large $n$, hence

$$R \leq \liminf_{n \to \infty} \frac{1}{n} \log \log N_n$$

(236)

$$\leq \limsup_{n \to \infty} \frac{1}{n} (\log \tilde{M}_n + \log |U| + \log n)$$

(237)

$$\leq S_0(G^*).$$

(238)

Since $G^*$ can be arbitrarily close to $G_s(R)$, we conclude that for any $R$ such that $G^*(R) > 0$,

$$R \leq \lim_{g \uparrow G^*(R)} S_0(g).$$

(239)

Finally, we finish the proof using the monotonicity. The $R = 0$ case is trivial so we assume $R > 0$ below. From (239),

$$R = \lim_{r \to R} \lim_{r \uparrow R} S_0(g).$$

(240)

Note that we have shown $e(\cdot)$ is positive in a neighborhood of $R$; this function must be strictly decreasing on this interval because as argued in the proof of [18, Corollary 2.5.4], the sphere packing exponent function $E_{sp}(R, Q_U, Q_{X|U})$ is strictly decreasing in $R$ in any interval where it is finite and positive. Then $G^*(\cdot)$ is also strictly decreasing on this neighborhood of $R$. Thus for any $r < R$ we have $G^*(r) > G^*(R)$ and so

$$\lim_{g \uparrow G^*(r)} S_0(g) \leq S_0(G^*(R))$$

since $S_0(\cdot)$ is non-increasing. Taking $r \uparrow R$ on both sides of (241) and using (240) gives the desired bound (225). □

We do not expect the bound in Theorem 47 to be tight in general. Indeed, in the case of identity channels, we can take the $X^n$ in Corollary 21 to be equiprobable on the set of sequences whose empirical distribution is equiprobable on $U$ if $n$ is a multiple of $|U|$, and let $X^n$ have an arbitrary distribution otherwise. Then Corollary 21 gives

$$S(G) = [\log |U| - G]^{+}, \quad \forall G \in (0, \log |U|)$$

(242)

so that the achievable bound in Theorem 30 is tight. However, the converse bound in Theorem 47 is not tight. To see this, consider an $R \in [0, \log |U|)$ and let $G := G^*(R) > 0$. Note that since $e(\cdot)$ defined in (221) is non-increasing,

$$e(R + 2G) = G \geq e(\log |U|)$$

(243)

implies that

$$R \leq \log |U| - 2G.$$ 

(244)

Therefore the bound $S_0(G) \geq R$ given by Theorem 47 is not tight, in view of (242) and (244).

Theorem 47 indicates that $S(G)$ is continuous at $G = 0$ for any DMC. Moreover, the following observation, which is immediate from the operational definition of identification coding, is sometimes useful for computations:

**Proposition 49:** Fix a DMC $Q_{X|U}$ and let $Q_{X|U'}$ be the DMC obtained by restricting the input to a subset $U' \subseteq U$. Then the outer bound of Theorem 47 for $Q_{X|U}$ is contained in the outer bound for $Q_{X|U'}$.

Now pick $U' = \{0, 1\}$ in Example 43, so that $Q_{X|U'}$ is a binary symmetric channel with crossover probability $\delta$. We have:

**Proposition 50:** The channel in Example 43 satisfies

$$S(d(a^+\|\delta)) \geq R$$

(245)

for any $0 < R < 1 - h(\delta)$, where $d(\cdot\|\cdot)$ is the binary divergence function and $a^+$ is the solution to the following equation in the range $a \in [\delta, 1/2]$.

$$R = 1 - h(a) - d(a\|\delta).$$

(246)

The bound in Proposition 50 and the achievability bound Theorem 30 are illustrated in Figure 4.

**Proof:** Clearly

$$e(R) \geq E_{sp}(R, Q_{X|U}^*, Q_{X|U}') = \min_{a: 1 - h(a) \leq R} d(a\|\delta)$$

(247)

(248)

where $Q_{X|U}^*$ is the equiprobable distribution. Thus the $G$ satisfying

$$\min_{a: 1 - h(a) \leq R + 2G} d(a\|\delta) = G,$$

(249)

denoted as $G_0$, satisfies $G_0 < G^*(R)$. Since $R < 1 - h(\delta)$, it is clear that $G_0 > 0$ and the minimum in (249) is achieved when

$$1 - h(a) = R + 2G.$$ 

(250)

Substituting $G = d(a\|\delta)$ into (250) we obtain the equation (for $a$) in (246). Then $G_0 = d(a^+\|\delta)$, Theorem 47 and Proposition 49 imply (245). □

**V. RESOLVABILITY UNDER OTHER DISTANCE MEASURES**

So far we have seen the tradeoffs between $G$ and $R$ for approximating either a fixed or the worst-case output distribution in $E_{\gamma}$. In this section we argue that most of these tradeoffs are insensitive to the distance metric.
A. Excess Relative Information

First, observe that the bounds relating $E_\gamma$ and the excess relative information (48) immediately imply that our asymptotic results on $E_\gamma$-resolvability (Corollaries 21, 23, 29 and Theorems 30, 33, 47) continue to hold if $E_{\gamma_n}(Q_{X^n}[c_m])\|\pi_{X^n})$ in Definition 14 is replaced by $F_{\gamma_n}(Q_{X^n}[c_m])\|\pi_{X^n}).$

B. Relative Entropy

Next we upper-bound the relative entropy between the output distribution and the target distribution, which essentially relies on the inequality (49).

Theorem 51: Fix $\pi_X$ and $Q_{UX} = Q_{UX|U}$. Let $U^M := (U_1, \ldots, U_M)$ be i.i.d. according to $Q_U$. Define for any $(c_1, \ldots, c_M) \in U^M$,

$$Q_{X[c^M]} := \frac{1}{M} \sum_{m=1}^{M} Q_{X|U=c_m}. \quad (251)$$

Then for any $\tau_0 \geq 0$, $a \in \mathbb{R}$ and $\beta > 1$,

$$\mathbb{E}[D(Q_{X[c^M]}\|\pi_X)] \leq \tau_0 + \mathbb{E} \left[ l_{Q_{X[c^M]}(X|U)}(X|U) - \tau_0 - \log \frac{M}{2} \right]^+$$

$$+ \beta \mathbb{E} \left[ l_{Q_{X[c^M]}(X)}(X) - \tau_0 + a \right]^+ + \frac{2\beta}{\beta - 1} \exp(-a) \quad (252)$$

where $(U, X) \sim Q_{UX|U}$.

Proof: Setting $\epsilon \leftarrow \frac{\tau}{2}$, $\gamma \leftarrow \exp(\tau)$ and $\gamma_2 \leftarrow \exp(\tau_0 + a + \frac{1}{\beta}(\tau - \tau_0))$ in (127), we obtain

$$\mathbb{P} \left[ l_{Q_{X[c^M]}(X)}(\hat{X}) > \tau \right]$$

$$\leq \mathbb{P} \left[ l_{Q_{X[c^M]}(X)}(\hat{X}) > \tau_0 - a + \frac{1}{\beta}(\tau - \tau_0) \right]$$

$$+ \mathbb{P} \left[ l_{Q_{X[c^M]}(X)}(X|U) > \log \frac{M}{2} + \tau \right]$$

$$+ 2 \exp \left( \tau_0 - a - \tau + \frac{1}{\beta}(\tau - \tau_0) \right) \quad (253)$$

where $\hat{X} \sim Q_{X[c^M]}$ conditioned on $U^M = c^M$. Integrating both sides with respect to $\tau$ and using (49), we find

$$\mathbb{E}[D(Q_{X[c^M]}\|\pi_X)] \leq \int_0^\infty \mathbb{P} \left[ l_{Q_{X[c^M]}(X)}(\hat{X}) > \tau \right] d\tau \quad (254)$$

$$\leq \tau_0 + \int_0^\infty \mathbb{P} \left[ l_{Q_{X[c^M]}(X)}(\hat{X}) > \tau \right] d\tau \quad (255)$$

$$\leq \text{R.H.S. of (252)}. \quad (256)$$

Recall that a sequence of nonnegative random variables converging to zero in probability is uniformly integrable if and only if the sequence also converges to zero in expectation.

