Abstract: The detection of a deterministic signal in additive noise is often accomplished by the use of linear matched filters, for that particular pair of signal and noise. In the case of minimal distortion or other types of modeling uncertainties, it is possible to have exact knowledge of the input signal and/or noise characteristics available. In this case it is of interest to design a robust matched filter: i.e., a noncausal filter with a guaranteed level of worst performance for the expected class of deviations from the nominal model. The problem is known to be equivalent to the search for the least favorable pair of signal and noise. i.e., the one whose matched filter gives the lowest signal-to-noise ratio. In this paper we investigate this problem in the case of discrete-time processing. We consider various types of uncertainty classes of signal and noise that arise naturally in practical cases and results in closed-form solutions. Moreover, conditions are given for the near-minimality of white noise and the robustness of the nominal filter.

1. Introduction

The linear system that maximizes the output signal-to-noise ratio at some instant of time when the input is a deterministic signal embedded in additive random noise is known as the matched filter for this pair of signal and noise. If the noise is a Gaussian process, then the output of the filter at the instant in which the signal-to-noise ratio is maximized provides a sufficient statistic for any likelihood-ratio detection of the input signal. Since the power of the noise at the output of the linear filter depends on the second-order statistics of the input noise, a complete specification of the signal and autocorrelation of the noise is necessary and sufficient in order to arrive the corresponding matched filter. Due to modeling uncertainties or changing operating environments it is possible that the second-order characterization of the noise is not completely known. Also, channel nonlinearities lead to distort the signal in an unpredictable (or difficult to ascertain) way. In these cases it is interesting to design a robust matched filter, i.e., a filter that gives the optimum in some sense—behavior within an uncertainty region. In [1] it is shown that, under some of these restrictions, the design of the robust matched filter in the worst case - the most widely used in robust decision problems - is equivalent to finding the least-favorable pair of signal and noise possible. In general, this is a minimization problem numerically solvable; in this paper we present some analytical solutions for the increasingly important case of discrete-time processing [2]. In Section 1, we introduce the necessary background by review of the results of [1] for robust matched filters and we present the adequate formulation for the discrete-time case. In Sections 3 and 4 we consider the situations with uncertainties only in signal and only in noise, respectively, for major uncertainty classes. In Section 3 we deal with the simultaneous occurrence of uncertainties in both signal and noise.

2. Problem Formulation

A general formulation of the matched filter design problem that allows the description of the input pair of signal and noise in various ways has been given in [1]. Let a signal quantity (in the time or frequency domain) be a ∈ X, a noise quantity (e.g., covariance matrix or autocorrelation function) be n ∈ X, and a filter quantity (e.g., impulse response or transfer function) be h ∈ X, where X is a Hilbert space with inner product (,·) and H is a space of bounded, linear, positive operators mapped X to itself. The real valued functional defined by

$\phi(h,a) = \langle h(a), a \rangle$  \quad (2.1)

represents the output signal-to-noise ratio of the filter at some time instant, for the usual descriptions of signal and noise, here in continuous or discrete-time. Note that, in order for this definition to make sense, the quantity n should represent a second-order characterization of the noise. By direct application of the Schwarz inequality, the filter matched to n and n is given by:

$\phi(h) = \frac{1}{2} n(h, n)$ \quad (2.2)

if $\langle h, n \rangle = 0$. In general, $\phi(h, n)$ is the least-favorable class of signal and noise, respectively, the robust (matched) filter $f_n$ is the one that has the best performance for the worst-case pair of signal and noise, i.e.,

$\phi_n = \inf \{ \phi(h,a) \} \quad (2.3)$

such that $\langle h, n \rangle = 0$. This is the minimization problem that leads to the least-favorable pair of signal and noise. If $\langle h, n \rangle = 0$, the least-favorable pair of signal and noise is

$\phi_n = \inf \{ \phi(h,a) \} \quad (2.4)$

such that $\langle h, n \rangle = 0$. This is the minimization problem that leads to the least-favorable pair of signal and noise. If $\langle h, n \rangle = 0$, the least-favorable pair of signal and noise is
In order to get \( h_2 \) we must find a saddle-point solution for the game of (2.3), summarizing the results proved in [1], under some mild continuity conditions we have the following:

\[ \begin{align*}
& \text{Let } \Pi \text{ and } \Pi' \text{ be convex, } (a_n, \mathbf{h}(a)) \text{ and } \\
& h_2 = u_2^* a_2, \text{ the triple } (a_n, h_2, h_3) \text{ is a saddle-point} \\
& \text{solution of (2.3) (i.e., } h_2 = h_2) \text{ if and only if one of the following equivalent conditions holds:} \\
& a) \quad \mathbf{h}(a_n) = a_n h_2, \\
& \Pi(\mathbf{h}(a_n)) = \mathbf{h}(a_n) \Pi, \\
& \Pi(\mathbf{h}(a_n)) = \mathbf{h}(a_n) \Pi' \\
& b) \quad \mathbf{Z}(a_n, h_2, h_3), h_2 > (a_n, h_2) \text{ for all } \\
& (a_n) \Pi' \Pi, \\
& c) \quad (a_n, h_2) \text{ is least favorable for matched filtering } \\
& \text{in } \mathbf{h}\Pi' \Pi. \\
\end{align*} \]

Note that (a) and (c) imply that the filter matched to the least favorable \( \mathbf{p} \) achieves its worst performance where the true signal and noise are the least favorable ones. In the sequel, we shall make use of the following decomposition of Condition b):

\[ \begin{align*}
& \mathbf{Z}(a_n, h_2, h_3) = \mathbf{Z}(a_n, h_2) \\
& \text{for all } a_n \neq \mathbf{h}_2 \\
& (a_n, h_2) > (a_n, h_2) \quad \text{for all } a_n \neq \mathbf{h}_2. \\
\end{align*} \]

The least favorable signal \( a_n \) depends on \( a_n \) and is given by the following:

\[ \begin{align*}
& a_n = \operatorname{argmin} (s_n, a_n) = s_n - \sigma^2 h_2, \\
& \sigma^2 h_2 = \Delta. \\
\end{align*} \]

Since the covariance matrix of the noise is known, according to Lemma 1 and 2, \( h_2 \) is least favorable if and only if

\[ \begin{align*}
& (s_n, h_2) = a_n (s_n, h_2), \\
& a_n \neq \mathbf{h}_2. \\
\end{align*} \]

Let \( \mathbf{h}_2 \) be the least favorable signal.

\[ \begin{align*}
& s_n = \mathbf{n}, \quad \sigma^2 = \mathbf{h}_2, \\
& s_n = \mathbf{h}_2. \\
\end{align*} \]

We can get an alternative expression for the robust filter, with (3.2) and (3.3):

\[ \begin{align*}
& h_2 = \mathbf{n}_0 + \sigma^2 \mathbf{h}_2, \\
& \mathbf{n}_0 = \mathbf{h}_2. \\
\end{align*} \]

Equation (3.7) shows that the robust filter is the filter matched to the nominal signal and the input noise with an added component of white noise of covariance \( \sigma^2 \). Note that in general, the matrix \( \mathbf{n}_0 \) is computed recursively from (3.3) and (3.7). Further simplification of the result can be obtained in particular cases like the following:

**Proposition 3:**

The nominal - filter matched to \( (s_n, h_2) \) is robust for deviations from \( s_n \) defined by the class \( \mathcal{A}_2 \) (3.1) if and only if \( s_n \) is an eigenvector of \( \mathbf{n}_0 \).

**Proof:**

First note from (2.1) that the performance of filter \( h_2 \) is not affected by scaling the impulse responses by a constant, so the nominal filter is robust if and only if there exists \( k > 0 \) such that

\[ \begin{align*}
& (s_n + \sigma^2 \mathbf{h}_2) = k s_n, \\
& \mathbf{n}_0 = \mathbf{h}_2. \\
\end{align*} \]

This is equivalent to

\[ \begin{align*}
& (s_n + \sigma^2 \mathbf{h}_2) = \mathbf{n}_0, \\
& (s_n + \sigma^2 \mathbf{h}_2) = \mathbf{n}_0 = \mathbf{h}_2. \\
\end{align*} \]

We consider here two classes of uncertainties described by a bound on the \( \mathcal{A}_2 \) and \( \mathcal{A}_3 \) norm of the deviation of the signal from a given nominal \( s_n \).

\[ \begin{align*}
& \mathcal{A}_2 \text{ Uncertainty} \\
& \mathcal{A}_3 \text{ Uncertainty} \\
& (s_n, \mathbf{h}_2) = (s_n, \mathbf{h}_2) \\
& \mathbf{n}_0 = \mathbf{h}_2. \\
\end{align*} \]
\( A = \{x \in \mathbb{R}^n : \max \{ |x|_1, |x|_2 \} \leq \delta, \ i = 0, \ldots, k-1 \} \). (3.10)

