Near–Far Resistance of Multiuser Detectors in Asynchronous Channels

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Abstract—**We consider an asynchronous code-division multiple-access environment in which the receiver has knowledge of the signature waveforms of all the users. Under the assumption of white Gaussian background noise, we compare detectors by their worst case bit error rate in a low background noise near–far environment where the received energies of the users are unknown to the receiver and are not necessarily similar.**

Conventional single-user detection in a multiuser channel is not near–far resistant, while the substantially higher performance of the optimum multiuser detector requires exponential complexity in the number of users. We explore suboptimal demodulation schemes which exhibit a low order of complexity while not exhibiting the impairment of the conventional single-user detector. Attention is focused on linear detectors, and it is shown that there exists a linear detector whose bit-error-rate is independent of the energy of the interfering users. Moreover it is shown that the near–far resistance of optimum multiuser detection can be achieved by a linear detector. The optimum linear detector for worst-case energies is found, along with existence conditions, which are always satisfied in the models of practical interest.

I. Introduction

The near–far problem is the principal shortcoming of current radio networks using direct-sequence spread-spectrum multiple-access (DS/SSMA) communication systems. Those systems achieve multiple-access capability by assigning a distinct signature waveform to each user from a set of waveforms with low mutual crosscorrelations. Then, when the sum of the signals modulated by several asynchronous users is received, it is possible to recover the information transmitted by correlating the received process with replicas of the assigned signature waveforms. This demodulation scheme is conventionally used in practice, and its performance is satisfactory if two conditions are satisfied: first, the assigned signals need to have low crosscorrelations for all possible relative delays between the data streams transmitted by the asynchronous users, and second the powers of the received signals cannot be very dissimilar. If either of these conditions is not fulfilled, then the bit-error-rate and the antijamming capability of the conventional detector are degraded substantially. The reason why system performance is unacceptable when the received energies are dissimilar even with good (i.e., quasiorthogonal) signal constellations, is that the output of each correlator or matched filter contains a spurious component which is linear in the amplitude of each of the interfering users. Thus, as the multiuser interference grows, the bit-error-rate increases until the conventional detector is unable to recover the messages transmitted by the weak users.

Due to the severe reduction of the multiple-access capability and the increase of vulnerability to hostile sources caused by the near–far problem and its ubiquity in networks with dynamically changing topologies (such as mobile radio), its alleviation has been a target of researchers in the area for several years. However, success has been very limited and the only remedies implemented in practice have been to use power control or to design signals with more stringent crosscorrelation properties, which as we have noted, does not eliminate the near–far problem.

The viewpoint of this paper is that the near–far problem is not an inherent shortcoming of DS/SSMA systems, but of the conventional single-user detector. The optimum multiuser detector was obtained in [1] and was shown to be near–far resistant in the sense that a (very good) performance level can be guaranteed regardless of the relative energy of the transmitters. The optimum multiuser detector consists of a bank of matched filters and a Viterbi algorithm whose complexity is exponential in the number of users. In decentralized applications (where each receiver is only interested in demodulating the data sent by one transmitter), it is possible to drastically reduce the complexity of the optimum receiver (without compromising performance) by neglecting all but the comparatively powerful interferers. However, in this paper we propose a receiver (which we refer to as the decorrelating receiver) whose complexity is only linear in the number of users, and whose bit-error-rate is independent of the powers of the interferers at the receiver. Moreover, the decorrelating receiver achieves optimum near–far resistance (in a sense to be defined precisely in the sequel). The only requirement is the knowledge of the signature waveforms of the interfering users, and, in particular, no knowledge of the received energies is required, in contrast to the optimum receiver.

This paper generalizes the results obtained in [7] in the case of synchronous code-division multiple-access channels. Other recent attempts to derive detectors for multiuser channels include [9]–[11].

The multiple-access channel model considered in this paper is spelled out in Section II, as well as the general structure of the proposed detector. In Section III, we present the performance measure of interest, the near–far resistance and we show that the near–far resistance of the optimum multiuser detector can be achieved by a linear detector (the decorrelating detector), which is explicitly obtained in Section IV, as well as its implementable version as a linear time-invariant system. Section V gives a numerical comparison of the error probabilities of the decorrelating receiver and the conventional receiver in a scenario of practical interest.

II. Multiuser Communication Model

Let the receiver input signal be

$$r(t) = S(t, \boldsymbol{b}) + n(t) \tag{2.1}$$

where $n(t)$ is white Gaussian noise with power spectral density σ^2 and

$$S(t, \boldsymbol{b}) = \sum_{i=-M}^{M} \sum_{k=1}^{K} b_k(i) \sqrt{w_k(i)} \tilde{s}_k(t - iT - \tau_k) \tag{2.2}$$

is the element of $\mathcal{L}_2$ (the Hilbert space of square-integrable functions) which contains the information sequence $\boldsymbol{b} = \{\boldsymbol{b}(i) = [b_1(i), \cdots, b_K(i)], b_k(i) \in \{-1, 1\}, k = 1, \cdots, K; i =$

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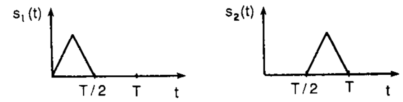


Fig. 1. Example of signature waveforms which can violate the LIA.

$-M, \cdots, M\}$, $\tilde{s}_k(t)$ is the normalized signature waveform of user k and is zero outside the interval $[0, T]$, and $w_k(i)$ is the received energy of user k in the ith time slot. Let $N = 2M + 1$ be the length of the transmitted sequence. Without loss of generality it is assumed that the users are numbered such that their delays satisfy $0 \le \tau_1 \le \cdots \le \tau_K < T$. The normalized signal $\tilde{S}(t, b)$ is the receiver input signal corresponding to unit energies.