14For a real-valued random variable $A$, we write $\mathbb{E}[A]^+$ as an abbreviation for $\mathbb{E}[(A)^+]$.

This implies that given a sequence of real-valued random variables $A_n$,

$$\mathbb{E}[A_n]^+ = o(n) \quad (257)$$

provided that $\lim_{n \to \infty} \mathbb{P} \left[ \frac{1}{n} A_n > \epsilon \right] = 0$ for any $\epsilon > 0$ (i.e. the limsup in probability [1] of $\frac{1}{n} A_n$ does not exceed 0) and that $\frac{1}{n} A_n$ is uniformly integrable. Therefore the $\mathbb{E}[\cdot]^+$ terms in (252) can be easily analyzed in the asymptotic setting by setting $A_n$ to be translates of the relative information functions. If we change the definitions of achievable triple/pair by replacing (58) and (59) with

$$\limsup_{n \to \infty} \frac{1}{n} D(Q_{X^n}[c_{m^n}]\|\pi_{X^n}) \leq G \quad (258)$$

then we see the achievability parts of Corollary 21, 23 and Theorem 30, 33 continue to hold. In particular, the relative entropy counterpart of Theorem 33 implies that the bound in [1, Th. 12] is not asymptotically tight. The intuition behind this fact has been explained in Section IV-A, following Corollary 29.

Using the strong converses of $E_\gamma$-resolvability and the upper-bound on relative entropy (50), we immediately obtain converses of resolvability in relative entropy. That is, if the definition of achievable triple/pair is changed by replacing (58) and (59) with (258), the the converse parts of Corollary 21, 23 and Theorem 33 continue to hold.

Unfortunately, we only have a “$\frac{1}{2}$-converse” instead of a strong converse for the worst-case $E_\gamma$-resolvability (see Remark 48). Therefore we don’t have a nice counterpart of Theorem 47: there is a loss of factor 2 when (50) is applied.

C. Smooth Rényi Entropy

Most of the asymptotic resolvability results also hold for smooth Rényi divergences automatically: suppose we change the definitions of achievable triple/pair by replacing (58) and (59) with

$$\limsup_{n \to \infty} \frac{1}{n} D^{\alpha}(Q_{X^n[\epsilon]}\|\pi_{X^n}) \leq G \quad (259)$$

where $\epsilon \in (0, 1)$ and $\alpha$ are fixed. Then the achievability parts of Corollary 21, 23 and Theorem 30, 33 continue to hold for $\alpha \in (0, 1)$, which is immediate from the bound (53) and the achievability results for $E_\gamma$. The converse parts of Corollary 21, 23 and Theorem 33, 47 also continue to hold for $\alpha \in (0, 1)$, because by Proposition 13-2) we only need to consider $\alpha = 0$, then (55) and the $(1 - \epsilon)$-converses for $E_\gamma$ imply the desired result.

If (58) and (59) in the definitions of achievable triple/pair are changed to

$$\limsup_{n \to \infty} \frac{1}{n} D^{\epsilon}(Q_{X^n[\epsilon]}\|\pi_{X^n}) \leq G \quad (260)$$

where $\epsilon \in (0, 1)$ and $\alpha$ are fixed, then the achievability parts of Corollary 21, 23 and Theorem 30, 33 can still be established for $\alpha \in (1, \infty]$ because of the monotonicity (Proposition 13-2)) and the fact that $E_\gamma$ directly corresponds to $D^{\epsilon}$ (Proposition 13-6)). The converse parts of Corollary 21, 23 and Theorem 33 also continue to hold.
for $\alpha \in (1, \infty]$, in view of (54) and the $\epsilon$-converses for $E_\gamma$; Theorem 47 continues to hold for $\epsilon \in (0, \frac{1}{2})$, by the “$\frac{1}{2}$-converse” (Remark 48) for $E_\gamma$.

VI. APPLICATION TO LOSSY SOURCE CODING

The simplest application of the new resolvability result (in particular, the softer-covering lemma) is to derive a one-shot achievability bound for lossy source coding, which is most fitting in the regime of low rate and exponentially decreasing success probability. The method is applicable to general sources. In the special case of i.i.d. sources, it recovers the “success exponent” in lossy source coding originally derived by the method of types [18] for discrete memoryless sources. The achievability bound in [11] can be viewed as the “success exponent” in lossy source coding originally derived by the method of types [18] for discrete memoryless sources. In the special case of i.i.d. sources, it recovers the

\begin{equation}
\lim_{n \to \infty} \frac{1}{n} \log \frac{1}{\mathbb{P}[d^n(U^n, X^n) \leq d]} = G(R, d) \tag{270}
\end{equation}

where

\begin{equation}
G(R, d) := \min_{Q} [D(Q || P) + R(Q, d) - R]^+. \tag{271}
\end{equation}

A weaker achievability result than (271) was proved in [51, p. 168], whereas the final form (271) is given in [18, p. 158, Example 6] based on method of types. Here we can easily prove the achievability part of (271) using Theorem 52 and Corollary 29 by setting $Q_X$ to be the minimizer of (271) and $Q_{UX}$ to be such that

\begin{equation}
\mathbb{E}[d(U, X)] \leq d, \tag{272}
\end{equation}

\begin{equation}
I(Q_U, Q_{UX}) \leq R. \tag{273}
\end{equation}

Then $\gamma_n = \exp(nE)$ with

\begin{equation}
E > D(Q_X || \pi_X) + [I(Q_U, Q_{UX}) - R]^+, \tag{274}
\end{equation}

ensures that

\begin{equation}
\mathbb{P}[d^n(\hat{U}^n, \hat{X}^n) \leq d] \geq \frac{1}{2} \exp(-nE) \tag{275}
\end{equation}

for $n$ large enough, by the law of large numbers.

Remark 53: In the i.i.d. setting, let $R(\pi_X, d)$ be the rate-distortion function when the source has per-letter distribution $\pi_X$. The distortion function for the block is derived from the per-letter distortion by

\begin{equation}
d^n(u^n, x^n) := \frac{1}{n} \sum_{i=1}^{n} d(u_i, x_i). \tag{269}
\end{equation}

Let $(\hat{X}^n, \hat{U}^n)$ be the source-reconstruction pair distributed according to $\pi_{X^n|U^n}$. If $0 \leq R < R(\pi_X, d)$, the maximal probability that the distortion does not exceed $d$ converges to zero with the exponent

\begin{equation}
\lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}[d^n(\hat{U}^n, \hat{X}^n) \leq d] = G(R, d) \tag{270}
\end{equation}

where

\begin{equation}
G(R, d) := \min_{Q} [D(Q || P) + R(Q, d) - R]^+. \tag{271}
\end{equation}

A weaker achievability result than (271) was proved in [51, p. 168], whereas the final form (271) is given in [18, p. 158, Example 6] based on method of types. Here we can easily prove the achievability part of (271) using Theorem 52 and Corollary 29 by setting $Q_X$ to be the minimizer of (271) and $Q_{UX}$ to be such that

\begin{equation}
\mathbb{E}[d(U, X)] \leq d, \tag{272}
\end{equation}

\begin{equation}
I(Q_U, Q_{UX}) \leq R. \tag{273}
\end{equation}

Then $\gamma_n = \exp(nE)$ with

\begin{equation}
E > D(Q_X || \pi_X) + [I(Q_U, Q_{UX}) - R]^+, \tag{274}
\end{equation}

ensures that

\begin{equation}
\mathbb{P}[d^n(\hat{U}^n, \hat{X}^n) \leq d] \geq \frac{1}{2} \exp(-nE) \tag{275}
\end{equation}

for $n$ large enough, by the law of large numbers.

