LEAST FAVORABLE SIGNALS AND NOISE FOR DISCRETE TIME ROBUST MATCHED FILTERING

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Abstract. The detection of a deterministic signal in additive noise is often accomplished by the well known linear matched filter for that particular pair of signal and noise. In the presence of channel distortion or other types of modelling uncertainties, it is possible that no exact knowledge of the actual input signal and/or noise characteristics is available. In this case it is of interest to design a robust matched filter; i.e., a nonadaptive filter with a guaranteed level of worst performance for the expected class of deviations from the nominal model. This problem is known to be equivalent to the search for the least favorable pair of signal and noise, i.e. the one whose matched filter gives the lowest signal-to-noise ratio. In this paper we investigate this problem in the case of discrete-time processing. We consider various types of uncertainty classes of signal and noise that arise naturally in practical cases and result in closed-form solutions. Moreover, conditions are given for the least favorability of white noise and the robustness of the nominal filter.

1. INTRODUCTION

The linear system that maximizes the output signal-to-noise ratio at some instant of time when the input is a deterministic signal embedded in additive random noise is known as the matched filter for this pair of signal and noise. If the noise is a Gaussian process, then the output of this filter at the instant in which the signal-to-noise ratio is maximized provides a sufficient statistic for any likelihood-ratio detection of the input signal. Since the power of the noise at the output of the linear filter depends on the second-order statistics of the input noise, a complete specification of the signal and autocorrelation of the noise is necessary and sufficient in order to derive the corresponding matched filter. Due to modelling uncertainties or changing operating environments it is possible that the second order characterization of the noise is not completely known. Also, channel nonlinearities tend to distort the signal in an unpredictable (or difficult to ascertain) way. In these cases it is interesting to design a robust matched filter, i.e. a filter that gives the optimum - in some sense - behavior within the uncertainty region. In [1] it is shown that, under some mild restrictions, the design of the robust matched filter in the maximin sense - the most widely used in robust decision problems - is equivalent to finding the least-favorable pair of signal and noise possible. In general, this is a minimization problem numerically solvable; in this paper, we present some analytical solutions for the increasingly important case of discrete-time processing [2]. In Section 2, we introduce the necessary background by a review of the results of [1] for robust matched filters and we present the adequate formulation for the discrete time case. In Sections 3 and 4 we consider the situations with uncertainties only in signal and only in noise, respectively, for major uncertainty classes. In Section 5 we deal with the simultaneous occurrence of uncertainties in both signal and noise.

2. PROBLEM FORMULATION

A general formulation of the matched filter design problem that allows the description of the input pair of signal and noise in various ways has been given in [1]. Let a signal quantity (in the time or frequency domain) be $s \in \mathcal{H}$, a noise quantity (e.g., covariance matrix or autocorrelation function) be $n \in \bar{\mathcal{H}}$, and a filter quantity (e.g. impulse response or tranfer function) be $h \in \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and $\bar{\mathcal{H}}$ is a space of bounded, linear, positive operators mapping $\mathcal{H}$ to itself. The real valued functional defined by

$$\rho(h;s,n) = |\langle h,s\rangle|^2/\langle h,nh\rangle \tag{2.1}$$

represents the output signal-to-noise ratio of the filter at some time instant, for the usual descriptions of signal and noise, either in continuous or discrete time. Note that, in order for this definition to make sense, the quantity n should represent a second-order characterization of the noise. By direct application of the Schwarz inequality, the filter matched to s and n is given by:

$$\operatorname*{argmax}_{h \in \mathcal{H}} \rho(h;s,n) = n^{-1}s \quad . \tag{2.2}$$

If $\mathcal{S} \subset \mathcal{H}$, $\mathcal{N} \subset \bar{\mathcal{H}}$ are the allowable classes of signal and noise respectively, the robust (maximin) filter h_R, is the one that has the best performance for the worst-case pair of signal and noise, i.e.

$$h_R = \operatorname*{argmax}_{h \in \mathcal{H}}\{ \inf_{(s,n)\in \mathcal{S}\times\mathcal{N}} \rho(h;s,n)\} \quad . \tag{2.3}$$