Define the vector space $L = \{x = [x(-M), \cdots, x(M)] = [[x_1(-M), \cdots, x_K(-M)], \cdots, [x_1(M), \cdots, x_K(M)]]^T, \; x_k(i) \in \mathbb{R}, k = 1, \cdots, K, i = -M, \cdots, M\}$, (each element of which can be equivalently viewed as a sequence of N $(K*1)$-vectors or as one single $(NK*1)$-vector), and define the (k, i)th unit vector $u^{k,i}$ in L with components $u_j^{k,i}(l) = \delta_{kj}\delta_{li}$. Note that the set of possible transmitted sequences b is a subset of L, obtained by restricting the components of the vector x to take on the values ± 1. Let $\langle \cdot , \cdot \rangle$ denote the usual inner product on $\mathcal{L}_2$, i.e., the integral of the product over the region of support, with induced norm $\| \cdot \|$. Henceforth, we make the following assumption on $\tilde{S}(t, b)$.

1) Linear Independence Assumption (LIA):

$$\forall \, v \in L, \, v \neq 0 \Rightarrow \left\| \tilde{S}(t, v) \right\| \neq 0. \tag{2.3}$$

In other words, no matter what the received energies are, the received signal does not vanish everywhere if at least one of the users has transmitted a symbol. This condition fails to hold only in pathological nonpractical cases with very heavy crosscorrelation between the signals, such as the two-user example in Fig. 1. There if the delay between the users is $T/2$, the received signal can be identically zero although transmissions have been made [this happens if, for all i, $b_2(i) = -b_1(i)$]. It is shown in Appendix II that such a situation will arise with probability zero if the *a priori* unknown delays are uniformly distributed, which is the case in the asynchronous channel used by noncooperating users. Basically, in order to violate the LIA, a subset of the users must be effectively synchronous and the modulating signals of this subset have to be heavily correlated. The LIA will be in effect in the rest of the paper. If it is removed all the given results can be generalized in a manner analogous to the treatment of the synchronous transmission case [7].

The sampled output of the normalized matched filter for the ith bit of the kth user, $i = -M, \cdots, M$, is

$$y_k(i) = \int_{iT+\tau_k}^{iT+T+\tau_k} r(t)\tilde{s}_k(t - iT - \tau_k)\, dt \tag{2.4}$$

$$= \int_{-\infty}^{\infty} S(t, b)\tilde{s}_k(t - iT - \tau_k)\, dt + \int_{-\infty}^{\infty} n(t)\tilde{s}_k(t - iT - \tau_k)\, dt \tag{2.5}$$

where the second equality is valid since the signals are zero outside $[0, T]$. It is well established (e.g., [1]) that the whole sequence y of outputs of the bank of K matched filters, with components $y_k(i)$ given by (2.5), for $k = 1, \cdots, K, i = -M, \cdots, M$, is a sufficient statistic for decision on the most likely transmitted information sequence b. The multiuser demodulation problem which needs to be solved at the receiver is to recover the transmitted sequence $b \in L$ from the sequence $y \in L$. Motivated by the state of the art—where the choice lies between the optimum multiuser detector, which is of exponential complexity and the ad hoc single user detector whose performance degrades to zero for sufficiently high interference energy—we define a class of simple detectors and optimize performance within this class, to obtain an acceptable error probability versus complexity tradeoff.

A *linear detector* for bit i of user k is characterized by $v^{k,i} \in L$. The decision of the detector is given by the polarity of the inner product of $v^{k,i}$ and the vector y of matched filter outputs, which is equal to

$$\sum_{l=-M}^{M} \sum_{j=1}^{K} v_j^{k,i}(l) y_j(l) = \int_{-\infty}^{\infty} \tilde{S}(t, wb)\tilde{S}(t, v^{k,i})\, dt + n_{k,i} \tag{2.6}$$

$$= \langle \tilde{S}(t, wb), \tilde{S}(t, v^{k,i}) \rangle + n_{k,i} \tag{2.7}$$

where for any information sequence b, wb will denote the sequence of amplitudes $wb = \{[\sqrt{w_1(i)}b_1(i), \cdots, \sqrt{w_K(i)}b_K(i)], i = -M, \cdots, M\}$. $n_{k,i}$ is the noise component at the output of the cascade of matched filter, sampler and detector, hence is a Gaussian zero-mean random variable with variance given by

$$E[n_{k,j}^2] = \sum_{l,i} v_k(l) v_j(i) \int_{-\infty}^{\infty} \sigma^2 \tilde{s}_k(t - lT - \tau_k)\tilde{s}_j(t - iT - \tau_j)\, dt$$

$$= \sigma^2 \left\| \tilde{S}(t, v^{k,i}) \right\|^2. \tag{2.8}$$

The receiver decides on the ith bit of the kth user according to the rule

$$\hat{b}_k(i) = \operatorname{sgn} \sum_{l=-M}^{M} \sum_{j=1}^{K} v_j^{k,i}(l) y_j(l) \tag{2.9}$$

$$= \operatorname{sgn}(\langle \tilde{S}(t, wb), \tilde{S}(t, v^{k,i}) \rangle + n_{k,i}). \tag{2.10}$$

Wherever it is clear from the context, the superscripts k, i will be omitted.