Remark 54: Since the $E_{\gamma}$ metric reduces to total variation distance when $\gamma = 1$, Theorem 52 generalizes the likelihood source encoder based on the standard soft-covering/resolvability lemma [11]. In [11], the error exponent for the likelihood source encoder at rates above the rate-distortion function is analyzed using the exponential decay of total variation distance in the approximation of output statistics, and the exponent does not match the optimal exponent found in [18]. It is also possible to upper-bound the success exponent of the total variation distance-based likelihood encoder at rates below the rate-distortion function by analyzing the exponential convergence to 2 of total variation distance in the approximation of output statistics; however that does not yield the optimal exponent (271) either. This application illustrates one of the nice features of the $E_{\gamma}$-resolvability method: it converts a large deviation analysis into a law of large numbers analysis, that is, we only care about whether $E_{\gamma}$ converges to 0, but not the speed, even when dealing with error exponent problems.
and for any \( u^M \in U^M \), define
\[
A_{u^m} := \bigcup_{m=1}^M \mathcal{F}_{u^m}.
\]

Now fix a \( U \)-codebook \( c^M \) and observe that
\[
\gamma P_V(A_{c^M}) \geq P_V(U^{c^M}(A_{c^M}) - E_\gamma(P_{V[c^M]}\|P_V)) \geq \frac{1}{M} \sum_{m=1}^M P_V(U = c_m) - E_\gamma(P_{V[c^M]}\|P_V)
\]
(280)
where we recall that
\[
P_{V[c^M]} := \frac{1}{M} \sum_{m=1}^M P_V(U = c_m).
\]

(280) is from the definition of \( E_\gamma \) and (281) is because \( \mathcal{F}_{c^M} \subseteq \bigcup_{m=1}^M \mathcal{F}_{c_m} = A_{c^M} \). Denote by \( \Gamma(c_1, \ldots, c_M) \) the right side of (281), which is trivially upper-bounded by 1. Next, we show that
\[
\mathbb{P}\left[ \bigcap_{m=1}^{M.L} \left( \{(U_m, V_l) \notin \mathcal{F}\} \right) \right] \leq 1 - \Gamma(c_{m}) + e^{-\frac{1}{\gamma}}
\]
(283)
which is trivial when \( \Gamma(c_{m}) < 0 \). In the case of \( \Gamma(c_{m}) \in [0, 1] \),
\[
\mathbb{P}\left[ \bigcap_{m=1}^{M.L} \left( \{(U_m, V_l) \notin \mathcal{F}\} \right) \right] \leq \left[ 1 - \Gamma(c_{m}) \right]^{L}
\]
(284)
\leq \left[ 1 - \Gamma(c_{m}) + e^{-\frac{1}{\gamma}} \right]^{L}
(285)
where (284) is from the definition of \( A_{c^M} \), and (285) is from (281). The last step (286) uses the basic inequality
\[
\left( 1 - \frac{p \alpha}{M} \right)^M \leq 1 - p + e^{-a}
\]
(287)
for \( M, \alpha > 0 \) and \( 0 \leq p \leq 1 \), which has been useful in the proofs of the basic covering lemma (see [25, 26, 58]). Integrating both sides of (283) over \( c^M \) with respect to \( P_U \times \cdots \times P_U \),
\[
\mathbb{P}\left[ \bigcap_{m=1}^{M.L} \left( \{(U_m, V_l) \notin \mathcal{F}\} \right) \right] \leq P_{U,V}(\mathcal{F}) + \mathbb{E}[E_\gamma(P_{V[U]}\|P_V)] + e^{-\frac{1}{\gamma}},
\]
(288)
where we have used the fact that \( \mathbb{E}[P_{V[U]}(\mathcal{F})_{U^m} ] = P_{U,V}(\mathcal{F}) \) for each \( m \). Applying the “softer-covering lemma” as in Remark 28, the middle term on the right side of (288) is upper-bounded by
\[
\mathbb{P}\left[ \left( V; U : (V,U) \geq \frac{\log \frac{M}{2}}{2} + \frac{2}{\gamma} \right) \right] \leq 2 \exp(-\tau)
\]
(289)
and the result follows by \( \gamma \leftarrow 2L \exp(-\tau) \). \( \square \)
Remark 56: From the above derivation we see that for the proof of the basic covering lemma \((M = 1)\) case we will need the “softest-covering lemma” (the case of one codeword) rather than the soft-covering lemma (case of \(\gamma = 1\) and \(L > 1\) codewords). However, it is still possible to prove the basic covering lemma using the soft-covering lemma using a different argument; see the discussion in [57], which is essentially based on the idea in [10].

Lemma 57 below is a strengthened version of the one-shot-mutual covering lemma, which improves Lemma 55 in terms of the error exponent. The proof of Lemma 57 essentially combines the proof the achievability part of resolvability and the proof Lemma 55, and the improvement results from not treating the two steps separately. The proof is not as conceptually simple as Lemma 55 since the complexities are no longer buried under the softer-covering lemma.

Lemma 57: Under the same assumptions as Lemma 55,
\[
P \left[ \bigcap_{m=1, l=1}^{M, L} \{ (U_m, V_l) \notin \mathcal{F} \} \right] \leq P \left[ (U, V) \notin \mathcal{F} \text{ or } \exp(u; Y(U; V)) > ML \exp(-\gamma) - \delta \right] + \frac{\min(M, L) - 1}{\delta} + e^{-\exp(\gamma)}. \tag{290} \]

for all \(\delta, \gamma > 0\) and event \(\mathcal{F}\).

Proof: See Lemma 1 and Remark 4 in the conference version [59].

Remark 58: An advantage of Lemma 57 over Lemma 55 is that the upper-bound in the former contains a probability of a union of two events, rather than the sum of the probability of the two events. This yields a strict improvement in the second order rate analysis. Moreover, by setting \(\delta \downarrow 0\) and \(M = 1\) we exactly recover the basic one-shot covering lemma in [26].

In terms of the second order rates, the one-shot Marton’s inner bound for broadcast obtained from our one-shot mutual covering lemma (21, Th. 10) is equivalent to the achievability bound claimed in [60, Th. 4] based on the stochastic likelihood encoder. However, although it is not demonstrated explicitly in [21, Th. 10], we can improve the analysis of [21, Th. 10] by using various nuisance parameters rather than a single \(\gamma\), to obtain a one-shot Marton’s bound which gives strictly better error exponents than [60, Th. 4]. The reason for such improvement is that the third term in (290) is doubly exponential and the second term converges to zero with a large exponent. On the other hand, the approach of [60, Th. 4] has the advantage of being easily extendable to the case of more than two users (which would correspond to a multivariate mutual covering lemma).

VIII. APPLICATION TO WIRETAP CHANNELS

Our final application of the \(E_\gamma\)-resolvability (in particular, the softer-covering lemma) is in the wiretap channel, whose setup is as depicted in Figure 5. The receiver and the eavesdropper observe \(Y \in Y\) and \(Z \in Z\), respectively. Given a codebook \(c^{ML}\), the input to \(P_{YZ|X}\) is \(c_{\text{col}}\) where \(w \in \{1, \ldots, M\}\) is the message to be sent, and \(l\) is equiprobably chosen from \(\{1, \ldots, L\}\) to randomize the eavesdropper’s observation. We call such a \(c^{ML}\) an \((M, L)\)-code. Moreover, the eavesdropper’s observation has the distribution \(\pi_Z\) when no message is sent. In this setup, we don’t need to assume a prior distribution on the message/non-message. We wish to design the codebook such that the receiver can decode the message (reliability) whereas the eavesdropper cannot detect whether a message is sent nor guess which message is sent (security). For general wiretap channels the performance may be enhanced by appending a conditioning channel \(Q_{X|U}\) at the input of the original channel [6]. In that case the same analysis can be carried out for the new wiretap channel \(Q_{YZ|U}\). Thus the model in Figure 5 entails no loss of generality.

In Wyner’s setup (see for example [20]), secrecy is measured in terms of the conditional entropy of the message given the eavesdropper observation. In contrast, we measure secrecy in terms of the size of the list that the eavesdropper has to declare for the message to be included with high probability. Practically, the message \(W\) is the compressed version of the plaintext. Assuming that the attacker knows which compression algorithm is used, the plaintext can be recovered by running each of the items in the eavesdropper output list through the decompressor and selecting the one that is intelligible.

