Linear and Linear-Conjugate/Linear receivers for CDMA Systems with long codes in Flat-Fading Channels *

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Abstract

We consider linear multiuser detection for synchronous DS/CDMA systems with long codes over flat-fading channels. Based on the observation that in DS/CDMA systems the multiaccess interference has a nonzero pseudo-autocorrelation function, we introduce a new nonlinearly-constrained cost function, whose minimization leads to two new linear multiuser detectors, which exploit the information contained in the pseudo-autocorrelation of the observables, and which generalize the classical decorrelating and minimum mean square error receivers.

Since the resulting structures cannot be implemented with acceptable complexity we resort to a reduced-rank filter approach in order to come up with low-complexity receiver implementations. Our study is supplemented by an asymptotic analysis (i.e. $K = \beta N$ with β fixed and $K \to \infty$ with K number of the users and N processing gain), carried out in order to indicate directions towards a simpler (albeit admittedly suboptimum) receiver design and to demonstrate the superiority of the new detectors with respect to classical linear detection structures. Our claims are also corroborated by computer simulations.

1 Introduction

The Direct-Sequence Code Division Multiple Access (DS-CDMA) technique has been selected as the basic technology for the realization of the air interface of most 3G wireless cellular networks [4, 1]. As a consequence there is a great interest in the design and analysis of advanced high-performance receivers for such system. Since the seminal work by Verdù [14], who showed that multiaccess interference (MAI), if explicitly accounted for at the design stage, is not a performance-limiting factor, and derived the optimum multiuser detector, huge research efforts have been devoted to the design of suboptimal, reduced-complexity, multiuser detectors. Among them, the most popular ones are the decorrelating detector [10] and the minimum mean square error (MMSE) receiver [11]: interestingly, both receivers have a complexity which is linear in the users number and achieve optimum performance in terms of near-far resistance. More recently, in [20], it is shown that the tools of subspace tracking theory can be efficiently applied to deal with blind adaptive multiuser detection.

All of the above papers, however, come up with detection structures which exploit only the information contained in the autocorrelation function of the observables. While this is the optimum strategy when dealing with proper complex random processes, it turns out to be suboptimal in situations where the disturbance is an improper complex random process[1]. Since, as shown in [18], the MAI can be modeled as an improper complex noise, it is expected that designing receiving structures capable of exploiting the information contained in the pseudo-autocorrelation function of the observables would improve performance [3].

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[1]A complex random process $n(t)$ is said to be *proper* if its pseudo-autocorrelation function $R_n(t,u) \triangleq \mathrm{E}\left[n(t)n(u)\right]$ is zero $\forall t, u$ [18, 12].

In this work we deal with the problem of linear multiuser detection under such a relevant scenario: for simplicity we will focus on a symbol-synchronous channel, although an extension to the asynchronous situation along the lines of [8] is possible; a new cost function is introduced, whose constrained minimization leads to new versions of the decorrelating and the MMSE receivers; we also tackle the problem of low-complexity receiver implementation resorting to a reduced-rank filter approach. As to performance analysis, the superiority of the new receivers with respect to the classical MMSE and decorrelating schemes is shown through both theoretical considerations and curves of Error-Probability and Near-Far resistance. The performance analysis is supplemented by an asymptotic analysis, (wherein the number of the users K is large i.e. $K = \beta N$ with β fixed and $K \to \infty$), carried out in order to demonstrate the superiority of the new detectors with respect to classical linear detection structures. Some asymptotic results are also used to indicate directions towards simpler (albeit admittedly suboptimum) receiver designs.

2 System Model

We consider a synchronous DS/CDMA system with K active users, employing long spreading codes and operating over a frequency-flat fading channel. The baseband equivalent of the received signal is:

$$r(t) = \sum_{\ell=-\infty}^{\infty} \sum_{k=1}^{K} \alpha_k(\ell)\sqrt{\mathcal{E}_k} b_k(\ell) s_k^{\ell}(t - \ell T_b) + n(t) \tag{1}$$

where

- $\mathcal{E}_k$ is the energy of the $k-$th user;
- $\{b_k(\ell)\}_{\ell=-\infty}^{+\infty}$ represents the bit stream of the $k-$th user, modeled as a sequence of independent and identically distributed binary variables, each taking on values in the set $\{-1, 1\}$;
- T_b is the bit interval duration;
- $s_k^{\ell}(t)$ is the *signature* waveform assigned to the $k-$th user in the $\ell-$th signaling interval, expressed as

$$s_k^{\ell}(t) = \sum_{n=0}^{N-1} s_{k,n}^{\ell} u_{T_c}(t - nT_c)$$

  with N the processing gain, $\boldsymbol{s}_k^{\ell} = [s_{k,0}^{\ell}, \ldots, s_{k,N-1}^{\ell}]$ the $k-$th spreading code in the $\ell-$th signaling interval, T_c the chip interval, and $u_{T_c}(\cdot)$ is a chip waveform with zero autocorrelation at multiples of T_c and unitary energy. The spreading code $\boldsymbol{s}_k^{\ell}$ is normalized to have unit norm and its elements[2] are $s_{k,n}^{\ell} \in \{-1/\sqrt{N}, 1/\sqrt{N}\} \quad n = 0, \ldots, N-1$.

- $\alpha_k(\ell)$ is a complex channel gain. The complex-valued fading parameters $\alpha_k(\ell)$ are zero-mean independent from user to user and follow a common distribution (for all k and ℓ) such that $\mathrm{E}[\alpha_k(\ell)\alpha_n^*(\ell)] = \delta_{nk}$ and $\mathrm{E}[\alpha_k(\ell)\alpha_n(\ell)] = 0$ with $(\cdot)^*$ denoting conjugate.

Let us assume that we want to demodulate user "1" , and that a decision about $b_1(\ell)$ is made by processing the observation in the interval $[\ell T_b, (\ell+1)T_b]$. Chip-matched filtering the received waveform in the interval $[\ell T_b, (\ell+1)T_b]$ we obtain the following N-dimensional vector sequence of observables:

$$\begin{aligned} \boldsymbol{r}(\ell) &= b_1(\ell)\sqrt{\mathcal{E}_1}\boldsymbol{s}_1^{\ell}\alpha_1(\ell) + \sum_{k=2}^{K} b_k(\ell)\sqrt{\mathcal{E}_k}\boldsymbol{s}_k^{\ell}\alpha_k(\ell) + \boldsymbol{n}(\ell) = \\ &= b_1(\ell)\sqrt{\mathcal{E}_1}\boldsymbol{s}_1^{\ell}\alpha_1(\ell) + \boldsymbol{S}_1^{\ell}\boldsymbol{A}_1\boldsymbol{B}_1(\ell)\boldsymbol{\alpha}_1(\ell) + \boldsymbol{n}(\ell) \end{aligned} \tag{2}$$

[2]Similar results can be obtained for non-binary sequences.

where $\boldsymbol{A}_1 = \operatorname{diag}\left(\sqrt{\mathcal{E}_2}, \ldots, \sqrt{\mathcal{E}_K}\right)$, $\boldsymbol{B}_1(\ell) = \operatorname{diag}\left(b_2(\ell), \ldots, b_K(\ell)\right)$, $\boldsymbol{S}_1^\ell$ is a $N \times (K-1)$ dimensional matrix whose columns are the signatures $\left\{\boldsymbol{s}_k^\ell\right\}_{k=2}^K$ and $\boldsymbol{\alpha}_1(\ell) = [\alpha_2(\ell), \ldots, \alpha_K(\ell)]$ is the $K-1$ dimensional fading coefficient vector. In equation (2) the first term on the right-hand-side (RHS) represents the contribution from the bit to be decoded, while the other terms represent the contributions from MAI ($\boldsymbol{z}(\ell)$) and the thermal noise ($\boldsymbol{n}(\ell)$), respectively. Finally, notice that $\boldsymbol{n}(\ell)$ is a white complex Gaussian vector with covariance matrix $2\mathcal{N}_0 \boldsymbol{I}_N$, with $\boldsymbol{I}_N$ the identity matrix of order N. The covariance matrix of the observables conditioned on the fading vector $\boldsymbol{\alpha}_1(\ell)$ has the following expression:

$$\boldsymbol{M}_{\boldsymbol{rr}|\boldsymbol{\alpha}_1}(\ell) \stackrel{\triangle}{=} \mathrm{E}\left[\boldsymbol{r}(\ell)\boldsymbol{r}^H(\ell)|\boldsymbol{\alpha}_1(\ell)\right] = \mathcal{E}_1|\alpha_1(\ell)|^2 \boldsymbol{s}_1^\ell \boldsymbol{s}_1^{\ell H} + \sum_{k=2}^K \mathcal{E}_k |\alpha_k(\ell)|^2 \boldsymbol{s}_k^\ell \boldsymbol{s}_k^{\ell H} + 2\mathcal{N}_0 \boldsymbol{I}_N \tag{3}$$

with $(\cdot)^H$ denoting conjugate transpose. The pseudocovariance matrix conditioned on $\boldsymbol{\alpha}_1(\ell)$ [12] is given by:

$$\boldsymbol{M}'_{\boldsymbol{rr}|\boldsymbol{\alpha}_1}(\ell) \stackrel{\triangle}{=} \mathrm{E}\left[\boldsymbol{r}(\ell)\boldsymbol{r}^T(\ell)|\boldsymbol{\alpha}_1(\ell)\right] = \mathcal{E}_1(\alpha_1(\ell))^2 \boldsymbol{s}_1^\ell \boldsymbol{s}_1^{\ell T} + \sum_{k=2}^K \mathcal{E}_k (\alpha_k(\ell))^2 \boldsymbol{s}_k^\ell \boldsymbol{s}_k^{\ell T} \tag{4}$$

with $(\cdot)^T$ denoting transpose. It is thus seen that, since $\boldsymbol{M}'_{\boldsymbol{rr}|\boldsymbol{\alpha}_1}$ is nonzero, the baseband equivalent of the CDMA signals is an improper random processes. This is also a straightforward, although not trivial, consequence of the fact that, since the radio-frequency CDMA signals, conditioned with respect to the fading coefficients, are not wide-sense stationary (WSS), the correlation properties of the corresponding complex envelopes are specified by four real functions, or, equivalently, by two complex functions. Likewise, the projections of these complex envelopes onto an orthonormal system are specified by two complex matrices, $\boldsymbol{M}_{\boldsymbol{rr}|\boldsymbol{\alpha}_1}$ and $\boldsymbol{M}'_{\boldsymbol{rr}|\boldsymbol{\alpha}_1}$, both of which being nonzero.

For future reference, we denote by $\mathcal{R}(\boldsymbol{A})$ the column space of $\boldsymbol{A}$, by $\mathcal{C}^N$ the space of the complex $N-$tuples on the complex field $\mathcal{C}$ with the usual internal and external operations, and by $\mathcal{S}(\ell) \stackrel{\triangle}{=} \mathcal{R}\left(\boldsymbol{S}_1^\ell\right)$ the *interference subspace* , namely the subspace of $\mathcal{C}^N$ spanned by the MAI (i.e. by $\{\boldsymbol{s}_k^\ell\}_{\ell,k}$ with $k = 2, \ldots, K$).

3 Detector Design

Any *linear* one-shot detector for user "1" implements a decision rule based on the projection of $\boldsymbol{r}(\ell)$ on a given vector:

$$\widehat{b}_1(\ell) = \operatorname{sgn}\left[\Re\left\{\boldsymbol{c}_1^H(\ell)\boldsymbol{r}(\ell)\right\}\right] \tag{5}$$

where $\operatorname{sgn}(\cdot)$ denotes the signum function, $\Re\{\cdot\}$ denotes real part, while the vector $\boldsymbol{c}_1(\ell) \in \mathcal{C}^N$ is a suitable direction, dictated by some optimality criterion, by the complexity constraints and by the information available at the receiver.

The most popular optimization criteria lead to the decorrelator and the MMSE techniques. If the desired signal does not belong to $\mathcal{S}(\ell)$, following [10], the decorrelator can be obtained as the unique solution to the following constrained maximization problem:

$$\widetilde{\boldsymbol{c}}_1(\ell) = \arg \max_{s.t. \boldsymbol{c}_1^H \boldsymbol{z}(\ell) = 0} \left\{ \frac{\Re\left\{\alpha_1(\ell)\boldsymbol{c}_1^H \boldsymbol{s}_1^\ell\right\}}{\|\boldsymbol{c}_1\|} \right\} \tag{6}$$

Likewise, the classical MMSE detector, according to [11, 6, 20], may be obtained as the solution to the problem

$$\widetilde{\boldsymbol{c}}_1(\ell) = \arg\min \mathrm{E}\left[\left|b_1(\ell) - \boldsymbol{c}_1^H \boldsymbol{r}(\ell)\right|^2\right] \tag{7}$$

or, alternatively, to the following constrained minimization of the output interference plus thermal noise energy:

$$\widetilde{\boldsymbol{c}}_1(\ell) = \arg \min_{s.t. \boldsymbol{c}_1^H \boldsymbol{s}_1^\ell = 1} \mathrm{E}\left[\left|\boldsymbol{c}_1^H (\boldsymbol{z}(\ell) + \boldsymbol{n}(\ell))\right|^2\right] \tag{8}$$

In spite of their being usually treated separately, the decorrelator (6), and MMSE receiver (8), can be subsumed under a single framework. In fact, both of them are solutions to a single general non linear constrained-optimization problem, viz.,

$$\begin{cases} \widetilde{\boldsymbol{c}}_1(\ell) = \arg \min_{\boldsymbol{c}_1 \in \mathcal{D}(\ell)} \|\boldsymbol{c}_1\| \\ \mathcal{D}(\ell) \stackrel{\triangle}{=} \left\{ \boldsymbol{c}_1 \in \mathcal{C}^N : \; \mathrm{E}_{\chi(\ell)} \left[\Re^2 \left\{\boldsymbol{c}_1^H \boldsymbol{w}(\ell)\right\}\right] = \min, \;\; s.t. \; \Re\left\{\alpha_1(\ell)\boldsymbol{c}_1^H \boldsymbol{s}_1^\ell\right\} = 1 \right\} \end{cases} \tag{9}$$

for different choices of the vector $\boldsymbol{w}(\ell)$. In (9) $\chi(\ell)$ is the vector of all parameters in $\boldsymbol{w}(\ell)$ which are not known to the receiver, and with respect to which we take the expectation. Specifically, here we are interested in two choices:

1. $\boldsymbol{w}(\ell) = \boldsymbol{z}(\ell) \stackrel{\triangle}{=} \boldsymbol{S}_1^\ell \boldsymbol{A}_1 \boldsymbol{B}_1(\ell) \boldsymbol{\alpha}_1(\ell)$

2. $\boldsymbol{w}(\ell) = \boldsymbol{z}(\ell) + \boldsymbol{n}(\ell)$

which yield, as the solution to (9), a family of decorrelators and a family of MMSE receivers, respectively. Since the expectation in (9) is with respect to all parameters in $\boldsymbol{w}(\ell)$ not known to the receiver, the solution to (9) depends on the prior information available at the receiver. To illustrate further, let us consider the following situations:

[**a**] The receiver has prior knowledge of the fading coefficients $\{\alpha_1(\ell)\}_\ell$, but not of the realizations of the fading coefficients of the other users which are modeled as random variables.

[**b**] The receiver has prior knowledge of the realizations of the fading coefficients of all users $\{\alpha_k(\ell)\}_{\ell,k}$.

In both situations we assume that in each signaling interval the receiver knows the signatures of all users. Finally notice that the operator $\Re\left\{\boldsymbol{x}^H \boldsymbol{y}\right\}$ (with $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{C}^N$) is not an admissible inner product in $\mathcal{C}^N$. Thus, unlike with the conventional optimization criteria (6) and (8), the constrained minimization problem (9) is not linearly constrained, and hence its solution requires special attention.

3.1 Unconditional linear receivers

First, let us focus on the situation [a]. In this case to solve the constrained minimization problem (9) is equivalent to solving:

$$\widetilde{\boldsymbol{c}}_1(\ell) = \arg \min_{s.t. \; \boldsymbol{c}_1 \in \mathcal{D}(\ell)} \|\boldsymbol{c}_1\| \tag{10}$$

with $\mathcal{D}(\ell) \equiv \left\{ \boldsymbol{c}_1 \in \mathcal{C}^N : \; \mathrm{E}_{\chi(\ell)} \left[\boldsymbol{c}_1^H \boldsymbol{w}(\ell)\right] = \min, \;\; \alpha_1(\ell)\boldsymbol{c}_1^H \boldsymbol{s}_1^\ell = 1\right\}$.