We say that a pair $(s_L,n_L) \in \mathcal{S}\times\mathcal{N}$ is least favorable for matched filtering in $\mathcal{S}$ and $\mathcal{N}$ if

$$(s_L,n_L) = \operatorname*{argmin}_{(s,n)\in \mathcal{S}\times\mathcal{N}} \langle s, n^{-1}s\rangle \quad . \tag{2.4}$$

In order to get h_R we must find a saddle-point solution for the game of (2.3). Summarizing the results proved in [1], under some mild continuity conditions we have the following:

<u>Lemma 1</u>

If $\mathcal{S}$ and $\mathcal{N}$ are convex, $(s_L, n_L) \in \mathcal{S} \times \mathcal{N}$ and $h_L = n_L^{-1} s_L$, the triple $(h_L; s_L, n_L)$ is a saddle-point solution of (2.3) (i.e. $h_L = h_R$) if and only if one of the following equivalent conditions holds:

a) $$\rho(h_L; s_L, n_L) = \min_{(s,n)\in \mathcal{S}\times\mathcal{N}} \rho(h_L; s, n) \tag{2.5}$$

b) $$2\mathrm{Re}\{\langle s, h_L\rangle\} - \langle h_L, nh_L\rangle \geq \langle s_L, h_L\rangle \text{ for all } (s,n)\in \mathcal{S}\times\mathcal{N} \tag{2.6}$$

c) (s_L, n_L) is least favorable for matched filtering in $\mathcal{S}$ and $\mathcal{N}$.

Note that a) and c) imply that the filter matched to the least favorable pair achieves its worst performance where the true signal and noise are the least favorable ones. In the sequel, we will make use of the following decomposition of Condition b):

<u>Lemma 2</u>

$$\mathrm{Re}\{\langle s, h_L\rangle\} \geq \langle s_L, h_L\rangle \tag{2.7}$$

for all $s \in \mathcal{S}$ and

$$\langle h_L, (n_L - n)h_L\rangle \geq 0 \tag{2.8}$$

for all $n \in \mathcal{N}$, if and only if (2.6) holds for all $(s,n) \in \mathcal{S}\times\mathcal{N}$.

<u>Proof</u>

First, note that since n_L^{-1} is a positive operator, $\langle s, n_L^{-1}s\rangle$ is real. Then it is easy to see that restricting (2.6) to $n = n_L$ and $x = s_L$ we get (2.7) and (2.8) respectively. On the other hand it is straightforward to see that (2.7) and (2.8) imply (2.6) for all pairs.

In the discrete time case, let $\mathcal{K} = \mathbb{R}^k$, $s = [s_o, \ldots, s_{k-1}]^T$, $h = [h_o, \ldots, h_{k-1}]^T$, where $h_i = \tilde{h}_{k-i-1}$, and $s_i, \tilde{h}_i$ are the values of signal and filter impulse response respectively at the i-th sample. The inner product is defined as the usual scalar product: $\langle a, b\rangle = a^T b$; and the noise descriptor is $n \in \bar{\mathcal{K}} \subset \mathbb{R}^{k\times k}$, a positive definite symmetric matrix, representing the covariance matrix of the zero-mean input noise. It is easy to see that with these definitions, (2.1) gives the power of the filter output due to the signal, divided by the variance of the filter output due to the noise, at the k-1 <u>th</u> sample.

3. SIGNAL UNCERTAINTY

We consider here two classes of uncertainties described by a bound on the ℓ_2 and ℓ_∞ norm of the deviation of the signal from a given nominal s_o.

3.1. ℓ_2 Uncertainty

Let n_0 be the covariance matrix of the input noise, and $\mathcal{S}_1$ the class of allowable signals defined by:

$$\mathcal{S}_1 = \{s \in \mathbb{R}^k, \|s - s_o\| \leq \Delta\} \tag{3.1}$$

where $\|x\|^2 = \langle x, x\rangle$. The least favorable signal s_L depends on n_0 and is given by the following:

<u>Proposition 1</u>

$$s_L = \operatorname*{argmin}_{s\in\mathcal{S}_1} \langle s, n_0^{-1}s\rangle = s_o - \sigma^2 h_L \tag{3.2}$$

with

$$\sigma^2\|h_L\| = \Delta \tag{3.3}$$

and $h_L = n_0^{-1}s_L$.