We need the following definitions to quantify the eavesdropper ability to detect/decode messages.

Definition 59: For a fixed codebook and channel \(P_{Z|X}\), we say the eavesdropper can perform \((A, T, \epsilon)\)-decoding if upon observing \(Z\), it outputs a list which is either empty or of size \(T\), such that
- \(\mathbb{P}\{\text{list} \neq \emptyset | \text{no message} \} \leq A^{-1}\).
- There exists \(\epsilon_m \in [0, 1], m = 1, \ldots, M\) satisfying \(\epsilon = \frac{1}{M} \sum_{m=1}^{M} \epsilon_m\) such that
\[
\mathbb{P}\{m \notin \text{list} | W = m\} \leq \epsilon_m. \tag{291} \]

Although the decoder in Definition 59 is reminiscent of erasure and list decoding [61], for the former it is possible that actually no message is sent, and we treat the undetected and detected errors together.

The logarithm of \(T\) can be intuitively understood as the equivocation \(H(W|Z)\) [22]. However, \(\log T\) can be much smaller than \(H(W|Z)\): a distribution can have 99% of its mass supported on a very small set, and yet have an arbitrarily large entropy.

The quantity \(A > 0\) characterizes how well the eavesdropper can detect that no message is sent, which is related the notion of stealth communication [23], [24]. The “non-stealth” is measured by \(D(P_Z \parallel \pi_Z)\) in [23], and is measured by \(|P_Z - \pi_Z|\) in [24]. Although both the relative entropy and total variation are related to error probability hypothesis testing,
their results cannot be directly compared with ours, since they are interested in the regime where non-stealth vanishes while the transmission rate is below the secrecy capacity. In contrast, we are mainly interested in the regime where A grows exponentially (so that the “non-stealth” in their definition grows) in the blocklength, but the transmission rate is above the secrecy capacity.

The asymptotic version of the eavesdropper achievability is as follows.

Definition 60: Fix a sequence of codebooks and a eavesdropper channel \( (P_{Z|X})_{n=1}^{\infty} \). The rate pair \((\alpha, \tau)\) is \(\epsilon\)-achievable by the eavesdropper if there exist sequences \((A_n)\) and \((T_n)\) with
\[
\lim inf \frac{1}{n} \log A_n \geq \alpha \quad (292)
\]
\[
\lim sup \frac{1}{n} \log T_n \leq \tau \quad (293)
\]
such that for sufficiently large \(n\), the eavesdropper can achieve \((A_n, T_n, \epsilon)\)-decoding.

By the diagonalization argument [46, p. 56], the set of \(\epsilon\)-achievable \((\alpha, \tau)\) is closed.

An \((M, L, Q_X)\)-random code is defined as the ensemble of the codebook \(c^{ML}\) where each codeword \(c_{mol}\) is i.i.d. chosen according to \(Q_X\), \(w \in \{1, \ldots, M\}\), \(m \in \{1, \ldots, M\}\). We shall focus on random codes, for which reliability is guaranteed by channel coding theorems, so we only need to consider the security condition.

First, we extend the notions of achievability to the case of a random ensemble of codes by taking the average: we say for a random ensemble of codes the eavesdropper can perform \((A, T, \epsilon)\)-decoding, if there exists \(\epsilon(c^{ML})\) such that for each \(c^{ML}\) the eavesdropper can perform \((A, T, \epsilon(c^{ML}))\)-decoding (in the sense of Definition 59), and the average of \(\epsilon(c^{ML})\) with respect to the codebook distribution is upper-bounded by \(\epsilon\).

Similarly, Definition 60 can be extended to random codes. Then, the following is our main result which characterizes the set of eavesdropper achievable pairs for stationary memoryless channels.

Theorem 61: Fix any \(Q_X\), \(R\), \(R_L\) and \(0 < \epsilon < 1\). Consider \((\exp(nR), \exp(nR_L))\), \(Q_X^{\epsilon}\)-random codes and stationary memoryless channel with per-letter conditional distribution \(Q_{ZX}\). Then the pair \((\alpha, \tau)\) is \(\epsilon\)-achievable by the eavesdropper if and only if
\[
\begin{align*}
\alpha &\leq D(Q_Z||\pi_Z) + I(Q_X, P_{Z|X}) - R - R_L^{+}; \\
\tau &\geq R - I(Q_X, P_{Z|X}) - R_L^{+}.
\end{align*}
\]
(294)

where \(Q_X \rightarrow Q_{ZX} \rightarrow Q_Z\).

From the noisy channel coding theorem, the supremum randomization rate \(R_L\) such that the sender can reliably transmit messages at the rate \(R\) is \(I(Q_X, P_{Y|X}) - R\). The larger \(R_L\) the less reliably the eavesdropper can decode, so the optimal encoder chooses \(R_L\) as close to this supremum as possible. Thus Theorem 61 implies the following result.

Theorem 62: Given a stationary memoryless wiretap channel with per-letter conditional distribution \(P_{YZ|X}\), there exists a sequence of codebooks such that messages at the rate \(R\) can be reliably transmitted to the intended receiver and that \((\alpha, \tau)\) is not \(\epsilon\)-achievable, for any \(\epsilon \in (0, 1)\), by the eavesdropper if there exists some \(Q_X\) such that
\[
R < I(Q_X, P_{Y|X})
\]
(295)

and either
\[
a > D(Q_Z||\pi_Z) + I(Q_X, P_{Z|X}) - I(Q_X, P_{Y|X})^{+}
\]
(296)
or
\[
\tau < R - I(Q_X, P_{Z|X}) - I(Q_X, P_{Y|X}) + R^{+}.
\]
(297)

Remark 63: In general the sender-receiver want to minimize \(\alpha\) and maximize \(\tau\) obeying the tradeoff (296), (297) by selecting \(Q_X\). In the special case where \(\alpha\) has no importance and \(R\) is larger than the secrecy capacity \(C := sup_{Q_X}(I(Q_X, P_{Y|X}) - I(Q_X, P_{Z|X})),\) we see from (297) that the supremum \(\tau\) is \(C\). The formula for the supremum of \(\tau\) is the same as the equivocation measure defined as \(\sup_{W}(H(W|Z^n))\) [22], but technically our result does not follow directly from the lower bound on equivocation, since it may be possible that the a posterior distribution of \(W\) is concentrated on a small list but has a tail spread over an exponentially large set, resulting a large equivocation.

The next two subsections prove the “only if” and “if” parts of Theorem 61, respectively.

A. Converse for the Eavesdropper

The “only if” part (the eavesdropper converse) of Theorem 61 follows by applying the following non-asymptotic bounds to different regions and invoking Corollary 29.

Theorem 64: In the wiretap channel, fix an arbitrary distribution \(\mu_Z\). Suppose the eavesdropper can either detect that no message is sent upon observing \(z \in D_0\) with
\[
\mu_Z(D_0) \geq 1 - A^{-1}
\]
(298)
for some \(A \in [1, \infty)\), or outputs a list of \(T(z)\) messages upon observing \(z \notin D_0\) that contains the actual message \(m \in \{1, \ldots, M\}\) with probability at least \(1 - \epsilon_m\), for some \(\epsilon_m \in [0, 1]\). Define the average quantities
\[
T := \frac{1}{\mu_Z(D_0)} \int_{D_0} T(z) \mu_Z(z),
\]
(299)
\[
\epsilon := \frac{1}{M} \sum_{m=1}^{M} \epsilon_m.
\]
(300)

Then for any \(\gamma \in [1, +\infty)\),
\[
\frac{1}{\gamma} \geq \frac{1}{A} (1 - \epsilon - E_{\gamma}(P_Z||\pi_Z)),
\]
(301)
where we recall that \(\pi_Z\) is the non-message distribution, \(P_Z := \sum_{m=1}^{M} P_{Z|W=m}\) and \(P_{Z|W=m}\) is the distribution of the eavesdropper observation for the message \(m\) (assuming an arbitrary codebook is used). Moreover,
\[
\frac{T}{MA} \geq \frac{1}{\gamma} (1 - \epsilon - \frac{1}{M} \sum_{m=1}^{M} E_{\gamma}(P_{Z|W=m}||\mu_Z)).
\]
(302)

We will choose \(\mu_Z = P_Z\) when we use Theorem 64 to prove Theorem 61, although (302) holds for any \(\mu_Z\).
From the eavesdropper viewpoint, a larger $A$ and a smaller $T$ is more desirable since it will then be able to find out that no message is sent with smaller error probability or narrow down to a smaller list when a message is sent. This observation agrees with (301) and (302): a smaller $\gamma$ implies a higher degree of approximation, and hence higher indistinguishability of output distributions which is to the eavesdropper disadvantage.