2) Matrix Notation: It is convenient to introduce the following compact notation. Define the $K*K$ normalized signal crosscorrelation matrices $R(l)$ whose entries are given by

$$R_{kj}(l) = \int_{-\infty}^{\infty} \tilde{s}_k(t - \tau_k)\tilde{s}_j(t + lT - \tau_j)\, dt. \tag{2.11}$$

Then, since the modulating signals are zero outside $[0, T]$

$$R(l) = 0 \; \forall \; |l| > 1, \tag{2.12}$$

$$R(-l) = R^T(l), \tag{2.13}$$

and, if the users are numbered according to increasing delays, $R(1)$ is an upper triangular matrix with zero diagonal. Also let $W(l) = \operatorname{diag}([\sqrt{w_1(l)}, \cdots, \sqrt{w_K(l)}])$. With this notation the matched filter outputs for $l = \{-M, \cdots, M\}$ can be written in vector form as (cf., [8])

$$y(l) = R(-1)W(l+1)b(l+1) + R(0)W(l)b(l) + R(1)W(l-1)b(l-1) + n(l), \tag{2.14}$$

as can be seen for each component by inserting (2.1) into (2.4). We adopt the convention that $b(-M-1) = b(M+1) = 0$. $n(l)$ is the matched filter output noise vector, with autocorrelation matrix given by

$$E[n(i)n^T(j)] = \sigma^2 R(i - j). \tag{2.15}$$

The entries of the matrices $R(i)$, $i = -1, 0, 1$ are obtained at the receiver by cross-correlating appropriately delayed replicas of the normalized signature waveforms according to (2.11). Note that no additional complexity is hereby required of the receiver, since knowledge of the normalized signature waveforms and the capability to lock onto the respective delays are necessary for matched filtering and sampling at the instant of maximal signal-to-noise ratio.

In contrast to (2.5) the asynchronous nature of the problem is clearly transparent in (2.14). To make this notation more compact

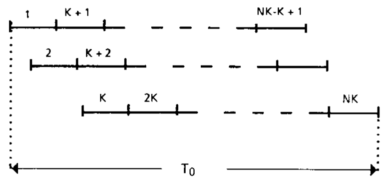


Fig. 2. Equivalent synchronous transmitted sequence.

we define the $NK*NK$ symmetric block-Toeplitz matrix $\mathcal{R}$ and the $NK*NK$ diagonal matrix $\mathcal{W}$, as follows:

$$\mathcal{R} = \begin{pmatrix} \boldsymbol{R}(0) & \boldsymbol{R}(-1) & 0 & \cdots & 0 \\ \boldsymbol{R}(1) & \boldsymbol{R}(0) & \boldsymbol{R}(-1) & & \vdots \\ 0 & \boldsymbol{R}(1) & \boldsymbol{R}(0) & \ddots & 0 \\ \vdots & & \ddots & \ddots & \boldsymbol{R}(-1) \\ 0 & \cdots & 0 & \boldsymbol{R}(1) & \boldsymbol{R}(0) \end{pmatrix}, \quad (2.16)$$

$$\mathcal{W} = \operatorname{diag}([\sqrt{w_1(-M)}, \cdots, \sqrt{w_K(-M)}, \cdots, \sqrt{w_1(M)}, \cdots, \sqrt{w_K(M)}]). \quad (2.17)$$

In this notation the matched filter output vector $\boldsymbol{y}$ depends on $\boldsymbol{b}$ via, from (2.14)

$$\boldsymbol{y} = \mathcal{R}\mathcal{W}\boldsymbol{b} + \boldsymbol{n}. \quad (2.19)$$

The matrix $\mathcal{R}$ can be interpreted as the cross-correlation matrix for an equivalent synchronous problem where the whole transmitted sequence is considered to result from $N*K$ users, labeled as shown in Fig. 2, during one transmission interval of duration $T_e = N*T + \tau_K - \tau_1$. Then the results presented here for finite transmission length can be derived via analysis of synchronous multiuser communication, as done in [7]. However, the approach taken in this paper is more general and gives more insight into the nature of the problem. The limit $N \to \infty$ is considered in Section IV-B.

The decision made on the ith bit of the kth user at the output of the detector $\boldsymbol{v}$ is:

$$\hat{b}_k(i) = \operatorname{sgn} \boldsymbol{v}^T \boldsymbol{y} = \operatorname{sgn} \boldsymbol{v}^T(\mathcal{R}\mathcal{W}\boldsymbol{b} + \boldsymbol{n}). \quad (2.20)$$

As for the inner product, for all $\boldsymbol{x}, \boldsymbol{y}$ in L

$$\langle \tilde{S}(t, \boldsymbol{x}), \tilde{S}(t, \boldsymbol{y}) \rangle = \boldsymbol{x}^T \mathcal{R} \boldsymbol{y}. \quad (2.21)$$

It can be seen from (2.21) and from (2.3) that $\mathcal{R}$ is positive definite.