<u>Proof</u>

Since the covariance matrix of the noise is known, according to Lemmas 1 and 2 s_L is least favorable if and only if

$$\langle s_L, h_L\rangle = \min_{s\in\mathcal{S}_1} \langle s, h_L\rangle \tag{3.4}$$

but by (3.2) and (3.3) we have for $s \in \mathcal{S}_1$

$$\langle s - s_L, h_L\rangle = \langle s - s_o, h_L\rangle + \Delta\|h_L\| \tag{3.5}$$

where the right side is nonnegative by the Schwarz inequality:

$$|\langle s - s_o, h_L\rangle| \leq \|h_L\|\,\|s - s_o\| \leq \Delta\|h_L\| \quad . \tag{3.6}$$

Since $s_L \in \mathcal{S}_1$, s_L is the least favorable signal.

We can get an alternative expression for the robust filter, with (3.2) and (3.3):

$$h_L = (n_0 + \sigma^2 I)^{-1} s_o \quad . \tag{3.7}$$

Equation (3.7) shows that the robust filter is the filter matched to the nominal signal and the input noise with an added component of white noise of covariance σ^2. Note that in general, σ is computed recursively from (3.3) and (3.7). Further simplification of the result can be obtained in particular cases like in the following:

<u>Proposition 2</u>

The - nominal - filter matched to (s_o, n_0) is robust for deviations from s_o defined by the class $\mathcal{S}_1$ (3.1) if and only if s_o is an eigenvector of n_0.

<u>Proof</u>

First note from (2.1) that the performance of filter h is not affected by scaling the impulse response by a constant, so the nominal filter is robust if and only if there exists $k \geq 0$ such that

$$(n_0 + \sigma^2 I)^{-1} s_o = k n_0^{-1} s_o \quad . \tag{3.8}$$

but this is equivalent to

$$[(1-k)/\sigma^2] s_o = n_0^{-1} s_o \tag{3.9}$$

and since n_0^{-1} is invertible, (3.9) holds if and only if s_o is an eigenvector of n_0.

3.2 ℓ_∞ Uncertainty

Let $\mathcal{S}_2$, the class of allowable signals, be defined by

$$\mathcal{S}_2 = \{s \in \mathbb{R}^k,\ \max|s_i - s_{oi}| \leq,\ i = 0,\ldots,k-1\}. \quad (3.10)$$

In order to get the least favorable signal in this class, according to (2.4) we have to find out the solution to the minimization problem:

$$\underset{s \in J_0 \times \ldots \times J_{k-1}}{\text{argmin}} \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} (n_0^{-1})_{ij}\, s_i s_j \quad (3.11)$$

with $J_i = [s_{oi} - \Delta,\ s_{oi} + \Delta]$. In some special cases an analytical result is achievable:

<u>Proposition 3</u>

If the samples of the noise are uncorrelated the least-favorable signal s_L in $\mathcal{S}_2$ is given by:

$$s_{Li} = \begin{cases} 0 & -\Delta \leq s_{oi} \leq \Delta \\ s_{oi} - \Delta & \Delta < s_{oi} \\ s_{oi} + \Delta & s_{oi} < -\Delta \end{cases} \quad (3.12)$$

<u>Proof</u>

If the noise samples are uncorrelated we have:

$$n_0^{-1} = \text{diag}(\ell_1,\ldots,\ell_k) \quad (3.13)$$

with $\ell_i > 0$. For all $i = 0,\ldots,k-1$, it is easy to see that:

$$s_i s_{Li} \geq (s_{Li})^2 \quad (3.14)$$

for any $s \in \mathcal{S}_2$. Since $h_{Li} = \ell_i s_{Li}$, this implies:

$$\langle s_L, h_L \rangle = \min_{s \in \mathcal{S}_2} \langle s, h_L \rangle \quad (3.15)$$

Therefore, by Lemmas 1 and 2, s_L is the least favorable signal.

<u>Proposition 4</u>

If there exists an element $s_L \in \mathcal{S}_2$ such that

$$s_{Li} = \begin{cases} s_{oi} - \Delta & \text{if } h_{Li} > 0 \\ s_{oi} + \Delta & \text{if } h_{Li} < 0 \end{cases} \quad (3.16)$$

with $h_L = n_0^{-1} s_L$, then s_L is the least favorable signal.