In order to get the least favorable signal in this class, according to (3.6) we have to find out the solution to the minimization problem:

\[
\min_{x \in A} \sum_{i=1}^{k} \sum_{j=1}^{k} (e_{ij} x_{ij}) \]  \( s.t. \quad e_{ij} x_{ij} = \delta \quad i, j \leq k \). (3.11)

with \( J = \{0, \delta, \delta + 1\} \). In some special cases an analytical result is achievable:

**Proposition 3**

If the samples of the noise are uncorrelated the least-favorable signal \( h_0 \) in \( A_0 \) is given by:

\[
h_0 = \begin{cases} \delta \quad & \text{if } \delta > 0 \\ 0 \quad & \text{if } \delta \leq 0 \end{cases}
\]  \( \text{s.t.} \quad s \in \mathbb{R}^k \). (3.12)

**Proof**

If the noise sample are uncorrelated we have:

\[
s_0 = \text{diag}(A_1, \ldots, A_d) \]  \( \text{s.t.} \quad s \in \mathbb{R}^k \). (3.13)

with \( A_0 > 0 \). For all \( i = 0, \ldots, k-1 \) it is easy to see that:

\[
e_{ij} h_0 = \sum_{i=1}^{k} A_i \]  \( \text{s.t.} \quad s \in \mathbb{R}^k \). (3.14)

for any \( s \in \mathbb{R}^k \). Since \( h_0 = \Delta \), this implies:

\[
\left( s^T h_0 \right) = \min \{ s^T h \} \quad s \in \mathbb{R}^k \]  \( \text{s.t.} \quad s \in \mathbb{R}^k \). (3.15)

Therefore, by Lemma 1 and 2, \( h_0 \) is the least favorable signal.

**Proposition 4**

If there exists an element \( s \in \mathbb{R}^k \) such that:

\[
\left( s^T h \right) < \delta \]  \( \text{s.t.} \quad s \in \mathbb{R}^k \). (3.16)

with \( h = \eta \), then \( h_0 \) is the least favorable signal.

**Proof**

The expression (3.10) and (3.16) imply that for any \( s \in \mathbb{R}^k \) and \( i = 0, \ldots, k-1 \):

\[
\left( s^T h \right) > \Delta \]  \( \text{s.t.} \quad s \in \mathbb{R}^k \). (3.17)

Since this is sufficient in order to have (3.15), \( h_0 \) is the least favorable signal in \( A_0 \). This result suffers from the same inconvenience that the one in Lemma 2, namely \( h_0 \) depends on \( h \), and therefore the solution must be reached recursively. However we can assure the existence of a solution of the type of (3.16) and its direct computation under the condition of the following:

**Proposition 5**

If the maximum deviation from the nominal signal in each sample is bounded by:

\[
\Delta < \min \{ |c_{ij}| \} \quad \text{for } i = 0, \ldots, k-1 \]  \( \text{s.t.} \quad c_{ij} \). (3.18)

the least favorable signal \( h_0 \) in \( A_0 \) is given by:

\[
h_0 = \begin{cases} \delta \quad & \text{if } h_0 > 0 \\ 0 \quad & \text{if } h_0 < 0 \end{cases}
\]  \( \text{s.t.} \quad s \in \mathbb{R}^k \). (3.19)

where \( h_0 = \eta \), \( h_0 \) is the nominal matched filter.

**Proof**

With \( h_0 = \eta \), \( h_0 \) is the nominal matched filter.

\[
|h_0| \leq |b_{ij}| \]  \( \text{s.t.} \quad s \in \mathbb{R}^k \). (3.20)

since \( s_0 \in \mathbb{R}^k \), the absolute value of the components of \( s_0 \) is bounded by \( \Delta \). Thus:

\[
|b_{ij}| \leq \Delta \]  \( \text{s.t.} \quad s \in \mathbb{R}^k \). (3.21)

and by (3.18),

\[
|h_0| \leq |b_{ij}| \]  \( \text{s.t.} \quad s \in \mathbb{R}^k \). (3.22)

which is sufficient in order that:

\[
\text{sgn}(h_0) = \text{sgn}(b_{ij}) \]  \( \text{s.t.} \quad s \in \mathbb{R}^k \). (3.23)

Therefore, by Proposition 3, \( h_0 \) given by (3.19) is the least favorable signal in \( A_0 \).

4. Noise Uncertainty

Here we suppose that the nominal signal \( s_0 \) is truly present at the input, but the actual covariance matrix \( \Sigma \) is allowed to differ from the nominal \( \Sigma_0 \). The first is a general result useful for different classes of uncertainties.