1. Assuming that $\boldsymbol{w}(\ell) = \boldsymbol{z}(\ell)$, the solution to (10) is given by the following expression:

$$\boldsymbol{c}_{\mathrm{UD}}(\ell) = \gamma_{UD} \alpha_1(\ell) \left(\boldsymbol{M}_{\boldsymbol{zz}}(\ell) + \mathcal{E}_1 |\alpha_1(\ell)|^2 \boldsymbol{s}_1^\ell \boldsymbol{s}_1^{\ell H}\right)^+ \boldsymbol{s}_1^\ell \tag{11}$$

where $\boldsymbol{M}_{\boldsymbol{zz}}(\ell) = E_{\chi(\ell)}[\boldsymbol{z}(\ell)\boldsymbol{z}(\ell)^H] = \boldsymbol{S}_1^\ell \boldsymbol{A}_1^2 \boldsymbol{S}_1^{\ell H}$ is the covariance matrix of the interference vector $\boldsymbol{z}(\ell)$ with $\chi(\ell) = [\boldsymbol{B}_1(\ell), \boldsymbol{\alpha}_1(\ell)]$, $(\cdot)^+$ denotes pseudo-inverse [15] and

$\gamma_{\rm UD} = 1/(\alpha_1(\ell)\boldsymbol{s}_1^{\ell H}\boldsymbol{M}_{\boldsymbol{zz}}^{+}(\ell)\boldsymbol{s}_1^{\ell})$. Notice that when $\boldsymbol{s}_1^{\ell} \notin \mathcal{S}(\ell)$, $\min \mathrm{E}\left[\boldsymbol{c}_1^H\boldsymbol{z}(\ell)\right]$ is equal to zero. As a consequence to solve the problem (10) for $\boldsymbol{w}(\ell) = \boldsymbol{z}(\ell)$ is equivalent to solving (6); thus (11) coincides with the classical Decorrelator detector given by (6). If, instead, $\boldsymbol{s}_1^{\ell} \in \mathcal{S}(\ell)$, then (11) depends also on the energy and on the variances of the fading coefficients $\{\alpha_k(\ell)\}_{\ell,k}$. We refer to (11) to as the *Unconditional* Decorrelator (UD).

2. When, instead, $\boldsymbol{w}(\ell) = \boldsymbol{z}(\ell) + \boldsymbol{n}(\ell)$, problem (10) admits as its solution the classical *Unconditional* MMSE (UM) receiver (8):

$$\boldsymbol{c}_{\rm UM}(\ell) = \gamma_{\rm UM}\alpha_1(\ell)\boldsymbol{M}_{\boldsymbol{rr}}^{-1}(\ell)\boldsymbol{s}_1^{\ell} \tag{12}$$

where $\boldsymbol{M}_{\boldsymbol{rr}}(\ell) = E_{\chi(\ell)}[\boldsymbol{r}(\ell)\boldsymbol{r}(\ell)^H]$ is the covariance matrix of the received vector $\boldsymbol{r}(\ell)$ with $\chi(\ell) = [\boldsymbol{B}_1(\ell), \boldsymbol{\alpha}_1(\ell), \boldsymbol{n}(\ell)]$ and $\gamma_{\rm UM} = 1/(\alpha_1(\ell)\boldsymbol{s}_1^{\ell H}\boldsymbol{M}_{\boldsymbol{rr}}^{-1}\boldsymbol{s}_1^{\ell})$.

3.2 Conditional linear receivers

Let us now consider situation [b], wherein the receiver has prior knowledge of the *realizations* of all fading coefficients, rather than of their ensemble features only. This might be the case of a base station in a cellular network. Consider the following results:

Proposition 1 *Assuming that $\boldsymbol{w}(\ell) = \boldsymbol{z}(\ell)$, define the $N \times 2N-$dimensional matrices defined as $\boldsymbol{F} = [\boldsymbol{I}\,\boldsymbol{0}]$ and $\boldsymbol{F}' = [\boldsymbol{0}\,\boldsymbol{I}]$; then the solution to the constrained problem (9) is written as:*

$$\boldsymbol{c}_{\rm LCD}(\ell) = \gamma_{\rm LCD}\boldsymbol{F}\left(\mathbf{Q}^{a}_{\boldsymbol{\alpha}_1}(\ell)\boldsymbol{F}^H\boldsymbol{s}_1^{\ell}\alpha_1(\ell) + \mathbf{Q}^{a*}_{\boldsymbol{\alpha}_1}(\ell)\boldsymbol{F}'^H\boldsymbol{s}_1^{\ell}\alpha_1^*(\ell)\right) \tag{13}$$

wherein $\mathbf{Q}^{a}_{\boldsymbol{\alpha}_1}(\ell)$ is the projector onto $\mathcal{N}\begin{pmatrix} \boldsymbol{M}_{\boldsymbol{zz}|\boldsymbol{\alpha}_1}(\ell) & \boldsymbol{M}'_{\boldsymbol{zz}|\boldsymbol{\alpha}_1}(\ell) \\ \boldsymbol{M}'^{*}_{\boldsymbol{zz}|\boldsymbol{\alpha}_1}(\ell) & \boldsymbol{M}^{*}_{\boldsymbol{zz}|\boldsymbol{\alpha}_1}(\ell) \end{pmatrix}$ with $\mathcal{N}(\boldsymbol{A})$ denoting the null space of the matrix $\boldsymbol{A}$ and finally $\gamma_{\rm LCD}$ ensures that $\Re\left\{\alpha_1(\ell)\boldsymbol{c}_{LCD}^H\boldsymbol{s}_1^{\ell}\right\} = 1$.

Solution (13) is referred to as the Linear-Conjugate Decorrelator (LCD).

Proposition 2 *Assuming $\boldsymbol{w}(\ell) = \boldsymbol{z}(\ell) + \boldsymbol{n}(\ell)$, the solution to the constrained minimization problem (9) is given by:*

$$\boldsymbol{c}_{\rm LCM}(\ell) = \gamma_{\rm LCM}\alpha_1(\ell)\boldsymbol{H}_{\boldsymbol{r}}^{-1}(\ell)\boldsymbol{s}_1^{\ell} - \alpha_1^*(\ell)\boldsymbol{M}_{\boldsymbol{rr}|\boldsymbol{\alpha}_1}^{-1}(\ell)\boldsymbol{M}'_{\boldsymbol{rr}|\boldsymbol{\alpha}_1}(\ell)\left(\boldsymbol{H}_{\boldsymbol{r}}^*(\ell)\right)^{-1}\boldsymbol{s}_1^{\ell} \tag{14}$$

with:

$$\boldsymbol{M}_{\boldsymbol{rr}|\boldsymbol{\alpha}_1}(\ell) = \mathrm{E}_{\chi(\ell)}[\boldsymbol{r}(\ell)\boldsymbol{r}^H(\ell)|\boldsymbol{\alpha}_1(\ell)], \qquad \boldsymbol{M}'_{\boldsymbol{rr}|\boldsymbol{\alpha}_1}(\ell) = \mathrm{E}_{\chi(\ell)}[\boldsymbol{r}(\ell)\boldsymbol{r}^T(\ell)|\boldsymbol{\alpha}_1(\ell)],$$

$$\boldsymbol{H}_{\boldsymbol{r}}(\ell) = \left(\boldsymbol{M}_{\boldsymbol{rr}|\boldsymbol{\alpha}_1}(\ell) - \boldsymbol{M}'_{\boldsymbol{rr}|\boldsymbol{\alpha}_1}(\ell)\left(\boldsymbol{M}^*_{\boldsymbol{rr}|\boldsymbol{\alpha}_1}(\ell)\right)^{-1}\boldsymbol{M}'^{*}_{\boldsymbol{rr}|\boldsymbol{\alpha}_1}(\ell)\right),$$

$\chi(\ell) = [\boldsymbol{B}_1(\ell), \boldsymbol{n}(\ell)]$ and γ_{LCM} such that $\Re\left\{\alpha_1(\ell)\boldsymbol{c}_{LCM}^H\boldsymbol{s}_1^{\ell}\right\} = 1$.

We refer to (14) as to the Linear-Conjugate MMSE (LCM) receiver. Notice that, unlike the situation [*a*], the proposed LCM receiver (14) does not coincide with the classical Conditional MMSE receiver with the same prior knowledge. The expression of the latter, apart from an irrelevant positive factor, is given by:

$$\boldsymbol{c}_{\rm CM}(\ell) = \alpha_1(\ell)\boldsymbol{M}_{\boldsymbol{rr}|\boldsymbol{\alpha}_1}^{-1}(\ell)\boldsymbol{s}_1^{\ell} \tag{15}$$

and can be obtained by solving (8) where the expectation in (8) is with respect to $\chi(\ell) = [\boldsymbol{B}_1(\ell), \boldsymbol{n}(\ell)]$. We refer to (15) as to Conditional MMSE (CM).