<u>Proof</u>

The expressions (3.10) and (3.16) imply that for any $s \in \mathcal{S}_2$ and $i = 0,\ldots,k-1$,

$$h_{Li} s_i \geq h_{Li} s_{Li} \quad . \quad (3.17)$$

Since this is sufficient in order to have (3.15), s_L is the least favorable signal in $\mathcal{S}_2$. This result suffers from the same inconvenience that the one in Lemma 2, namely s_L depends on h_L, and therefore the solution must be reached recursively. However we can assure the existence of a solution of the type of (3.16) and its direct computation under the condition of the following:

<u>Proposition 5</u>

If the maximum deviation from the nominal signal in each sample is bounded by:

$$\Delta < \min_i |h_{0i}| / \max_j \sum_m |(n_0^{-1})_{jm}| \quad (3.18)$$

the least favorable signal s_L in $\mathcal{S}_2$ is given by

$$s_{Li} = \begin{cases} s_{oi} - \Delta & \text{if } h_{0i} > 0 \\ s_{oi} + \Delta & \text{if } h_{0i} < 0 \end{cases} \quad (3.19)$$

where $h_0 = n_0^{-1} s_o$ is the nominal matched filter.

<u>Proof</u>

With $h_L = n_0^{-1} s_L$ and $u_L = s_L - s_o$, for $i = 0,\ldots,k-1$

$$|h_{Li} - h_{0i}| = |(n_0^{-1} u_L)_i| \quad (3.20)$$

since $s_L \in \mathcal{S}_2$, the absolute value of the components of u_L is bounded by Δ. Thus

$$|(n_0^{-1} u_L)_i| \leq \Delta \max_j \sum_m |(n_0^{-1})_{jm}| \quad (3.21)$$

and by (3.18),

$$|h_{Li} - h_{0i}| < |h_{0i}| \quad (3.22)$$

which is sufficient in order that

$$\text{sgn}(h_{Li}) = \text{sgn}(h_{0i}) \quad . \quad (3.23)$$

Therefore, by Proposition 4, s_L given by (3.19) is the least favorable signal in $\mathcal{S}_2$.

4. NOISE UNCERTAINTY

Here we suppose that the nominal signal s_o is truly present at the input, but the actual covariance matrix n is allowed to differ from the nominal n_0. The first is a general result useful for different classes of uncertainties.

<u>Lemma 3</u>

If the uncertainty class $\mathcal{N}$ is independent of the nominal signal s_o, then n_L is the least favorable noise for every $s_o \in \mathbb{R}^k$ if and only if n_L is a maximal element of $\mathcal{N}$.

<u>Proof</u>

Note from Lemma 2 that when there is no uncertainty in the signal, n_L is the least favorable noise if and only if

$$\langle h_L, (n_L - n) h_L \rangle \geq 0 \quad (4.1)$$

for all $n \in \mathcal{N}$.

By definition, n_L is a maximal element of $\mathcal{N}$, if and only if

$$n_L \geq n \quad (4.2)$$

for all $n \in \mathcal{N}$. This is equivalent to (4.1) holding for all $s_o \in \mathbb{R}^k$ since for every $h_L \in \mathbb{R}^k$ there exists $s_o = n_L h_L$.

As an application of this previous Lemma, we have a result on the least favorability of white noise.

Proposition 6

Suppose $n_L = cI \in \mathcal{N}$; this is the least favorable covariance in $\mathcal{N}$ if and only if $\|n\|_2 \leq c$ for all $n \in \mathcal{N}$, where $\|n\|_2$ denotes the Euclidean norm of n.

Proof

According to Lemma 3, cI is the least favorable covariance in $\mathcal{N}$ if and only if it is the maximal element of the class, i.e. for all $n \in \mathcal{N}$, $x \in \mathbb{R}^k$, $x \neq 0$:

$$x^T(n_0-n)x = c\|x\|^2 - x^T nx \geq 0 \tag{4.3}$$

$$(x^T nx)/\|x\|^2 \leq c \quad . \tag{4.4}$$

Equivalently, by Rayleigh's principle [3],

$$\rho(n) \leq c \tag{4.5}$$

where $\rho(n)$ denotes the spectral radius of n (maximum absolute value of its eigenvalues), but $\|n\|_2 = \rho(n)$ since n is symmetric.