III. Near-Far Resistance

The main performance measure we are interested in is the bit-error-rate in the high signal-to-background noise region. Thus, even though the background thermal noise is not neglected, our main focus will be on the underlying performance degradation due to multiple-access interference. This performance degradation is conveniently quantified by the *asymptotic efficiency* which was introduced in [1]-[2], and is defined as follows. Let $P_k(\sigma)$ denote the bit-error-rate of the kth user when the spectral level of the background white Gaussian noise is σ^2, and let $e_k(\sigma)$ be such that $P_k(\sigma) = Q(\sqrt{e_k(\sigma)}/\sigma)$.[1]

Then, $e_k(\sigma)$ is actually the energy that the kth user would require to achieve bit-error-rate $P_k(\sigma)$ in the same white Gaussian channel but without interfering users. Hence, we refer to $e_k(\sigma)$ as the *effective energy* of the kth user, and the *efficiency* or ratio between the effective and actual energies $e_k(\sigma)/w_k$ is a number between 0 and 1 which characterizes the performance loss due to the existence of other users in the channel. Thus, the *asymptotic efficiency* (for high SNR) of a transmitter whose bit-error-rate curve and energy are given by $P_k(\sigma)$ and w_k, respectively, is

$$\eta_k = \lim_{\sigma \to 0} \frac{e_k(\sigma)}{w_k} = \sup \left\{ 0 \le r \le 1;\ \lim_{\sigma \to 0} P_k(\sigma)/Q\left(\frac{\sqrt{r w_k}}{\sigma}\right) < \infty \right\} \quad (3.1)$$

where the last equation follows immediately upon substitution of $P_k(\sigma)$ by its expression in terms of the effective energy. In order to visualize intuitively the asymptotic efficiency, note that the logarithm of the bit-error-rate $P_k(\sigma)$ decays asymptotically with the same slope as the logarithm of the bit-error-rate of a single-user with energy $\eta_k w_k$. Therefore, if $\lim_{\sigma \to 0} P_k(\sigma) > 0$, (i.e., there is an irreducible probability of error even in the absence of background noise), then the asymptotic efficiency is zero. Conversely, nonzero asymptotic efficiency implies that the bit-error-rate goes to zero (as $\sigma \to 0$) exponentially in $1/\sigma^2$.

While asymptotic efficiency and low-noise bit-error-rate are equivalent performance measures, asymptotic efficiency has the advantage of being analytically tractable and of resulting in explicit expressions for the detectors we are interested in. For example, while the probability of error of the optimum multiuser detector does not admit an explicit expression, its asymptotic efficiency is given by [2]

$$\eta_{k,i} = \frac{1}{w_k(i)} \min_{\epsilon \in Z_k} \|\tilde{S}(t, w\epsilon)\|^2 \quad (3.2)$$

where Z_k is the set of error-sequences $\epsilon = \{\epsilon(i) \in \{-1, 0, 1\}^K,\ i = -M, \cdots, M,\ \epsilon_k(i) = 1\}$ that affect the ith bit of the kth user. It was shown in [3] (see also [15]) that the numerical computation of the asymptotic efficiency of optimum multiuser detection given by (3.2) is an NP-complete combinatorial optimization problem.

In an environment where the transmission energies change in time, e.g., if the transmitters are mobile, a performance measure of interest for any detector is its kth *user near-far resistance*, $\overline{\eta_{k,i}}$, which is defined for each detector as its worst case asymptotic efficiency for bit i of user k over all possible energies of the other (interfering and noninterfering) bits, i.e.,

$$\overline{\eta_{k,i}} = \inf_{\substack{w_j(l) \ge 0 \\ (j,l) \ne (k,i)}} \eta_{k,i}. \quad (3.3)$$

In our definition of near-far resistance we model the most general case where the energies of the users are allowed to be time-dependent. This captures the worst case operating conditions of the detector, which are, for example, encountered in mobile radio communication, due to positioning and tracking variations. In the case where the energies are constrained to be arbitrary but nonvarying the present near-far resistance is a lower bound. That case is not amenable to closed-form analysis, since one has to deal with a combinatorial optimization problem.

For illustration consider the two-user case. If the user energies are constant over time, i.e., $w_1(i) = w_1$, $w_2(i) = w_2$, the asymptotic efficiency of the optimal multiuser detector given by (3.2) reduces to [2]:

$$\eta_1 = \min\left\{1,\ 1 + \frac{w_2}{w_1} - 2\max\{|\rho_{12}|, |\rho_{21}|\}\frac{\sqrt{w_2}}{\sqrt{w_1}},\ 1 + 2\frac{w_2}{w_1} - 2(|\rho_{12}| + |\rho_{21}|)\frac{\sqrt{w_2}}{\sqrt{w_1}}\right\}$$

and hence

$$\eta_{\min} \triangleq \min_{\substack{w_2 \\ w_1 \text{ const.}}} \eta_1 = \min\left\{1 - \rho_{12}^2,\ 1 - \rho_{21}^2,\ 1 - \rho_{12}^2 - \rho_{21}^2 + \frac{(|\rho_{12}| - |\rho_{21}|)^2}{2}\right\}, \quad (3.4)$$

[1] $Q(x) = \int_x^\infty (1/\sqrt{2\pi}) e^{-v^2/2}\, dv$.

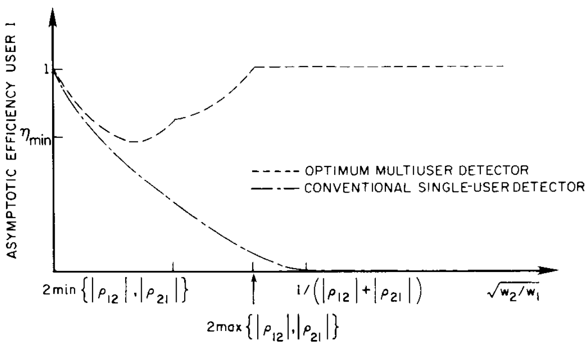


Fig. 3. Asymptotic efficiencies in the two-user case for infinite transmitted sequence length, when the user energies are constant over time (here we chose $|\rho_{12}|, |\rho_{21}| = 0.3, 0.5$).

and analogously for user 2 where $\rho_{12} = R_{12}(0)$ and $\rho_{21} = R_{12}(1)$. The dependence of η_1 for constant energies on the energy ratio is shown in Fig. 3. Note that the optimal multiuser detector is near-far resistant, and in fact has an asymptotic efficiency of unity for sufficiently powerful interference ([2]). Note also that in this case three different error-sequences minimize (3.2) for different values of w_2/w_1, as can be seen from the discontinuity points of the derivative of η. The minimum of η over constant energies, $\eta_{\min}$, is an upper bound on the near-far resistance of optimum multiuser detection $\bar{\eta}$, which is the minimum asymptotic efficiency over unconstrained energies.