The receivers defined by (13) and (14) outperform the Unconditional Decorrelator (11) and the Unconditional and Conditional MMSE receiver, (12) and (15); the reason is that (13) and (14) fully exploit the information about the fading coefficients resorting to Linear/Conjugate processing, wherein not only the received vector but also its conjugate are processed.

Before proceeding further in our discussion, we provide some intuition as to why Linear-Conjugate techniques cannot be worse than classical linear optimization techniques, (6) and (8). The key point is that both decorrelator and MMSE receivers are *linear* receivers, achieving interference elimination by projecting the observables onto suitable subspaces. On the other hand, if the fading vector is known, the interference vector $\boldsymbol{z}(\ell)$ and the received vector $\boldsymbol{r}(\ell)$ do *not* represent complex envelopes of stationary radio-frequency signals. As a consequence, their correlation properties are described by a pair of complex covariance matrices: such information is *not* contained in the conditional covariance matrix of the vectors themselves, but in the pair of conditional matrices: $\boldsymbol{M}_{\boldsymbol{rr}|\boldsymbol{\alpha}_1}(\ell)$ and $\boldsymbol{M}'_{\boldsymbol{rr}|\boldsymbol{\alpha}_1}(\ell)$. Notice also that the unconditional pseudocovariance matrices $\mathrm{E}\left[\boldsymbol{z}(\ell)\boldsymbol{z}^T(\ell)\right]$ and $\mathrm{E}\left[\boldsymbol{r}(\ell)\boldsymbol{r}^T(\ell)\right]$ are zero (due to $\mathrm{E}\left[\alpha_k(\ell)\alpha_n(\ell)\right] = 0$), in keeping with the fact that, in the *ensemble* of the fading realizations, the vectors $\boldsymbol{z}(\ell)$ and $\boldsymbol{r}(\ell)$ *do* represent complex envelopes of stationary signals, whose correlation properties can thus be characterized through the conventional covariance matrix only. Thus, the conditional covariance matrix of the augmented observables contains additional information which is lost once an ensemble average is performed (see also [18]). It is worth noticing that the baseband solutions (13) and (14), (unlike the conventional (11) and (15)) achieve the same performance that radio-frequency (or intermediate-frequency) processing would be able to achieve.

3.2.1 Suboptimal receivers

Notice that both proposed receivers, (13) and (14), depend on the eigenvalues and eigenvectors of the covariance and pseudocovariance matrices of the interference vector $\boldsymbol{z}(\ell)$ conditioned on $\{\alpha_k(\ell)\}_{\ell,k}$. However, in long-code CDMA, the spreading codes as well as the corresponding covariance matrix $\boldsymbol{M}_{\boldsymbol{zz}}(\ell)$, like as the conditional covariance matrix $\boldsymbol{M}_{\boldsymbol{zz}|\boldsymbol{\alpha}_1}(\ell)$ and the pseudocovariance matrix $\boldsymbol{M}'_{\boldsymbol{zz}|\boldsymbol{\alpha}_1}(\ell)$, change in every symbol interval. This implies that the proposed optimal solutions, (13) and (14), have to be computed and updated symbol by symbol, thus possibly leading to a prohibitive computational effort in most cases. As a consequence, a less complex approach is desirable. To understand how to simplify the optimum structure, let us define the subset of $\mathcal{C}^{2N}$:

$$\mathcal{S} \triangleq \left\{ \boldsymbol{x}_a : \ \boldsymbol{x}_a = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{x}^* \end{pmatrix}, \boldsymbol{x} \in \mathcal{C}^N \right\}$$

It is easily seen that, with the following external operation (multiplication by the complex scalar γ):

$$\gamma \cdot \boldsymbol{x}_a = \begin{pmatrix} \gamma \boldsymbol{x} \\ \gamma^* \boldsymbol{x}^* \end{pmatrix}$$

and the usual component-wise vector sum as internal operation, $\mathcal{V} = \{\mathcal{S}, +, \cdot\}$ is a vector space on $\mathcal{C}$.

Since, on the other hand:

$$d^2(\boldsymbol{x}_a, \boldsymbol{y}_a) \triangleq \| \boldsymbol{x}_a - \boldsymbol{y}_a \|^2 = 2d^2(\boldsymbol{x}, \boldsymbol{y}) \qquad 2\Re\left\{\boldsymbol{x}^H \boldsymbol{y}\right\} = \boldsymbol{x}^H \boldsymbol{y} + \boldsymbol{x}^T \boldsymbol{y}^* = \boldsymbol{x}_a^H \boldsymbol{y}_a \tag{16}$$

then, by considering the new vector space $\mathcal{V}$ in lieu of $\mathcal{C}^N$ in (9), we can work with a linear constraint. As a consequence, define:

$$\mathbf{u}_{1a}^{\ell} = \begin{bmatrix} \alpha_1 \boldsymbol{s}_1^{\ell} \\ \alpha_1^* \boldsymbol{s}_1^{\ell} \end{bmatrix}, \quad \boldsymbol{U}_{1a}^{\ell} = \begin{bmatrix} \boldsymbol{S}_1^{\ell} \operatorname{diag}(\boldsymbol{\alpha}_1(\ell)) \\ \boldsymbol{S}_1^{\ell} \operatorname{diag}(\boldsymbol{\alpha}_1^*(\ell)) \end{bmatrix}, \tag{17}$$

$$\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a|\boldsymbol{\alpha}_1}(\ell) = \mathrm{E}_{\boldsymbol{B}_1(\ell)}\left[\boldsymbol{z}_a(\ell)\boldsymbol{z}_a^H(\ell)|\boldsymbol{\alpha}_1(\ell)\right] = \begin{pmatrix} \boldsymbol{M}_{\boldsymbol{z}\boldsymbol{z}|\boldsymbol{\alpha}_1}(\ell) & \boldsymbol{M}'_{\boldsymbol{z}\boldsymbol{z}|\boldsymbol{\alpha}_1}(\ell) \\ \boldsymbol{M}'^{*}_{\boldsymbol{z}\boldsymbol{z}|\boldsymbol{\alpha}_1}(\ell) & \boldsymbol{M}^{*}_{\boldsymbol{z}\boldsymbol{z}|\boldsymbol{\alpha}_1}(\ell) \end{pmatrix} \quad (18)$$

and denote by $\boldsymbol{U}_J^\ell$ the $2N \times J$ matrix whose columns form a linearly independent subset spanning the column space of matrix $\boldsymbol{U}_{1a}^\ell$. Then, with the definition $P_1(\ell) \triangleq \mathcal{E}_1|\alpha_1(\ell)|^2$, the solutions (13) and (14) to the constrained optimization problem (9) can be rewritten, except for an irrelevant positive factor, as:

$$\begin{aligned} \boldsymbol{c}_{\mathrm{LCD}} &= \boldsymbol{F}\Big(P_1(\ell)\mathbf{u}_{1a}^\ell\mathbf{u}_{1a}^{\ell H} + \boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a|\boldsymbol{\alpha}_1}(\ell)\Big)^\dagger \mathbf{u}_{1a}^\ell = \boldsymbol{F}\left(\boldsymbol{I} - \boldsymbol{U}_J^\ell(\boldsymbol{U}_J^{\ell H}\boldsymbol{U}_J^\ell)^{-1}\boldsymbol{U}_J^{\ell H}\right)\boldsymbol{s}_{1a}^\ell \\ \boldsymbol{c}_{\mathrm{LCM}} &= \boldsymbol{F}\Big(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a|\boldsymbol{\alpha}_1}(\ell) + 2\mathcal{N}_0\boldsymbol{I}_{2N}\Big)^{-1}\mathbf{u}_{1a}^\ell \end{aligned} \quad (19)$$

where $\boldsymbol{I}_{2N}$ is the $2N$-dimensional identity matrix. Based on (19), the $\boldsymbol{c}_{LCD}$ and the $\boldsymbol{c}_{LCM}$ vectors (13) and (14) can be expanded in an infinite power series of the conditional covariance matrix of the $2N-$dimensional vector $\boldsymbol{z}_a(\ell)$. As a consequence, the suboptimum approximation that we advocate consists of choosing $\widetilde{\boldsymbol{c}}_1(\ell)$ as corresponding to the minimum of (9) over a feasible constrained set. In this case after choosing the integer D depending on the allowable complexity, we obtain the parametric solutions:

$$\begin{aligned} \boldsymbol{c}_{\mathrm{SCD}}(\ell) &= \sum_{m=0}^{D-1} w_{\mathrm{SCD}}^m(\ell)\boldsymbol{U}_J^\ell\left(\boldsymbol{U}_J^{\ell H}\boldsymbol{U}_J^\ell\right)^m\boldsymbol{U}_J^{\ell H}\mathbf{u}_{1a}^\ell \\ \boldsymbol{c}_{\mathrm{SCM}}(\ell) &= \sum_{m=0}^{D-1} w_{\mathrm{SCM}}^m(\ell)\left(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a|\boldsymbol{\alpha}_1}(\ell) + 2\mathcal{N}_0\boldsymbol{I}\right)^m\mathbf{u}_{1a}^\ell \end{aligned} \quad (20)$$

where the vector weights, $\boldsymbol{w}_{(\cdot)}(\ell) = [w_{(\cdot)}^0(\ell), \ldots, w_{(\cdot)}^{D-1}(\ell)]$, are obtained by solving the constrained minimization problem (9). The optimal weight vectors are given by:

$$\boldsymbol{w}_{\mathrm{SCM}}(\ell)=\begin{pmatrix} \mathcal{H}_1^\ell, & \ldots, & \mathcal{H}_D^\ell \\ \vdots, & \ldots, & \vdots \\ \mathcal{H}_D^\ell, & \ldots, & \mathcal{H}_{(2D-1)}^\ell \end{pmatrix}^{-1}\begin{pmatrix} \mathcal{H}_1^\ell \\ \vdots \\ \mathcal{H}_D^\ell \end{pmatrix}, \quad \boldsymbol{w}_{\mathrm{SCD}}(\ell)=\begin{pmatrix} \mathcal{Q}_1^\ell, & \ldots, & \mathcal{Q}_D^\ell \\ \vdots, & \ldots, & \vdots \\ \mathcal{Q}_D^\ell, & \ldots, & \mathcal{Q}_{(2D-1)}^\ell \end{pmatrix}^{-1}\begin{pmatrix} \mathcal{Q}_1^\ell \\ \vdots \\ \mathcal{Q}_D^\ell \end{pmatrix} \quad (21)$$

where

$$\mathcal{H}_m^\ell \triangleq \mathbf{u}_{1a}^{\ell H}\left(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a|\boldsymbol{\alpha}_1}(\ell) + 2\mathcal{N}_0\boldsymbol{I}_{2N}\right)^m\mathbf{u}_{1a}^\ell, \qquad \mathcal{Q}_m^\ell \triangleq \mathbf{u}_{1a}^{\ell H}\left(\boldsymbol{U}_J^\ell\boldsymbol{U}_J^{\ell H}\right)^{m+2}\mathbf{u}_{1a}^\ell \quad (22)$$

We refer to (20) as Suboptimal Linear-Conjugate Decorrelator (SCD) and Suboptimal Linear-Conjugate MMSE detector (SCM). Eq. (21) implies that in order to compute the optimal weights (i.e. those solving the constrained minimization problem (9)) we have to invert a $D \times D$ Hankel matrix. The computational effort for on line inversion of the matrix in (21) is feasible when D is very small (this is equivalent, in order to obtain good performance, to saying that the number of users is not too large). If D is not too small the weight optimization approach cannot be adopted but we can resort to the design guidelines proposed at the end of this paper.

4 Performance Analysis

4.1 Nonasymptotic Analysis

The uncoded performance of DS/CDMA communication systems is usually assessed by evaluating their Bit-Error-Rate (BER), Near-Far Resistance and signal-to-noise ratio at the output of the receivers. As regards the newly proposed conditional receivers (13), (14) and (20), the BER can be computed as in the case of any linear receiver [15]. The statistical average with respect to the interference term $\boldsymbol{z}(\ell)$ (i.e. the average with respect to the MAI fading coefficients and bit patterns realizations) cannot be carried out in closed form, and numerical evaluation is called for. In order to numerically evaluate the performance of the proposed receivers, (13) and (14), and to carry out comparisons with the classical linear multiuser receivers, we focus, in the following, on a CDMA system employing Gold codes with spreading length $N = 31$ operating in Rayleigh fading. In Figure 1 we have represented the error probability versus the average received signal-to-noise ratio $\mathsf{SNR}= (\mathcal{E}_1)/(2\mathcal{N}_0)$, for the conditional MMSE receiver (15), labeled as "CM", for the newly proposed MMSE receiver (14), labeled as "LCM", and for the suboptimal linear-conjugate MMSE detector (20), labeled as "SCM" with $D = 10$. We assume $K = 25$ and interfering users with a power level of 5dB above the desired signal. The results clearly show the superiority of the new strategy, which largely outperforms the conditional MMSE receiver (15). For comparison purposes, we also report the error probability corresponding to an uncoded BPSK transmission over a MAI-free flat-fading channel [2].

Based on (19) and applying standard multiuser detection analysis techniques [15], it easy to see that the near-far resistance of (13) and (14) for a given set of fading coefficients $\{\alpha_k(\ell)\}_{\ell,k}$, can be written as:

$$\eta_1(\ell) = 1 - \frac{1}{2}\mathbf{u}_{1a}^{\ell H}\boldsymbol{U}_J^{\ell}(\boldsymbol{U}_J^{\ell H}\boldsymbol{U}_J^{\ell})^{-1}\boldsymbol{U}_J^{\ell H}\mathbf{u}_{1a}^{\ell} \tag{23}$$

On the other hand, it is well know that the conventional decorrelator (11), the unconditional MMSE receiver (12) and the conditional MMSE detector (15) have the same Near-Far resistance given by [15] :

$$\eta_1^C(\ell) = 1 - \boldsymbol{s}_1^{\ell H}\boldsymbol{S}_L^{\ell}\left(\boldsymbol{S}_L^{\ell H}\boldsymbol{S}_L^{\ell}\right)^{-1}\boldsymbol{S}_L^{\ell H}\boldsymbol{s}_1^{\ell} \tag{24}$$

with $\boldsymbol{S}_L^{\ell}$ the $N \times L$ matrix whose columns are a linearly independent subset spanning the column space of matrix $\boldsymbol{S}_1^{\ell}$. It can be shown that $\eta_1(\ell) \geq \eta_1^C(\ell)$, but the general proof is very long and tedious. In order to corroborate our claim, Figure 3 contrasts the classical linear detection structures (24) to the newly proposed receivers (23) in terms of near-far resistance, which is represented versus K. Again, it is seen that the new approach allows a noticeable performance improvement.

The signal-to-noise ratio for the Linear-Conjugate receivers (13) and (14) can be written as:

$$\begin{gathered}\mathsf{SNR}_{LCD}(\ell) = P_1\mathbf{u}_{1a}^{\ell H}\left(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a|\boldsymbol{\alpha}_1}(\ell) + P_1\mathbf{u}_{1a}^{\ell}\mathbf{u}_{1a}^{\ell H}\right)^{-1}\mathbf{u}_{1a}^{\ell}, \\ \mathsf{SNR}_{LCM}(\ell) = P_1\mathbf{u}_{1a}^{\ell H}\left(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a|\boldsymbol{\alpha}_1}(\ell) + 2\mathcal{N}_0\boldsymbol{I}\right)^{-1}\mathbf{u}_{1a}^{\ell}\end{gathered} \tag{25}$$

while for the classical linear detection structures (11), (12) and (15) we have:

$$\begin{gathered}\mathsf{SNR}_{UD}(\ell) = P_1\boldsymbol{s}_1^{\ell H}\left(\boldsymbol{M}_{\boldsymbol{z}\boldsymbol{z}}(\ell) + P_1\boldsymbol{s}_1^{\ell}\boldsymbol{s}_1^{\ell H}\right)^{-1}\boldsymbol{s}_1^{\ell}, \\ \mathsf{SNR}_{UM}(\ell) = P_1\boldsymbol{s}_1^{\ell H}(\boldsymbol{M}_{\boldsymbol{z}\boldsymbol{z}}(\ell) + 2\mathcal{N}_0\boldsymbol{I})^{-1}\boldsymbol{s}_1^{\ell}, \quad \mathsf{SNR}_{CM}(\ell) = P_1\boldsymbol{s}_1^{\ell H}\left(\boldsymbol{M}_{\boldsymbol{z}\boldsymbol{z}|\boldsymbol{\alpha}_1}(\ell) + 2\mathcal{N}_0\boldsymbol{I}\right)^{-1}\boldsymbol{s}_1^{\ell}\end{gathered} \tag{26}$$

Eqs. (25) and (26) for the signal-to-noise ratio and Eqs. (23) and (24) for the Near-Far resistance, describe the performance of the proposed receivers, (13) and (14), and of the conventional receivers (11) and (15), and can be computed for specific choices of the signature sequences and of the bit interval. However, it is not easy to obtain a qualitative insight directly from those formulas, as the gain in signal-to-noise ratio and in Near-Far Resistance achieved by Linear-Conjugate receivers with respect to the linear classical techniques cannot be directly seen.