**Lemma 3**

If the uncertainty class \( \Pi \) is independent of the nominal signal \( s_0 \), then \( h_0 \) is the least favorable noise for every \( s_0 \in \mathbb{R}^k \). If and only if \( s_0 \) is a maximal element of \( \Pi \).

**Proof**

Note from Lemma 2 that when there is no uncertainty in the signal, \( h_0 \) is the least favorable noise if and only if:

\[
\left( h_0, (s_0 - \eta) h_0 \right) > 0
\]  \( \text{s.t.} \quad s \in \mathbb{R}^k \). (4.1)

for all \( s \in \Pi \).

By definition, \( s_0 \) is a maximal element of \( \Pi \) if and only if:

\[
\eta = s_0
\]  \( \text{s.t.} \quad s \in \mathbb{R}^k \). (4.2)

for all \( s \in \Pi \). This is equivalent to (4.1) holding for all \( s_0 \in \mathbb{R}^k \) since for every \( h_0 \in h_0 \) there exists \( h_0 = \eta = s_0 \).

As an application of this previous lemma, we have a result on the least favorability of white noise.
Proposition 6

Suppose \( \eta_1 \) is \( \in \mathcal{P} \); this the least-favorable
coefficient in \( \mathcal{P} \) if and only if \( |\eta_2| \leq c \) for
all \( c \neq 0 \), where \( |\eta_2| \) denotes the Euclidean norm
of \( \eta_2 \).

Proof

According to Lemma 1, \( \eta_2 \) is the least-favorable
coefficient in \( \mathcal{P} \) if and only if it is the maximal
element of the class, i.e., for all \( c \neq 0 \), \( \mathcal{P} \),
\[ x(\eta_2) + c|x|^2 \leq 0 \] (6.2)
and
\[ x(\eta_2)|x|^2 < c \] (6.3)

Equivalently, by Rayleigh's principle [3],
\[ x(\eta_2)|x|^2 \geq c \] (6.4)

where \( x(\eta) = \text{the spectral radius of } \eta \) (minimum absolute value or its eigenvalue), but
\( |\eta_2|^2 \geq x(\eta) \) since \( x(\eta) \) is symmetric.

Here, in analogy with the \( \eta_2 \) signal uncertainty,
we deal with a specific set to class defined by
\[ \eta_2 = \beta x(\eta_2) \] (6.5)

where \( \beta \) is a positive constant, \( \eta_2 \) is the canonical
noise covariance matrix and the norm is any valid
matrix norm.

Proposition 7

The least-favorable noise in \( \mathcal{P} \) is
\[ \eta_2 = \beta x(\eta_2) + \text{noise} \] (6.6)

Proof

By means of Lemma 3, this is equivalent to
proof that \( \eta_2 \) is a maximal element of \( \mathcal{P} \). For
any \( x \neq \eta_2 \) and \( x \neq 0 \),
\[ x(\eta_2) - x(\eta_2 + 411) - x \] (6.7)
but by the Schwarz inequality
\[ x(\eta_2) - x(\eta_2 + 411) \leq \|x\| \|\eta_2 - \eta_2 - 411\| \] (6.8)

where the last inequality must hold for any type of
matrix norm [4]. Combining (6.4) and (6.10) we get
\[ x(\eta_2) - x(\eta_2) \leq \|x\| \|\eta_2 - \eta_2 - 411\| \] (6.9)

Finally, it is easy to see that \( \eta_2 \in \mathcal{P} \) since for
any \( x \neq 0 \), \( x \neq 0 \),
\[ x(\eta_2) = x(\eta_2) + x(x)^2 > 0 \] (6.11)

Conclusion

Several closed-form solutions have been presented
for the computation of the least-favorable
noise: matched filtering formulation (6.4) or
the discrete case. We note that signal
uncertainties that allow partitioning in
practical cases have been obtained. In the search
for least-favorable noise covariance matrices,
we have found conditions for the existence
of signal-dependent solutions and for least-favorable
ability of white noise. Also we have dealt with
the uncertainty class described by the norm
of the deviation in analogy with the \( \eta_2 \) class for
signals. In the case of uncertainties in both
signal and noise, we have shown how and when
the problem can be reduced to the previous case by
a cumulant decoupling of the least-favorable
condition. Thus the allowable signal and noise
factors in the interior of a ball around the momental,
the robust matched filter is found to be the one
matched to the canonical signal and canonical
cases, i.e., a white noise component. Therefore, in
the frequent event in which the canonical noise is
white, the canonical matched filter is robust.

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REFERENCES