The near-far resistance of the optimal multiuser detector is important since it is the least upper bound on the near-far resistance of any detector, and a measure of the relative performance of any suboptimal detector. From (3.2) and the definition of near-far resistance it is equal to

$$\overline{\eta_{k,i}} = \inf_{\substack{w_j(l)\geq 0\\ (j,l)\neq(k,i)}} \frac{1}{w_k(i)} \min_{\epsilon\in Z_k} \|\tilde{S}(t, w\epsilon)\|^2 \tag{3.5}$$

$$= \inf_{\substack{w_j(l)\geq 0\\ (j,l)\neq(k,i)}} \min_{\epsilon\in Z_k} \left\| \tilde{S}\left(t, \frac{1}{\sqrt{w_k(i)}} w\epsilon\right)\right\|^2 \tag{3.6}$$

$$= \inf_{\substack{y\in L\\ y_k(i)=1}} \|\tilde{S}(t, y)\|^2. \tag{3.7}$$

In Section IV, we obtain a closed-form expression for (3.7) as the reciprocal of the (k, i)th diagonal element (see footnote 2) of the inverse of $\mathcal{R}$. Hence, the near-far resistance of optimum multiuser resistance is guaranteed to be nonzero because of the linear independence assumption of (2.3), which ensures that $\mathcal{R}$ is invertible.

We now turn to the performance analysis of the linear detectors introduced above. The probability of error at decision upon $b_k(i)$ of the linear detector v is, from (2.10):

$$P_k(i) = P(\hat{b}_k(i) \neq b_k(i)) \tag{3.8}$$

$$= P(\langle \tilde{S}(t, wb), \tilde{S}(t, v)\rangle + n_{k,i} < 0 | b_k(i) = 1). \tag{3.9}$$

The equality follows since the hypotheses $+1, -1$ are assumed equally likely. Let B be the set of possible transmitted sequences. From (2.8) $n_{k,i}$ is a zero-mean Gaussian random variable with variance $\sigma^2 \|\tilde{S}(t, v)\|^2$, hence the probability of error in (3.9) is a sum of Q-functions, one for each possible interfering bit-combination. For $\sigma \to 0$ the Q-function with the smallest argument dominates the error probability, hence from (3.1), since the expression below can be shown (cf. [15]) to be upper bounded by 1, the asymptotic efficiency achieved by the linear detector v for the ith bit of the kth user is

$$\eta_{k,i}(v) = \frac{1}{w_k(i)} \max{}^2 \left\{0, \min_{\substack{b\in B\\ b_k(i)=1}} \frac{\langle \tilde{S}(t, wb), \tilde{S}(t, v)\rangle}{\|\tilde{S}(t, v)\|}\right\}. \tag{3.10}$$

Knowledge of the asymptotic efficiency of a linear detector is equivalent to knowledge of the worst case probability of error over the bit sequences of the interfering users, since this error probability, which is a Q-function, is set equal to $Q(\sqrt{\eta_{k,i}(v) w_k(i)}/\sigma)$ to obtain (3.10).

For illustration consider the conventional single-user detector in the two-user case. We have $v = u^{k,i}$ (recall that $u^{k,i}$ is the (k, i)th unit vector in the space L of linear detectors). If the user energies are constant over time, i.e., $w_1(i) = w_1$, $w_2(i) = w_2$, the asymptotic efficiency of the conventional single-user detector is found from (3.10) to be

$$\eta_1^c = \max{}^2 \left\{0, 1 - (|\rho_{12}| + |\rho_{21}|) \frac{\sqrt{w_2}}{\sqrt{w_1}}\right\} \tag{3.11}$$

and analogously for user 2. The dependence of η_1^c for constant energies on the energy ratio is shown in Fig. 3. Note that the asymptotic efficiency of the conventional single-user detector is zero for sufficiently high interference energy ($\sqrt{w_2}/\sqrt{w_1} > 1/(|\rho_{12}| + |\rho_{21}|)$). This implies that its near-far resistance is zero, which is what we want to remedy.

There are three quantities of interest in this communication environment, on the one hand the transmitted bit-sequence and the set of energies, both of which depend only on the transmitters and determine the *operating points* for the receiver, and on the other hand the data-processing detector v at the receiver, which we called a *linear detector*. In determining which linear detector to choose at the receiver a useful procedure is the *minimax* approach, in which the design goal is to optimize the worst case performance of the receiver over the class of operating points. Thus we are interested in finding the *maximin linear detector*, whose worst case performance over all allowable input sequences is the highest in the class of linear detectors. The following result quantifies the performance of the maximin detector, in the sequel denoted by v^*.