Next, in analogy with the ℓ_2 signal uncertainty, we deal with a specific deviation class defined by

$$\mathcal{N}_1 = \{n \in \mathbb{R}^{k\times k}, \|n-n_0\| \leq \epsilon,\ n > 0\} \tag{4.6}$$

where ϵ is a positive constant, n_0 is the nominal noise covariance matrix and the norm is any valid matrix norm.

Proposition 7

The least favorable noise in $\mathcal{N}_1$ is

$$n_L = n_0 + \epsilon I \tag{4.7}$$

Proof

By means of Lemma 3, this is equivalent to proof that n_L is a maximal element of $\mathcal{N}_1$. For any $n \in \mathcal{N}_1$ and $x \in \mathbb{R}^k$:

$$\begin{aligned} x^T(n_L-n)x &= x^T(n_0+\epsilon I - n)x \\ &= \epsilon\|x\|^2 + x^T(n_0-n)x \end{aligned} \tag{4.8}$$

but by the Schwarz inequality

$$|x^T(n_0-n)x| \leq \|x\|\ \|(n_0-n)x\| \tag{4.9}$$

$$\leq \|x\|^2\|n_0-n\| \tag{4.10}$$

where the last inequality must hold for any type of matrix norm [4]. Combining (4.8) and (4.10) we get

$$x^T(n_0-n)x \geq \|x\|^2(\epsilon - \|n_0-n\|) \geq 0 \ . \tag{4.11}$$

Finally, it is easy to see that $n_L \in \mathcal{N}_1$ since for any $x \in \mathbb{R}^k$, $x \neq 0$:

$$x^T n_L x = x^T n_0 x + \epsilon\|x\|^2 > 0 \ . \tag{4.12}$$

5. UNCERTAINTY IN SIGNAL AND NOISE

In this section we suppose that we receive some signal $s \in \mathcal{S}$ and some noise with covariance $n \in \mathcal{N}$. We must look for a least favorable pair (s_L,n_L) such that s_L solves (2.7) for all $s \in \mathcal{S}$ and n_L (2.8) for all $n \in \mathcal{N}$. If $\mathcal{S}$ and $\mathcal{N}$ are such that analytical solutions for the least-favorable signal and noise are available, we get a set of two equations in two unknowns: s_L and n_L, whose solution exists under the assumptions of Lemma 1.

Further simplification is obtained when one or both of the equations give s_L and/or n_L directly, i.e. s_L (n_L) does not depend on the input noise (signal). As an example consider the case in which $\mathcal{S} = \mathcal{S}_1$ (3.1) and $\mathcal{N} = \mathcal{N}_1$ (4.6). Recall that the least favorable signal and noise were given respectively by:

$$s_L = s_o - \sigma^2 h_L$$

$$n_L = n_0 + \epsilon I \ .$$

Therefore the robust matched filter $h_L = n_L^{-1}s_L$ is

$$h_L = (n_0 + (\epsilon+\sigma^2)I)^{-1}s_o \tag{5.1}$$

with $\sigma^2\|h_L\| = \Delta$. Note that if the nominal noise is white, the nominal matched filter ($h_0 = s_o$) is robust for uncertainties in signal and noise defined by $\mathcal{S}_1$ and $\mathcal{N}_1$ respectively.

5. CONCLUSIONS

Several closed-form solutions have been presented for the application of the maximin-robust matched filtering formulation of [1] to the discrete time case. Two types of signal uncertainties that find wide justification in practical cases have been studied. In the search for least favorable noise covariance matrices, we have found the conditions for the existence of signal-independent solutions and for least favorability of white noise. Also we have dealt with the uncertainty class described by the norm of the deviation in analogy with the ℓ_2 class for signals. In the case of uncertainties in both signal and noise, we have shown how and when the problem can be reduced to the previous ones by a concise decoupling of the least favorability condition. When the allowable signal and noise lie in the interior of a ball around the nominal, the robust matched filter is found to be the one matched to the nominal signal and nominal noise plus a white noise component. Therefore, in the frequent event in which the nominal noise is white, the nominal matched filter is robust.

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