Proof: To see (301),

$$\frac{1}{A} \geq \pi_Z(D_0^0)$$

(303)

$$\geq \frac{1}{\gamma} (P_Z(D_0^0) - E_\gamma (P_Z \| \pi_Z))$$

(304)

$$= \frac{1}{\gamma} \left( \frac{1}{M} \sum_{m=1}^{M} P_{Z|W=m}(D_0^0) - E_\gamma (P_Z \| \pi_Z) \right)$$

(305)

$$\geq \frac{1}{\gamma} \left( 1 - \epsilon - E_\gamma (P_Z \| \pi_Z) \right).$$

(306)

To see (302), let $D_m$ be the set of outputs $z \in Z$ for which the eavesdropper list contains $m \in \{1, \ldots, M\}$. Then

$$\frac{T}{MA} \geq \frac{T}{M} \mu_Z(D_0^0)$$

(307)

$$= \frac{1}{M} \int_{D_0^0} T(z) d\mu_Z(z)$$

(308)

$$= \frac{1}{M} \sum_{m=1}^{M} I(z \in D_m) d\mu_Z(z)$$

(309)

$$= \frac{1}{M} \sum_{m=1}^{M} \int I(z \in D_m) d\mu_Z(z)$$

(310)

$$= \frac{1}{M} \sum_{m=1}^{M} \mu_Z(D_m)$$

(311)

$$\geq \frac{1}{M} \sum_{m=1}^{M} \left( P_{Z|W=m}(D_m) - E_\gamma (P_{Z|W=m} \| \mu_Z) \right)$$

(312)

$$\geq \frac{1}{\gamma} \left( 1 - \epsilon - \frac{1}{M} \sum_{m=1}^{M} E_\gamma (P_{Z|W=m} \| \mu_Z) \right).$$

(313)

Next, we particularize Theorem 64 to the asymptotic setting.

Proof of “Only If” in Theorem 61:

- Fix an arbitrary

$$\alpha > D(Q_Z \| \pi_Z) + [I(Q_X, P_{Z|X}) - R - R_L]^+.$$  

(314)

We will show that $(\alpha, \tau)$ is not $\epsilon$-achievable by the eavesdropper for any $\tau > 0$ and $\epsilon \in (0, 1)$. Pick $\sigma > 0$ such that

$$\alpha > D(Q_Z \| \pi_Z) + [I(Q_X, P_{Z|X}) - R - R_L]^+ + 2\sigma.$$  

(315)

and define

$$\begin{cases}
A_n = \exp(n(\alpha - \sigma)) \\
T_n = \exp(n(\tau + \sigma)).
\end{cases}$$

(316)

Assuming the eavesdropper can perform $(A_n, T_n, \epsilon(c^{ML}))$-decoding for a particular realization of the codebook $c^{ML}$, then applying Theorem 64 with

$$\gamma_n = \exp(n(D(Q_Z \| \pi_Z) + [I(Q_X, P_{Z|X}) - R - R_L]^+ + \sigma)),$$  

(317)

we obtain

$$\exp(n(D(Q_Z \| \pi_Z) + [I(Q_X, P_{Z|X}) - R - R_L]^+ - \alpha + 2\sigma))$$  

(318)

$$\geq \frac{\gamma_n}{A_n}$$

$$\geq 1 - \epsilon(c^{ML}) - E_{\gamma_n}(P_{Z^n(c_m^{\epsilon,c^{ML})}} \| \pi_Z^n).$$

(319)

From (315), the above implies

$$E_{\gamma_n}(P_{Z^n(c_m^{\epsilon,c^{ML})}} \| \pi_Z^n) \geq 1 - \epsilon(c^{ML}) \frac{2}{2}.$$  

(320)

For sufficiently large $n$. By Corollary 29 and (317), the average of the left side converges to zero as $n \to \infty$, thus the average of the right side cannot be lower bounded by $\frac{1}{\epsilon c^{ML}}$.

- Fix an arbitrary

$$\tau < R - [I(Q_X, P_{Z|X}) - R_L]^+.$$  

(321)

We will show that $(\alpha, \tau)$ is not $\epsilon$-achievable by the eavesdropper for any $\tau > 0$ and $\epsilon \in (0, 1)$. Pick $\sigma > 0$ such that

$$\tau + 2\sigma < R - [I(Q_X, P_{Z|X}) - R_L]^+.$$  

(322)

and again define $A_n$ and $T_n$ as in (316). Assuming the eavesdropper can perform $(A_n, T_n, \epsilon(c^{ML}))$-decoding for a particular realization of the codebook $c^{ML}$, then applying Theorem 64 with

$$\mu_Z = Q_Z^n,$$  

(323)

$$\gamma_n = \exp(n([I(Q_X, P_{Z|X}) - R_L]^+ + \sigma)),$$  

(324)

and noting that $A_n \geq 1$, we obtain

$$\exp(n(\tau - R + [I(Q_X, P_{Z|X}) - R_L]^+ + 2\sigma))$$  

(325)

$$= \frac{T_n \gamma_n}{M_n}$$

$$\geq 1 - \epsilon(c^{ML}) - \frac{1}{M} \sum_{m=1}^{M} E_{\gamma_n}(P_{Z^n|W=m} \| Q_Z^n)$$  

(326)

$$= 1 - \epsilon(c^{ML}) - \frac{1}{M} \sum_{m=1}^{M} E_{\gamma_n}(P_{Z^n(c_m^{\epsilon,c^{ML})}} \| Q_Z^n)$$  

(327)

where $c_m^{L} := (c_m^{\epsilon})_L^{1+L}$. From (322), the above implies

$$\frac{1}{M} \sum_{m=1}^{M} E_{\gamma_n}(P_{Z^n|W=m} \| Q_Z^n) \geq 1 - \epsilon(c^{ML}) \frac{2}{2}.$$  

(328)

for sufficiently large $n$. Invoking Corollary 29, we see the average of the right side with respect to the codebook converges zero as $n \to \infty$, and in particular cannot be lower-bounded by $\frac{1}{\epsilon c^{ML}}$.