Proposition 1: There exists a linear detector (which is independent of the received energies) that achieves optimum near-far resistance (i.e., the near-far resistance of the optimum multiuser detector). ●

Proof: From (3.10) the asymptotic efficiency of the linear detector v is

$$\eta_{k,i}(v) = \frac{1}{w_k(i)} \max{}^2 \left\{0, \min_{\substack{b\in B\\ b_k(i)=1}} \frac{\langle \tilde{S}(t, wb), \tilde{S}(t, v)\rangle}{\|\tilde{S}(t, v)\|}\right\} \tag{3.12}$$

$$\times \min_{\substack{b\in B\\ b_k(i)=1}} \frac{1}{w_k(i)} \max{}^2 \left\{0, \frac{\langle \tilde{S}(t, wb), \tilde{S}(t, v)\rangle}{\|\tilde{S}(t, v)\|}\right\} \tag{3.13}$$

$$= \min_{\substack{b\in B\\ b_k(i)=1}} \frac{1}{w_k(i)} \max{}^2 \left\{0, \frac{b^T \mathcal{W}\mathcal{R}v}{\sqrt{v^T \mathcal{R} v}}\right\} \tag{3.14}$$

where in the last equality we have used the compact matrix notation of (2.21) for simplicity. We are interested in the linear detector with the highest worst case asymptotic efficiency, i.e., whose near-far resistance is

$$\overline{\eta_{k,i}(v^*)} = \sup_{\substack{v\in L\\ \|\tilde{S}(t,v)\|\neq 0}} \inf_{\substack{w_j(l)\geq 0\\ (j,l)\neq(k,i)}} \eta_{k,i}(v) \tag{3.15}$$

$$= \sup_{\substack{v\in L\\ v^T\mathcal{R}v\neq 0}} \underbrace{\inf_{\substack{w_j(l)\geq 0\\ (j,l)\neq(k,i)}} \min_{\substack{b\in B\\ b_k(i)=1}}} \frac{1}{w_k(i)} \max{}^2 \left\{0, \frac{b^T \mathcal{W}\mathcal{R}v}{\sqrt{v^T \mathcal{R} v}}\right\} \tag{3.16}$$

$$= \sup_{\substack{v \in L \\ y^T \mathcal{R} v \neq 0}} \inf_{\substack{y \in L \\ y_k(i)=1}} \max^2 \left\{0, \frac{y^T \mathcal{R} v}{\sqrt{v^T \mathcal{R} v}}\right\} \tag{3.17}$$

$$= \max^2 \left\{0, \sup_{\substack{v \in L \\ v^T \mathcal{R} v \neq 0}} \inf_{\substack{y \in L \\ y_k(i)=1}} \frac{y^T \mathcal{R} y}{\sqrt{v^T \mathcal{R} v}}\right\} \tag{3.18}$$

where we have set $y_j(l) = b_j(l)\sqrt{w_j(l)}/\sqrt{w_k(i)}$ for the third equality. Let $M(v, y)$ denote the penalty function $y^T \mathcal{R} v/\sqrt{v^T \mathcal{R} v}$ where the first argument is from the set of detectors and the second from the set of operating points, both specified in (3.18). We show in Appendix 1 that $M(v, y)$ has a saddle point, i.e.,

$$\sup_{\substack{v \in L \\ v^T \mathcal{R} v \neq 0}} \inf_{\substack{v \in L \\ y_k(i)=1}} \frac{y^T \mathcal{R} v}{\sqrt{v^T \mathcal{R} v}} = \inf_{\substack{y \in L \\ y_k(i)=1}} \sup_{\substack{v \in L \\ v^T \mathcal{R} v \neq 0}} \frac{y^T \mathcal{R} v}{\sqrt{v^T \mathcal{R} v}}, \tag{3.19}$$

which establishes the existence of v^* and hence

$$\overline{\eta_{k,i}(v^*)} = \max^2 \left\{0, \inf_{\substack{y \in L \\ y_k(i)=1}} \sup_{\substack{v \in L \\ v^T \mathcal{R} v \neq 0}} \frac{y^T \mathcal{R} v}{\sqrt{v^T \mathcal{R} v}}\right\} \tag{3.20}$$

$$= \max^2 \left\{0, \inf_{\substack{y \in L \\ y_k(i)=1}} \sqrt{y^T \mathcal{R} y}\right\} \tag{3.21}$$

$$= \inf_{\substack{y \in L \\ y_k(i)=1}} \|\tilde{S}(t, y)\|^2 \tag{3.22}$$

$$= \overline{\eta_{k,i}} \tag{3.23}$$

where the second equality is obtained in (A.1), the third line follows since $\mathcal{R}$ is nonnegative definite and the last equality was obtained in (3.7). ◇

The reason why the near-far optimum linear receiver achieves the same near-far resistance as the optimum receiver can be understood as follows. Let Ω be the set of multiuser signals modulated by all positive amplitudes, i.e., $\Omega = \{\tilde{S}(t, y),\ y \in L\}$ and let Ξ denote the subset of Ω such that the amplitude of the ith symbol of the kth user is fixed to 1, i.e., $\Xi = \{\tilde{S}(t, y),\ y \in L,\ y_k(i) = 1\}$ (note that Ξ is a convex set, and because of the LIA it does not include the origin). Since the penalty function in (3.18) is invariant to scaling of v and the operator $\mathcal{R}$ is positive definite, (3.18) can be rewritten as

$$\overline{\eta_{k,i}(v^*)} = \max^2 \left\{0, \sup_{\substack{v \in L \\ |\tilde{S}(t,v)|=1}} \inf_{\substack{y \in L \\ y_k(i)=1}} \langle \tilde{S}(t, y), \tilde{S}(t, v)\rangle\right\} \tag{3.24}$$

$$= \max^2 \left\{0, \sup_{\substack{v \in \Omega \\ |v|=1}} \inf_{y \in \Xi} \langle y, v\rangle\right\}. \tag{3.25}$$

Therefore the kth user decorrelating filter can be viewed as the unit-norm multiuser waveform whose minimum inner product with the elements of Ξ is highest. But since Ξ is a convex set, that signal is a scaled version of the closest vector in Ξ to the origin (Fig. 4), and its near-far resistance [cf. (3.22)] is the norm squared of that vector. But, as (3.7) indicates, the square of the distance from Ξ to the origin is precisely the near-far resistance of the optimum detector.