4.2 Asymptotic Analysis

In practice in the case of long sequences (and in other cases, e.g., with sequences distorted by random multipath, or in a network with random access) it is reasonable to assume that the spreading sequences are randomly and independently chosen. In this case, the performance measures of the receivers can be modeled as random variables, since they are a function of the spreading sequences. In the following we average the performance measures with respect to the random sequences and we also show that in the limiting regime $K \to \infty$, the random performance measures converge to deterministic quantities. The elements of the signatures $s^{\ell}_{k,n} \in \{-1/\sqrt{N}, 1/\sqrt{N}\} \quad n = 0, \ldots, N-1$ are chosen equally likely and independently for all (k, n, ℓ). Notice that similar results can be obtained for nonbinary sequences whose elements are independent and identically distributed (i.i.d.), with zero mean and variance $1/N$. The normalization ensures that $E[\|\boldsymbol{s}^{\ell}_k\|] = 1$; however in this more general case we need also the assumption $E[|s^{\ell}_{k,n}|^4] < \infty$. The ratio of the number of users to the number of dimensions is denoted by

$$\beta \triangleq K/N.$$

For further use notice that

$$\boldsymbol{S}^{\ell H}_{1a} \boldsymbol{S}^{\ell}_{1a} = \boldsymbol{\Phi}^*(\ell) \boldsymbol{S}^{\ell T}_1 \boldsymbol{\Gamma}(\ell) \boldsymbol{S}^{\ell}_1 \boldsymbol{\Phi}(\ell) + \boldsymbol{\Phi}(\ell) \boldsymbol{S}^{\ell T}_1 \boldsymbol{\Gamma}(\ell) \boldsymbol{S}^{\ell}_1 \boldsymbol{\Phi}(\ell)^*$$

where $\boldsymbol{\Phi}(\ell) \triangleq \mathrm{diag}(\arg(\alpha_2(\ell)), \ldots, \arg(\alpha_K(\ell)))$ and $\boldsymbol{\Gamma}(\ell) \triangleq \mathrm{diag}(|\alpha_2(\ell)|^2, \ldots, |\alpha_K(\ell)|^2)$.
Thus $\forall m$

$$\begin{aligned} \mathrm{E}\left[\mathrm{trace}\left(\boldsymbol{S}^{\ell}_{1a} \boldsymbol{S}^{\ell H}_{1a}\right)^m\right] &= \mathrm{E}\left[\mathrm{trace}\left(\boldsymbol{S}^{\ell H}_{1a} \boldsymbol{S}^{\ell}_{1a}\right)^m\right] \\ &= \mathrm{E}\left[\mathrm{trace}\left(\boldsymbol{\Phi}^* \boldsymbol{S}^{\ell T}_1 \boldsymbol{\Gamma} \boldsymbol{S}^{\ell}_1 \boldsymbol{\Phi} + \boldsymbol{\Phi} \boldsymbol{S}^{\ell T}_1 \boldsymbol{\Gamma} \boldsymbol{S}^{\ell}_1 \boldsymbol{\Phi}^*\right)\right] \end{aligned}$$

For later use, we recall here the following proposition [15]:

Proposition 3 *Let $\boldsymbol{A}$ be a $N \times K$ dimensional matrix of i.i.d. complex random variables with zero mean and variance $1/N$. Then the empirical distribution function of the eigenvalues of the $K-$dimensional random matrix $\boldsymbol{A}^H \boldsymbol{A}$, converges, as $K \to \infty$, to the cumulative distribution function of the probability density:*

$$f_\beta(x) = \left[1 - \frac{1}{\beta}\right]^+ \delta(x) + \frac{\sqrt{(x-a)^+(b-x)^+}}{2\pi\beta x} \tag{27}$$

where $z^+ = \max\{0, z\}$ and $a = (1-\sqrt{\beta})^2$ and $b = (1+\sqrt{\beta})^2$.

A generalization of Proposition 3 yields the following result [17]:

Proposition 4 *Let $\boldsymbol{A}$ be a $N \times K$ dimensional matrix defined as in Proposition 3. Let $\boldsymbol{Z}$ be a $K \times K$ random Hermitian nonnegative-definite matrix independent of $\boldsymbol{A}$ such that G^K, the empirical distribution of its eigenvalues, converges almost surely to a non-random limit G, as $K \to \infty$. Then almost surely the empirical distribution of the eigenvalues of the matrix $\boldsymbol{A}^H \boldsymbol{Z} \boldsymbol{A}$ converges almost surely to a non-random limit F whose Stieltjes transform $\mathcal{F}(z)$ with $z \in \{\mathcal{C}/\{0\}\}$ satisfies:*

$$\mathcal{F}(z) = \frac{1}{-z + \beta \int \frac{\nu dF(\nu)}{1+\nu\mathcal{F}(z)}} \tag{28}$$

Propositions 3 and 4 allow the following propositions to be derived:

Proposition 5 *Let $\boldsymbol{A}^H\boldsymbol{Z}\boldsymbol{A}$ have a bounded spectral radius and let $\boldsymbol{s}$ be a vector with i.i.d. entries with zero means and finite variances. Then*

$$\boldsymbol{s}^H(\boldsymbol{A}^H\boldsymbol{Z}\boldsymbol{A})^m\boldsymbol{s} \overset{a.s.}{\rightarrow} E[trace\,(\boldsymbol{A})^m] \triangleq \int_0^\infty \lambda(\boldsymbol{A}^H\boldsymbol{Z}\boldsymbol{A})G(\lambda)d\lambda$$

where $G(\lambda)$ its cumulative distribution function given in Proposition 4.

Proposition 6 *The empirical distribution function of the eigenvalues of the $K-$dimensional random matrix $\left(\boldsymbol{\Phi}^*\boldsymbol{S}_1^{\ell T}\boldsymbol{S}_1^{\ell}\boldsymbol{\Phi}+\boldsymbol{\Phi}\boldsymbol{S}_1^{\ell T}\boldsymbol{S}_1^{\ell}\boldsymbol{\Phi}^*\right)$ converges to the cumulative distribution function of the probability density:*

$$f_\beta(x) = \left[1-\frac{2}{\beta}\right]^+\delta(x) + \frac{\sqrt{(x-2a)^+(2b-x)^+}}{\pi\beta x} \tag{29}$$

with $a=\left(1-\sqrt{\beta/2}\right)^2$ and $b=\left(1+\sqrt{\beta/2}\right)^2$.

Proof: In order to prove Proposition 6 it is sufficient to show that:

$$\mathrm{E}\left[\mathrm{trace}\left(\boldsymbol{\Phi}^*\boldsymbol{S}_1^{\ell T}\boldsymbol{S}_1^{\ell}\boldsymbol{\Phi}+\boldsymbol{\Phi}\boldsymbol{S}_1^{\ell T}\boldsymbol{S}_1^{\ell}\boldsymbol{\Phi}^*\right)^m\right] = \mathrm{E}\left[\mathrm{trace}\left((\mathbf{T}\mathbf{T}^H)^m\right)\right] \qquad \forall m$$

with $\mathbf{T}$ a $2N-$dimensional matrix whose entries are i.i.d. complex-valued variables with variance $\frac{1}{\sqrt{N}}$ and zero mean. The proof can be obtained by applying some standard combinatorial techniques, but the general proof for every m is very long and tedious. Another proof relies upon a simple application of the following Lemma from free-probability theory [17, 5] as applied to random matrices:

Lemma 1 *Let $\boldsymbol{A}$ and $\{\mathbf{Q}_1,\boldsymbol{O}_1,\mathbf{Q}_2,\boldsymbol{O}_2\}$ be a set of n-dimensional random matrices with $\boldsymbol{A}$ independent of $\{\mathbf{Q}_1,\boldsymbol{O}_1,\mathbf{Q}_2,\boldsymbol{O}_2\}$ such that*

$$\mathbf{Q}_j\boldsymbol{O}_j=\boldsymbol{I} \qquad and \qquad E\left[trace\left(\mathbf{Q}_j\boldsymbol{O}_i\right)\right]=0 \;\; i\neq j$$

thenfor any $m=2,3,\ldots$ the quantity

$$\frac{1}{n}E[trace\,(\mathbf{Q}_1\boldsymbol{A}\boldsymbol{O}_1+\mathbf{Q}_2\boldsymbol{A}\boldsymbol{O}_2)^m]$$

is specified only by the moments $\frac{1}{n}E[trace\,(\mathbf{Q}_1\boldsymbol{A}\boldsymbol{O}_1)^m]$ and $\frac{1}{n}E[trace\,(\mathbf{Q}_2\boldsymbol{A}\boldsymbol{O}_2)^m]$.