□
B. Ensemble Tightness

The (eavesdropper) achievability part of Theorem 61 follows by analyzing the eavesdropper list decoding ability for different cases of the rates \((R, R_L)\). First, consider the following one-shot achievability bounds for channel coding with possibly no message sent:

**Theorem 65:** Consider a random transformation \(P_{Z|X} \) and a \((M, L, Q_X)-\)random code. Let \(Q_X \rightarrow F_{Y|X} \rightarrow Q_Y\), and let \(\pi_Z\) be the distribution of the eavesdropper observation when no message is sent. Define

\[
i_{Z,X}(z; x) := \log \frac{dP_{Z|X} = x}{d\pi_Z}(z); \tag{329}
\]

\[
i_{Z,X}(z; x) := \log \frac{dP_{Z|X} = x}{dQ_Z}(z). \tag{330}
\]

Let \(\delta, \beta, A, T > 0\). Then, there exist three list decoders such that for Decoder 1,

\[
\mathbb{E}_C \mathbb{P}[^{\text{error}} \text{no message}] \leq \frac{1}{A} \exp(-\delta), \tag{331}
\]

\[
\mathbb{E}_C \mathbb{P}[^{\text{error}} \text{message is } m] \leq \mathbb{P}[i_{Z,X}(Z; X) \leq \log(LMA)+\delta] \tag{332}
\]

\[
+ \mathbb{P}[i_{Z,X}(Z; X) \leq \log \frac{LM}{T}+\delta] \tag{333}
\]

\[
+ \frac{1}{1+\beta} + e^{-(\beta+1)} + \beta \exp(-\delta). \tag{334}
\]

Here an error in the case of no message means that a non-empty list is produced. An error in the case of message \(m\) means either the list does not contain \(m\), or the list size exceeds \(T\). For Decoder 2,

\[
\mathbb{E}_C \mathbb{P}[^{\text{error}} \text{no message}] \leq \frac{1}{A}; \tag{335}
\]

\[
\mathbb{E}_C \mathbb{P}[^{\text{error}} \text{message is } m] \leq \mathbb{P}[i_{Z,X}(Z; X) \leq \log \frac{LM}{T}+\delta] \tag{336}
\]

\[
+ \mathbb{P}[i_{Q_Z,i_{Z}}(Z) \leq \log A] \tag{337}
\]

\[
+ \frac{1}{1+\beta} + e^{-(\beta+1)} + \beta \exp(-\delta). \tag{338}
\]

where the error events are defined similarly to Decoder 1. Decoder 3 either output an empty list or a list of all messages, and

\[
\mathbb{E}_C \mathbb{P}[^{\text{error}} \text{no message}] \leq \frac{1}{A}; \tag{339}
\]

\[
\mathbb{E}_C \mathbb{P}[^{\text{error}} \text{message is } m] \leq \mathbb{P}[i_{Q_Z,i_{Z}}(Z) \leq \log A]. \tag{340}
\]

**Proof of Theorem 65:** See Appendix C.

Under various conditions, one out of the three decoders are asymptotically optimal. By choosing appropriate parameters \(\delta, \beta, A, T > 0\), it is clear that Theorem 65 implies the following:

\[\text{Corollary 66: Fix any } Q_X, R, R_L \text{ and } 0 < \epsilon < 1. \text{ Consider } (\exp(nR), \exp(nR_L), Q_{X^n})-\text{random codes and stationary memoryless channel with per-letter conditional distribution } Q_{Z|X}. \]

\[-\text{achievable by a Decoder 1}\] if

\[
\begin{cases}
D(Q_Z \parallel \pi_Z) + I(Q_X, P_{Z|X}) > R_L + R + \alpha; \\
I(Q_X, P_{Z|X}) > R + R_L - \tau.
\end{cases}
\]

\[-\text{achievable by a Decoder 2} \text{ if}\]

\[
\begin{cases}
I(Q_X, P_{Z|X}) > R_L + R - \tau; \\
\alpha < D(Q_Z \parallel \pi_Z).
\end{cases}
\]

\[-\text{achievable by a Decoder 3} \text{ if}\]

\[
\begin{cases}
\tau \geq \epsilon; \\
\alpha < D(Q_Z \parallel \pi_Z).
\end{cases}
\]

**Proof Sketch:** Consider the first case. To see the achievability of \((\alpha, \tau)\) satisfying (337), choose

\[
\delta_n := n^{-0.9}, \tag{341}
\]

\[
\beta_n := n, \tag{342}
\]

\[
A_n := \exp(n\alpha), \tag{343}
\]

Then the right sides of (331) and (332) converges to zero as \(n \to \infty\). The analyses of the other two cases are similar using the same choice of the parameters as above.

The eavesdropper's achievability ("if " part) of Theorem 61 then follows from Corollary 66 and an application of the standard diagonalization argument to show that the achievable region is closed (see [46]).

**IX. CONCLUSION AND FUTURE WORK**

This paper develops general bounds among various distance metrics, and shows that, in the memoryless case, the exponential growth of \(\gamma\) for the \(E_\gamma\) between the synthesized and target distributions to vanish is the same as the linear growth of the relative entropy or smooth Rényi \(\alpha\)-divergence (of order \(\alpha \neq 1\)). This implies that in the context of relative entropy gauged approximation the bound in [1, Th. 12] is not asymptotically tight. An intuitive explanation for this is that [1, Th. 12] uses random codebooks where the distribution of each codeword induces the target output distribution through the stationary memoryless channel. However, this is not necessary, and in fact the optimal choice of the codeword distribution generally does not induce the target distribution. This is in stark contrast to the conventional resolvability under total variation distance, where the codeword distribution must induce the target distribution to cause total variation between the output distribution and the target distribution to vanish.

We have seen three examples of the application of \(E_\gamma\)-resolvability in information theory. The essence is a change-of-measure: if \(E_\gamma(P \parallel Q)\) is small and \(P(A)\) is large,
then \( Q(A) \) is essentially lower bounded by \( \frac{1}{\gamma} \). There are two motivations for such a change-of-measure:

1) change one distribution to another distribution which is simpler to analyze. Consider the source coding example; in (265) we changed the real joint distribution of the source and the reproduction to \( P_{X|U} \), which is simpler to analyze since its average is the given joint distribution \( Q_{X|U} \). The application to one-shot mutual covering lemma (step (280)) also falls into this category.

2) change one distribution from a family to a common distribution. In the wiretap channel example, step (312) upper-bounds the conditional probability \( P_{Z|W=x}(D_m) \) for each message using the probability under a fixed distribution \( \mu_Z(D_m) \). The sum of \( \mu_Z(D_m) \) over \( m \) would be \( T \) since (on average) the space \( Z \) is covered by \( (D_m) \) for \( T \) times.

Other distance measures have been useful for change-of-measure in information theory. For example, \( \beta_n(P\|Q) \) is used in the converse proof for the error exponent of lossy compression [18]. The relative entropy can also play a similar role (e.g. proof of sphere packing bound in [18, Th. 2.5.3]): the Log-Sum Inequality implies that

\[
Q(A) \geq \exp \left( -\frac{D(P\|Q) + h(Q(A))}{P(A)} \right) \tag{344}
\]

where \( h(\cdot) \) is the binary entropy function. Thus if \( P(A) \) is close to 1 and \( D(P\|Q) \gg 1 \), then \( Q(A) \) is essentially lower bounded by \( \exp(-D(P\|Q)) \). Nevertheless, the \( E_\gamma \) metric is sometimes more desirable because of its nice properties and relations to other metrics (see Proposition 4 and 13). For example, \( \beta_n \) and \( D(\cdot\|\cdot) \) do not seem to share analogues of the triangle inequality (Proposition 4-3) of \( E_\gamma \).

In Section IV-D the achievability of identification coding is used to derive a converse for worst-case resolvability. The contrapositive of this argument is that achievability of worst-case resolvability would imply a converse for identification coding. Indeed, [1], [62] have initiated this approach to prove the strong converse of identification where resolvability in total variation distance is used; with achievability of resolvability in \( E_\gamma \) (e.g. Theorem 30) the minimum of the two exponents for ID coding over a general sequence of channels can be upper-bounded, in contrast to the approach of [2] specific to DMC. But for DMC, our preliminary study indicates that the resulting bound is not as tight.

Another potential application of the \( E_\gamma \)-based analysis, suggested by Yury Polyanskiy [36], is the study of the sphere-packing exponent in channel coding. In [63], the achievability of channel synthesis (under total variation distance) is used to prove the strong converse of channel coding. This combined with a standard change-of-measure argument (see [18, Th. 2.5.3]) yields the conventional sphere-packing bound. In [36] Polyanskiy pointed out that the sphere-packing bound might be improved if one could prove that channel synthesis under the more forgiving \( E_\gamma \) metric requires smaller communication rates. Since resolvability under total variation distance has been used to prove channel synthesis under total variation distance [10], it is natural to ask whether \( E_\gamma \)-resolvability can lead to an \( E_\gamma \)-channel synthesis result.

Unfortunately, it appears not to be the case, because of the asymmetry of \( E_\gamma(P\|Q) \) in \( P \) and \( Q \) (for \( \gamma > 1 \)).