Equation (3.7) leads to a nice intuitive interpretation of near-far resistance. Rewrite this equation, using the definition of $\tilde{S}(t, \cdot)$, as

$$\overline{\eta_{k,i}} = \inf_{\substack{y_j(l) \in R \\ (j,l) \neq (k,i)}} \left\| \tilde{s}_k(t - iT - \tau_k) + \sum_{(j,l) \neq (k,i)} y_j(l) \tilde{s}_j(t - lT - \tau_j) \right\|^2 \tag{3.26}$$

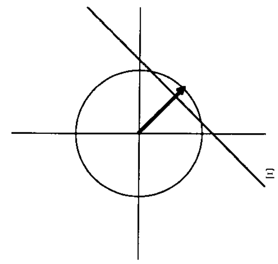


Fig. 4. Interpretation of near-far resistance. Vector in boldface corresponds to decorrelating filter.

Letting $\{y_j(l)\}$ vary over the admissible set, the second term above generates all points of a linear subspace which includes the origin, therefore the infimum in (3.26) is the distance of $\tilde{s}_k(t - iT - \tau_k)$ to this space, i.e.,

$$\overline{\eta_{k,i}} = d^2(\tilde{s}_k(t - iT - \tau_k),\ \text{span}\,\{\tilde{s}_j(t - lT - \tau_j),\ (j, l) \neq (k, i)\}) \tag{3.27}$$

where $d(a, b)$ denotes the Euclidean distance between the $\mathcal{L}_2$ elements a and b. In the synchronous case because the time-support is disjoint, the infimum in (3.26) is achieved when $y_j(l) = 0,\ l \neq i$, and (3.27) reduces to

$$\overline{\eta_k} = d^2(\tilde{s}_k(t),\ \text{span}\,\{\tilde{s}_j(t),\ j \neq k\}), \tag{3.28}$$

i.e., the kth user near-far resistance in a synchronous channel is the square of the *distance of the kth user signal to the space spanned by the signals of the interfering users*. Viewing the asynchronous problem in terms of the equivalent synchronous system with $N*K$ users and period NT, the near-far resistance of asynchronous communication allows for the same interpretation. Note, however, that the shifted versions $s_k(t - lT - \tau_k), l \neq i$ of the kth user signal affect the near-far resistance of the ith symbol of user k.

The following section characterizes the linear detector that achieves the optimum near-far resistance anticipated by Proposition 1.

IV. The Decorrelating Detector

We first assume N to be finite, as in the case in all communication environments, and characterize the linear filter which achieves the near-far resistance of optimum multiuser detection. This filter is nonstationary for finite N. The limit as $N \to \infty$ is then considered, yielding a stationary noncausal limiting filter, and hence, after appropriate truncation of the noncausal part, an approximation of the near-far optimal linear filter which can be implemented easily.

A. The Finite Sequence Length Case

Definition: A decorrelating detector $\boldsymbol{d}^{k,i}$ for the ith bit of the kth user is a linear detector for which

$$\mathcal{R}\boldsymbol{d}^{k,i} = \boldsymbol{u}^{k,i} \tag{4.1}$$

or equivalently, from (2.21), $\langle \tilde{S}(t, v), \tilde{S}(t, \boldsymbol{d}^{k,i})\rangle = v_k(i)$, for all v in L.

Existence: By the LIA, statement (4.2) below holds for all k, i. Hence, the following equivalences show the existence of the decorrelating detectors for each bit of each user.

$$\forall\, v \in L \text{ with } v_k(i) \neq 0\text{: } \|\tilde{S}(t, v)\| \neq 0 \tag{4.2}$$

$$\Leftrightarrow \forall\, v \in L \text{ with } v_k(i) \neq 0\text{: } v^T \mathcal{R} v \neq 0 \tag{4.3}$$

$$\Leftrightarrow \nexists\, v \in L \text{ with } v_k(i) \neq 0 \text{ s.t. } \mathcal{R} v = 0 \tag{4.4}$$

$\Leftrightarrow$ the $(k, i)^{th}$ column[2] *of* $\mathcal{R}$ is

[2]We refer to the (k, i)th row (or column) of a matrix of the dimension of $\mathcal{R}$ when we want to name the kth row (or column) within the ith block in vertical (horizontal) direction. This notation was adopted since $\mathcal{R}$ is block-Toeplitz.

$$\times \quad \text{linearly independent of the others} \tag{4.5}$$

$$\Leftrightarrow \exists d \text{ s.t. } \mathcal{R}d = u^{k,i}. \tag{4.6}$$

Properties:

i) The decorrelating detector for each bit of each user is invariant with respect to received energies and does not require knowledge thereof.

Proof: Since the elements of the matrix $\mathcal{R}$ are normalized crosscorrelation coefficients, the defining equation (4.1) is energy independent.

ii) The decorrelating detector eliminates the multiuser interference present in the respective matched filter output. (Hence its name).