Since $\boldsymbol{\Phi}\boldsymbol{\Phi}^*=\boldsymbol{I}_k$ and $\mathrm{E}[\mathrm{trace}\,(\boldsymbol{\Phi}\boldsymbol{\Phi})]=0$ from Lemma 1, the moments $\mathrm{E}\left[\mathrm{trace}\left(\boldsymbol{S}_{1a}^{\ell H}\boldsymbol{S}_{1a}^{\ell}\right)^m\right]$ depend only on

$$\mathrm{E}\left[\mathrm{trace}\left(\boldsymbol{\Phi}^*\boldsymbol{S}_1^{\ell T}\boldsymbol{\Gamma}\boldsymbol{S}_1^{\ell}\boldsymbol{\Phi}\right)^m\right] = \mathrm{E}\left[\mathrm{trace}\left(\boldsymbol{S}_1^{\ell H}\boldsymbol{\Gamma}\boldsymbol{S}_1^{\ell}\right)^m\right] = \mathrm{E}\left[\mathrm{trace}\left(\boldsymbol{\Sigma}_1^H\boldsymbol{\Gamma}\boldsymbol{\Sigma}_1\right)^m\right]$$

and on

$$\mathrm{E}\left[\mathrm{trace}\left(\boldsymbol{\Phi}\boldsymbol{S}_1^{\ell T}\boldsymbol{\Gamma}\boldsymbol{S}_1^{\ell}\Phi^*\right)^m\right] = \mathrm{E}\left[\mathrm{trace}\left(\boldsymbol{S}_1^{\ell H}\boldsymbol{\Gamma}\boldsymbol{S}_1^{\ell}\right)^m\right] = \mathrm{E}\left[\mathrm{trace}\left(\boldsymbol{\Sigma}_2^H\boldsymbol{\Gamma}\boldsymbol{\Sigma}_2\right)^m\right]$$

with $\boldsymbol{\Sigma}_i$ a $N\times(K-1)$ dimensional matrix with i.i.d. entries with zero mean and variance $1/N$. As a consequence:

$$\mathrm{E}\left[\mathrm{trace}\left(\boldsymbol{S}_{1a}^{\ell H}\boldsymbol{S}_{1a}^{\ell}\right)^m\right] = \mathrm{E}\left[\mathrm{trace}\left(\boldsymbol{\Sigma}_1^H\boldsymbol{\Gamma}\boldsymbol{\Sigma}_1+\boldsymbol{\Sigma}_2^H\boldsymbol{\Gamma}\boldsymbol{\Sigma}_2\right)\right]$$

For $\boldsymbol{\Gamma}=\boldsymbol{I}_K$ Proposition 3 concludes the proof of Proposition 6. □

It is well known [15, 16] that the Near-Far resistance $\eta_1^C(\ell)$ of detectors (11), (12) and (15) converges almost surely as $K\to\infty$ to:

$$\lim_{K\to\infty}\eta_1^C(\ell)=[1-\beta]^+ \tag{30}$$

Based on Proposition 6, we prove the following:

Proposition 7 *The average of $\eta_1(\ell)$ w.r.t. the signatures of all users is lower bounded by:*

$$E[\eta_1(\ell)] \geq \left[1 - \frac{K-1}{2N}\right]^+$$

Furthermore, the Near-Far resistance $\eta_1(\ell)$ for any $0 \leq \beta \leq \infty$ converges almost surely to:

$$\eta_1^\infty = \lim_{K\to\infty} \eta_1(\ell) = \left[1 - \frac{\beta}{2}\right]^+ \tag{31}$$

Proof: Based on (23) we have:

$$\begin{aligned} \mathrm{E}\left[\eta_1(\ell)\right] &= \mathrm{E}\left[1 - \frac{1}{2}\mathbf{u}_{1a}^{\ell H}\boldsymbol{U}_J^\ell(\boldsymbol{U}_J^{\ell H}\boldsymbol{U}_J^\ell)^{-1}\boldsymbol{U}_J^{\ell H}\mathbf{u}_{1a}^\ell\right] = \\ &= 1 - \mathrm{E}\left[\frac{1}{2N}\mathrm{trace}\left(\boldsymbol{U}_J^\ell(\boldsymbol{U}_J^{\ell H}\boldsymbol{U}_J^\ell)^{-1}\boldsymbol{U}_J^{\ell H}\right)\right] = 1 - \frac{\mathrm{E}\left[L_a\right]}{2N} \end{aligned} \tag{32}$$

where L_a is the number of non zero eigenvalues of the matrix $\boldsymbol{S}_{1a}^{\ell H}\boldsymbol{S}_{1a}^\ell$. Since the signatures are random, L_a is a random variable itself. Since $\mathrm{E}[L_a] \leq \min\{K-1, 2N\}$, we have that:

$$\mathrm{E}\left[\eta_1(\ell)\right] \geq \left[1 - \frac{K-1}{2N}\right]^+$$

Using (5) and on (6) and since $\boldsymbol{s}_{1a}$ is a independent of $\boldsymbol{U}_J^\ell$, we obtain:

$$\eta_1^\infty = \lim_{K\to\infty} \eta_1(\ell) = 1 - \frac{\mathrm{E}\left[L_a\right]}{2N} \tag{33}$$

where $\dfrac{L_a}{2N} = 1 - F_\lambda^N(0) = \dfrac{\beta}{2}$ is equal to the proportion of the K eigenvalues of the matrix $\boldsymbol{S}_{1a}^\ell\boldsymbol{S}_{1a}^{\ell H}$ that lie above zero, where $F_\lambda^N(x)$ is the empirical distribution function of the eigenvalues of the $2N$-dimensional random matrix $\boldsymbol{S}_{1a}^\ell\boldsymbol{S}_{1a}^{\ell H}$. Thus (31) follows from Lemma 1. □

Let us focus now on the (finite SNR) multiuser efficiency [15, 16] of the proposed receivers. Define a nonnegative random variable $|A|^2$ whose distribution is the limit distribution of $\left\{\frac{\mathcal{E}_k}{\mathcal{E}_1}|\alpha_k|^2,\ k = 1,\dots K\right\}$. For the classical linear detection structures (11), (12) and (15), the multiuser efficiency for $K \to \infty$ converges almost surely to the solution $\eta_{CM}^\infty(\mathsf{SNR})$ of [7, 13]:

$$\eta + \beta\mathrm{E}\left[\frac{|A|^2\ \mathsf{SNR}\,\eta}{1 + |A|^2\ \mathsf{SNR}\,\eta}\right] = 1 \tag{34}$$

One of our main results is:

Proposition 8 *The multiuser efficiency of the Linear-Conjugate receivers (13) and (14) converges almost surely to the solution $\eta_{LCM}^\infty(\mathsf{SNR})$ of:*

$$\eta + \frac{\beta}{2}E\left[\frac{2\,\mathsf{SNR}\,|A|^2\ \eta}{1 + 2\,\mathsf{SNR}\,|A|^2\ \eta}\right] = 1 \tag{35}$$

In Figure 4 we show the asymptotic multiuser efficiency of the Linear-Conjugate MMSE receiver versus the ratio $\beta = K/N$ for a fixed average received signal-to noise ratio $\mathsf{SNR} = 10dB$ and for equal powers. For comparison purposes the spectral efficiency of the conditional MMSE receiver (CM) for equal powers [13] is also given.