**Appendix A**

**Proof of Proposition 2**

We first show that it suffices to consider the case of \( |X| \leq 3 \). Put

\[
B_t := \left\{ x : \frac{dP}{dQ}(x) > t \right\}, \quad \forall t \geq 0. \tag{345}
\]

We partition the whole space as

\[
X = A_1 \cup A_2 \cup A_3 \cup A_4 \tag{346}
\]

with

\[
A_1 = B_1 \cap B_2; \quad A_2 = B_1^c \cap B_2; \quad A_3 = B_1 \cap B_3^c; \quad A_4 = B_1^c \cap B_3^c. \tag{347-350}
\]

Let \( Z = \{1, 2, 3, 4\} \) and \( z(x) \) be a function of \( x \) indicating which of the above nonempty sets \( x \) belongs to. Next observe that

\[
P_Z(1) > \lambda Q_Z(1), \quad P_Z(1) > Q_Z(1) \quad \text{if} \quad Q_Z(1) > 0; \tag{351}
\]

\[
P_Z(2) > \lambda Q_Z(2), \quad P_Z(2) \leq Q_Z(2) \quad \text{if} \quad \lambda > Q_Z(2); \tag{352}
\]

\[
P_Z(3) \leq \lambda Q_Z(3), \quad P_Z(3) > Q_Z(3) \quad \text{if} \quad Q_Z(3) > 0; \tag{353}
\]

\[
P_Z(4) \leq \lambda Q_Z(4), \quad P_Z(4) \leq Q_Z(4). \tag{354}
\]

For example, to see (352), consider

\[
P_Z(2) = \int 1[x \in A_2]dP_X(x) \leq \int 1[x \in A_2]dQ_X(x) = Q_Z(2). \tag{355-357}
\]

This establishes the second inequality in (352). The first inequality in (352) takes a little more effort since it is not strict. Define

\[
A_{2,t} := \left\{ x : \frac{dP}{dQ}(x) > t \quad \text{and} \quad \frac{dP}{dQ}(x) \leq \lambda t \right\}, \quad \forall t > 0. \tag{358}
\]

Then clearly, \( A_2 = \bigcup_{t > \lambda} A_{2,t} \). By continuity of measure, there exists \( t > \lambda \) such that

\[
Q(A_{2,t}) > \frac{1}{2} Q(A_2), \tag{359}
\]

so that

\[
P_Z(2) = \int 1[x \in A_2]dP(x) = \int 1[x \in A_{2,t}]dP(x) + \int 1[x \in A_2 \setminus A_{2,t}]dP(x) \geq \int 1[x \in A_{2,t}]dQ(x) + \int 1[x \in A_2 \setminus A_{2,t}]\lambda dQ(x) \geq \int 1[x \in A_{2,t}](t - \lambda)dQ(x) + \int 1[x \in A_2]\lambda dQ(x) > \frac{t - \lambda}{2} Q(A_2) + \lambda Q_Z(2). \tag{362-364}
\]
Thus (352) is established, and (351), (353), (354) can be proved similarly (Note that there is no condition in (354) because neither of the inequalities in (354) is strict). (351)-(354) imply that

\[ |P - Q| = |P_Z - Q_Z|; \]

\[ \mathbb{P} \left[ \frac{dP}{dQ} (X) > \lambda \right] = \mathbb{P} \left[ \frac{dP}{dQ} (Z) > \lambda \right]; \]

(365)

(366)

where \( Z = z(X) \). However depending on the value of \( \lambda \), one of the four sets defined in (347)-(350) is empty. Therefore we have shown that it suffices to consider the case of \( |X| \leq 3 \), so that verifying (13) and (14) becomes elementary. We begin with the proof of the upper bound in (13):

1) \( \lambda \geq \frac{1}{1 - \delta} \). The upper bound (13) follows from [27, Th. 9].

To verify its tightness, consider \( |X| = 2 \), and

\[ P := \left[ 1 - \frac{\lambda + \delta}{\lambda^+ - 1}, \frac{\lambda + \delta}{\lambda^+ - 1} \right]; \]

\[ Q := \left[ 1 - \frac{\delta}{\lambda^+ - 1}, \frac{\delta}{\lambda^+ - 1} \right]; \]

(367)

(368)

where \( \lambda^+ > \lambda \) ensures that \( P \) and \( Q \) are distributions, and \( \frac{\lambda + \delta}{\lambda^+ - 1} \) can be made arbitrarily close to \( \frac{\delta}{\lambda^+ - 1} \) as \( \lambda \rightarrow \lambda^+ \).

2) \( \lambda < \frac{1}{1 - \delta} \). The upper bound is trivial. Its tightness can be seen by choosing

\[ P := [0, 1]; \]

\[ Q := [\delta, 1 - \delta]. \]

(369)

(370)

The lower bound (14) is proved as follows:

1) \( \lambda > \frac{1}{1 - \delta} \).

The lower bound is trivial. Its tightness can be seen by choosing

\[ P := \left[ 1 - \frac{\lambda \delta}{\lambda - 1}, \frac{\lambda \delta}{\lambda - 1} \right]; \]

\[ Q := \left[ 1 - \frac{\delta}{\lambda - 1}, \frac{\delta}{\lambda - 1} \right]; \]

(371)

(372)

2) \( 1 < \lambda \leq \frac{1}{1 - \delta} \). Without loss of generality, we may assume that \( X = \{1, 2, 3\} \) and

\[ \frac{P(1)}{Q(1)} \leq 1; \]

\[ 1 < \frac{P(2)}{Q(2)} \leq \lambda; \]

\[ \lambda < \frac{P(3)}{Q(3)}; \]

(373)

(374)

(375)

Then the lower bound (14) follows from

\[ P(3) = 1 - P(1) - P(2); \]

\[ \geq 1 - P(1) - \lambda Q(2); \]

\[ = 1 - P(1) - \lambda [1 - Q(1) - Q(3)]; \]

\[ = 1 - P(1) - \lambda [1 - (P(1) + \delta) - Q(3)]; \]

\[ = 1 + (\lambda - 1)P(1) + \lambda Q(3) - \lambda + \lambda \delta \]

\[ \geq 1 - \lambda + \lambda \delta \]

(376)

(377)

(378)

(379)

(380)

(381)

and its tightness can be verified by considering the following distribution as \( \epsilon \rightarrow 0 \):

\[ P := [0, (1 - \delta - \epsilon)\lambda, 1 - (1 - \delta - \epsilon)\lambda]; \]

\[ Q := [\delta, 1 - \delta, \epsilon]. \]

(382)

(383)

3) \( 1 - \delta < \lambda \leq 1 \). The lower bound (14) follows from

\[ \mathbb{P} \left[ \frac{dP}{dQ} (X) > \lambda \right] \geq \mathbb{P} \left[ \frac{dP}{dQ} (X) > 1 \right] - \mathbb{P} \left[ \frac{dP}{dQ} (X) > 1 \right] \]

\[ = \delta \]

(384)

(385)

where \( \tilde{X} \sim Q \). The tightness can be seen by considering

\[ P := [1 - \delta, \delta]; \]

\[ Q := [1, 0]. \]

(386)

(387)

4) \( \lambda \leq 1 - \delta \).