So far we have only analyzed the unconstrained-complexity solutions to (9). Let us focus on the suboptimal structure receivers (20) that we obtain as solution to (9) when complexity constraints are imposed. As we have already noticed, the optimal weights (21) can be computationally intensive. Instead we propose an asymptotic weighting that assumes to operate in a limiting regime (i.e. the number of users and processing gain are large with fixed ratio and random spreading sequences). Using Proposition 5 we have:

$$\lim_{K\to\infty} \boldsymbol{s}_1^{\ell H} \left(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a|\boldsymbol{\alpha}_1}(\ell) + 2\mathcal{N}_0 \boldsymbol{I} \right)^m \boldsymbol{s}_1^{\ell} \overset{a.s.}{=} E\left[\text{trace} \left(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a|\boldsymbol{\alpha}_1} + 2\mathcal{N}_0 \boldsymbol{I} \right)^m \right]$$

$$\lim_{K\to\infty} \boldsymbol{s}_1^{\ell H} \left(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a|\boldsymbol{\alpha}}(\ell) \right)^m \boldsymbol{s}_1^{\ell} \overset{a.s.}{=} E\left[\text{trace} \left(\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a|\boldsymbol{\alpha}_1} \right)^m \right]$$

It follows that:

$$\boldsymbol{w}_{\text{SCM}}^{\infty} = \mathrm{E} \begin{bmatrix} (\Lambda + 2\mathcal{N}_0) & (\Lambda + 2\mathcal{N}_0)^2, & \ldots, & (\Lambda + 2\mathcal{N}_0)^D \\ (\Lambda + 2\mathcal{N}_0)^2 & (\Lambda + 2\mathcal{N}_0)^3, & \ldots, & (\Lambda + 2\mathcal{N}_0)^{D+1} \\ \vdots & \vdots, & \ldots, & \vdots \\ (\Lambda + 2\mathcal{N}_0)^D & (\Lambda + 2\mathcal{N}_0)^{D+1}, & \ldots, & (\Lambda + 2\mathcal{N}_0)^{2D-1} \end{bmatrix}^{-1} \mathrm{E} \begin{pmatrix} \Lambda \\ \Lambda^2 \\ \vdots \\ \Lambda^D \end{pmatrix} \tag{36}$$

$$\boldsymbol{w}_{\text{SCD}}^{\infty} = \mathrm{E} \begin{bmatrix} \Lambda^3 & \Lambda^4, & \ldots, & \Lambda^{D+2} \\ \Lambda^4 & \Lambda^5, & \ldots, & \Lambda^{D+3} \\ \vdots & \vdots, & \ldots, & \vdots \\ \Lambda^{D+2} & \Lambda^{D+1}, & \ldots, & \Lambda^{2D+1} \end{bmatrix}^{-1} \mathrm{E} \begin{pmatrix} \Lambda^3 \\ \Lambda^4 \\ \vdots \\ \Lambda^{D+2} \end{pmatrix} \tag{37}$$

where the expectations are with respect to Λ, whose distribution is the asymptotic eigenvalue distribution of $\boldsymbol{M}_{\boldsymbol{z}_a\boldsymbol{z}_a|\boldsymbol{\alpha}_1}$. The entries of the matrices given in (36) and (37) can be computed by applying some tools of combinatorial theory [19]. In fact it can be shown that $E[\lambda^m]$ is a function of β and $\overline{P^m} = E[P]$ where the average is with respect to cumulative distribution $F(P)$; namely we have:

$$E[\Lambda^m] = \sum_{n=1}^{m} \beta^n \sum_{(j_1,\ldots,j_n)\in Z_n^{(m)}} \mathcal{E}_1^n c(j_1,\ldots,j_n) \mathrm{E}\left[|A|^{2\ j_1} \right] \cdots \mathrm{E}\left[isr^{j_n} \right] \cdot \beta^n$$

where the sum is over all possible $j_1, \ldots, j_n > 0$ such that $j_1 + \cdots j_n = n$ and for all $1 \le n \le m$ [19]. We refer to (20), where the weights are given by (37), to as the Asymptotic Suboptimal Linear-Conjugate Decorrelator (ASD) and Asymptotic Suboptimal Linear-Conjugate MMSE detector (ASM). We refer to (20) as Suboptimal Linear-Conjugate Decorrelator (SCD) and Suboptimal Linear-Conjugate MMSE detector (SCM). In Figure (**??**) we have represented the error probability versus the average received signal-to-noise ratio SNR= $(\mathcal{E}_1)/(2\mathcal{N}_0)$, for the newly proposed MMSE receiver (14), labeled as "LCM", and for the asymptotic suboptimal linear-conjugate MMSE detector (20), labeled as "ASM" with $D = 15$. We assume $N = 100$ and $K = 30$ and interfering users with equal powers. For comparison purposes, we also report the error probability corresponding to an uncoded BPSK transmission over a MAI-free flat-fading channel.

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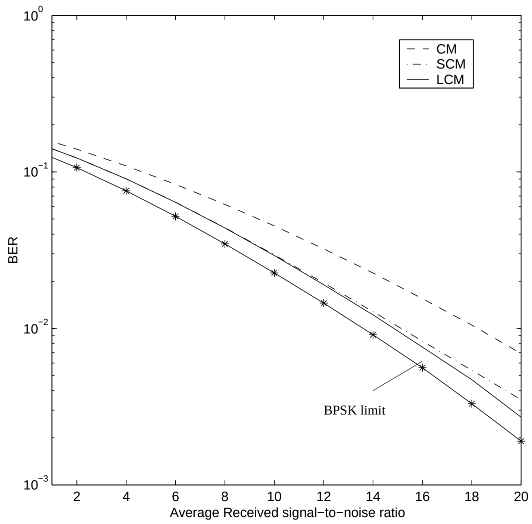

Figure 1: BER of the classical and of the newly proposed MMSE detectors, LCM and SCM, for $K = 25$.

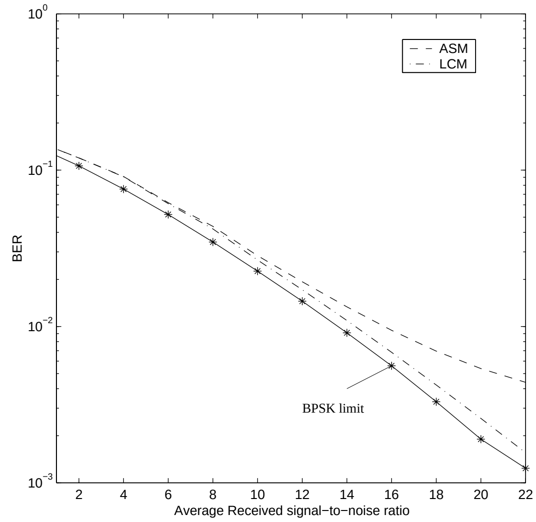

Figure 3: BER of the classical and of the newly proposed MMSE detectors, LCM and ASM, for $K = 30$ and $N = 100$.

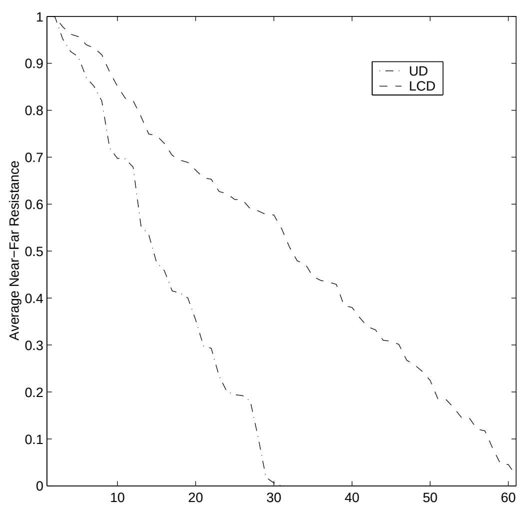

Figure 2: Near-Far resistance of the classical decorrelator and of the newly proposed Linear-Conjugate decorrelator, versus the users number.

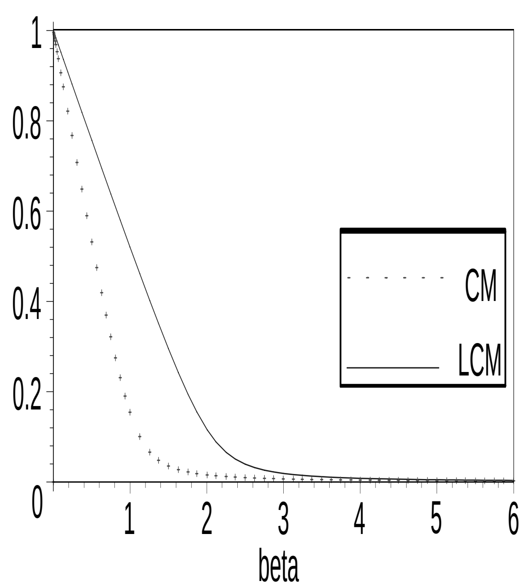

Figure 4: LCM multiuser efficiency for SNR = 10dB.