Without loss of generality, assume that \( X = \{1, 2, 3\} \) and

\[ \frac{P(1)}{Q(1)} \leq \lambda; \]

\[ \frac{P(2)}{Q(2)} \leq 1; \]

\[ \frac{P(2)}{Q(2)} > 1. \]

(388)

(389)

(390)

Then observe that

\[ 1 - P(1) - Q(3) - \delta = 1 - P(1) - P(3) \]

\[ = P(2); \]

\[ \leq Q(2); \]

\[ = 1 - Q(1) - Q(3); \]

\[ \leq 1 - \frac{1}{\lambda} P(1) - Q(3); \]

(391)

(392)

(393)

(394)

(395)

which upon rearrangements gives

\[ P(1) \geq \frac{\lambda \delta}{1 - \lambda}; \]

(396)

implying the lower bound (14). To see its tightness, consider

\[ P := \left[ \frac{\lambda \delta}{1 - \lambda}, 1 - \frac{\delta}{1 - \lambda}, \delta \right]; \]

\[ Q := \left[ \frac{\delta}{1 - \lambda}, 1 - \frac{\delta}{1 - \lambda}, 0 \right]. \]

(397)

(398)

APPENDIX B

PROOF OF PROPOSITION 13

1) The first inequality in (48) is evident from the definition. For the second inequality in (48), consider the event

\[ A := \{ x \in X : t_{\|Q(x)} > \log \gamma \}. \]

(399)
4) By considering the \( \frac{d\mu}{dQ} > \gamma \) and \( \frac{d\mu}{dQ} \leq \gamma \) cases separately, we can check the following (homogeneous) inequalities: for each \( \alpha < 1 \),
\[
\left| \frac{d\mu}{dQ} - \gamma \right| \geq \frac{d\mu}{dQ} - 2\gamma^{1-\alpha} \left( \frac{d\mu}{dQ} \right)^\alpha + \gamma,
\]
and for each \( \alpha > 1 \),
\[
\left| \frac{d\mu}{dQ} - \gamma \right| \leq -\frac{d\mu}{dQ} + 2\gamma^{1-\alpha} \left( \frac{d\mu}{dQ} \right)^\alpha + \gamma.
\]
Integrating with respect to \( dQ \) both sides of (405), we obtain
\[
|\mu - \gamma Q| \geq \mu(X) - 2\gamma^{1-\alpha} \int \left( \frac{d\mu}{dQ} \right)^\alpha dQ + \gamma
\]
and (51) follows by rearrangement. Integrating with respect to \( dQ \) on both sides of (406), we obtain
\[
|\mu - \gamma Q| \leq -\mu(X) + 2\gamma^{1-\alpha} \int \left( \frac{d\mu}{dQ} \right)^\alpha dQ + \gamma
\]
and (52) follows by rearrangement.

5) Immediate from the previous result, the definition of smooth Rényi divergence, and the triangle inequality (21).

6) \( \Rightarrow \): By assumption there exists a nonnegative finite measure \( \mu \) such that \( E_1(P\|\mu) \leq \epsilon \) and \( \mu \leq \gamma Q \). Then from Proposition 6,
\[
E_\gamma(P\|Q) \leq E_1(P\|\mu) + E_\gamma(\mu\|Q)
\]
\[
\leq \epsilon + 0.
\]
APPENDIX C

PROOF OF THEOREM 65

• Codebook generation: \((c_{ij})_{1 \leq i \leq M, 1 \leq j \leq L}\) according to \(Q_X^{\otimes M L}\).

• Decoders: Fix an arbitrary constant \(\delta > 0\). Upon observing \(z\), Decoder 1 outputs as a list all \(1 \leq i \leq M\) such that there exists \(1 \leq j \leq L\) satisfying

\[
\begin{align*}
&\left\{ i \in \mathbb{Z}; X(z; c_{ij}) > \log(L M A) + \delta \right\} \\
&\left\{ i \in \mathbb{Z}; X(z; c_{ij}) > \log(L M A) + \delta \right\}
\end{align*}
\]  

(423)

if there is at least one such an \(i\), or declares that no message is sent (i.e. outputs an empty list) if otherwise. Decoder 2 outputs as a list all \(1 \leq i \leq M\) such that there exists \(1 \leq j \leq L\) satisfying

\[
i_{Z,X}(z; c_{ij}) > \log\left(\frac{LM}{T}\right) + \delta
\]

(424)

if there exists at least one such \(i\) and in addition,

\[
i_{QZ|Z}(z) > \log A,
\]

(425)

or declares that no message is sent if otherwise. Decoder 3 outputs \(\{1, \ldots, M\}\) as the list if (425) holds (so that the list size equals \(M\)), or otherwise declares no message.

• Error analysis: we denote by \(\mathcal{L}\) the list of messages recovered by the eavesdropper.

Decoder 1:

\[
\mathbb{P}[\mathcal{L} \neq \emptyset | \text{no message}] 
\leq \mathbb{E}\left[\max_{1 \leq m \leq M, 1 \leq l \leq L} i_{Z,X}(\hat{Z}; X_{ml}) > \log(L M A) + \delta\right]
\]

(426)

\[
\leq \frac{1}{A} \exp(-\delta)
\]

(427)

where the probability is averaged over the codebook, \((X^{ML}, \hat{Z}) \sim Q_X^{\otimes ML} \times \pi_Z\), and (427) used the packing lemma [26]. Moreover

\[
\mathbb{P}[1 \notin \mathcal{L} \text{ or } \mathcal{L} = \emptyset | W = 1] 
\leq \mathbb{P}[i_{Z,X}(Z; X) \leq \log(L M A) + \delta] \\
+ \mathbb{P}\left[ i_{Z,X}(Z; X) \leq \log\left(\frac{LM}{T}\right) + \delta \right]
\]

(429)

where \(W\) denotes the message sent, and \((X, Z) \sim Q_{XZ}\) Further,

\[
\mathbb{P}[|\mathcal{L}| \geq T + 1 | W = 1] 
\leq \mathbb{P}\left[ |\mathcal{L}| \geq T + 1, \mathcal{L} \cap \left\{ 2, \ldots, \frac{BM}{T} + 1 \right\} = \emptyset \right] \\
+ \mathbb{P}\left[ \mathcal{L} \cap \left\{ 2, \ldots, \frac{BM}{T} + 1 \right\} = \emptyset \right] \\
\leq \left(1 - \frac{BM}{T}\right)^T
\]

(430)

and we used the change of measure. On the other hand,

\[
\mathbb{P}[1 \notin \mathcal{L} \text{ or } 1 \notin \mathcal{L} | W = 1] 
\leq \mathbb{P}[i_{QZ|Z}(\hat{Z}) > \log A]
\]

(439)

\[
\leq \frac{1}{A} \mathbb{P}[i_{QZ|Z}(Z) > \log A]
\]

(440)

where \(\hat{Z} \sim \pi_Z\) and \(Z \sim Q_Z\), and (440) used the change of measure. On the other hand,

\[
\mathbb{P}[1 \notin \mathcal{L} | W = 1] 
\leq \mathbb{P}[i_{QZ|Z}(Z) \leq \log\left(\frac{LM}{T}\right) + \delta]
\]

(442)

and

\[
\mathbb{P}[\mathcal{L} = \emptyset | W = 1] \leq \mathbb{P}[i_{QZ|Z}(Z) \leq \log A],
\]

(443)

Moreover, as in (433), we have

\[
\mathbb{P}[|\mathcal{L}| \geq T + 1 | W = 1] 
\leq \frac{1}{1 + \beta} + e^{-(\beta + 1)} + \beta \exp(-\delta)
\]

(444)
By union bound,
\[
P[\text{error} | \text{no message}] \leq \frac{1}{A}; \tag{445}
\]
and for each \( m = 1, \ldots, M \),
\[
P[\text{error} | W = m] \leq \mathbb{P}
\left[
\begin{aligned}
&\left| t_{Z,Y}(Z; X) \leq \log \frac{LM}{T} + \delta \right|
+ \mathbb{P}
\left[
\begin{aligned}
&\left| Q_{Z,Y}(Z) \leq \log L \right|
+ \frac{1}{1 + \beta} + e^{-(\delta+1)} + \beta \exp(-\delta).
\end{aligned}
\right]
\end{aligned}
\right]. \tag{446}
\]
- Decoder 3: The algorithm is similar to that of Decoder 2 and the result follows from (440) and (443).

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Our initial focus was on the alternative formulation of Wyner’s wiretap setup without using average information measures (Section VIII) as well as the excess information metric for channel resolvability, as in Theorem 26. We gratefully acknowledge Yury Polyanskiy for bringing the \( E_Y \) metric to our attention and sharing [36] (see Section IX).

**References**


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### References