Proof: From (2.20) the decision made on the ith bit of the kth user at the output of the decorrelating filter d is,

$$\hat{b}_k(i) = \text{sgn}\,(d^T \mathcal{R} \mathcal{W} b + d^T n) = \text{sgn}\,(\sqrt{w_k(i)} b_k(i) + d^T n). \tag{4.7}$$

Interestingly, this natural strategy, though not necessarily optimal for specific user-energies, is optimal with respect to the worst possible distribution of energies.

iii) The kth-user bit-error-rate of the decorrelating detector is independent of the energies of the interfering users $w_j(i)$, $j \neq k$, $i = -M, \cdots, M$.

Proof: It follows from (4.7) that the decision statistic that is compared to a zero threshold is independent of the energies of the interfering users.

iv) The efficiency of the decorrelating detector is independent of the energies and is given by

$$\eta^d_{k,i} = \max{}^2 \left\{ 0, \min_{\substack{b \in B \\ b_k(i)=1}} \frac{1}{\sqrt{w_k(i)}} \frac{\langle \tilde{S}(t, Wb), \tilde{S}(t, d)\rangle}{\|\tilde{S}(t, d)\|} \right\} \tag{4.8}$$

$$= \max{}^2 \left\{ 0, \min_{\substack{b \in B \\ b_k(i)=1}} \frac{1}{\sqrt{w_k(i)}} \frac{\sqrt{w_k(i)} b_k(i)}{\sqrt{d_k(i)}} \right\} \tag{4.9}$$

$$= \frac{1}{d_k(i)}, \tag{4.10}$$

which by i) is energy-independent.

v) The decorrelating detector is the worst case optimal linear detector, and achieves the near-far resistance of optimum multiuser detection.

Proof: The proof of Proposition 1 is constructive, hence the first part of v) was obtained as a byproduct in Appendix 1. Here is a shorter proof, using the following fact. Any single linear strategy which is not decorrelating has a near-far resistance of zero. This is shown as follows. The near-far resistance of a linear filter is (cf. (3.18)):

$$\overline{\eta_{k,i}(v)} = \max{}^2 \left\{ 0, \inf_{\substack{v \in L \\ y_k(i)=1}} \frac{v^T \mathcal{R} y}{\sqrt{v^T \mathcal{R} y}} \right\}. \tag{4.11}$$

Unless $\mathcal{R}v = u^{k,i}$ (note invariance of η to scaling of v) the value of the inf-term is $-\infty$. Hence any linear filter which is not decorrelating has a near-far resistance $\bar{\eta} = 0$. This fact together with the nonzero asymptotic efficiency (4.10) of the decorrelating detector establish optimality of the decorrelating detector within the class of linear filters. Therefore the second part of v) results from Proposition 1.

Note that since the asymptotic efficiency of the decorrelating detector is independent of energies (Property iv) it equals the near-far resistance. This gives us an explicit solution for the Hilbert space optimization problem we obtained for the near-far resistance of optimal multiuser detection in (3.7), namely,

$$\overline{\eta_{k,i}} = \eta^d_{k,i} = \frac{1}{d_k(i)} \tag{4.12}$$

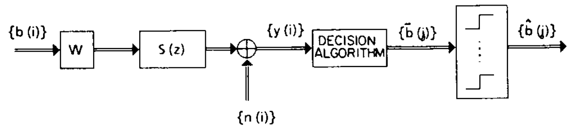


Fig. 5. Equivalent communication system.

and outlines an alternative proof for Proposition 1: we could have explicitly solved the above optimization problem by proceeding along the same lines as in Appendix 1, postulated the decorrelating detector by reasoning as in Fact under v), and shown that the asymptotic efficiency of the decorrelating detector and the near-far resistance of optimal multiuser detection are equal (see [7]). However, the game theoretic proof provides more insight into the nature of the solution.

Property iii) is of special importance. By this property the decorrelating detector does not become multiple-access limited, no matter how strong the multiple-access interference is. Also the decorrelating detector demodulates the data perfectly in the absence of noise, as can be seen from (4.7).

Characterization: We would now like to find an explicit expression for the decorrelating detector which we have up to now defined implicitly. It follows immediately from (4.1) and the uniqueness of the inverse of an invertible matrix that the decorrelating detector for the ith bit of user k is the (k, i)th row of the inverse of $\mathcal{R}$.

From the above and (4.10) the asymptotic efficiency of the decorrelating detector for the ith bit of user k is given by the (k, i)th diagonal element of the inverse of $\mathcal{R}$:

$$\eta^d_{k,i} = \frac{1}{\mathcal{R}^{-1}_{(k,i),(k,i)}}. \tag{4.13}$$

For the values of N encountered in practical applications, inverting a $NK * NK$ matrix is not possible. This issue is addressed in Section IV-B where we represent the decorrelating detector as a K-input K-output time-varying linear filter, and then show that in the limit as N tends to infinity the filter becomes time-invariant.

B. The Limiting Case $N \to \infty$

Proposition 2: As the length of the transmitted sequence increases ($N \to \infty$) the decorrelating detector approaches the K-input K-output linear time-invariant filter with transfer function

$$G(z) = [R^T(1)z + R(0) + R(1)z^{-1}]^{-1}. \tag{4.14}$$

●

Proof: From (2.14) and (2.13) the matched filter outputs for $l = \{-M, \cdots, M\}$ are

$$y(l) = R^T(1)W(l+1)b(l+1) + R(0)W(l)b(l) + R(1)W(l-1)b(l-1) + n(l) \tag{4.15}$$

where $b(-M-1) = b(M+1) = 0$. Taking z-transforms and letting N go to infinity we have

$$Y(z) = S(z)[WB](z) + N(z) \tag{4.16}$$

where $[WB](z)$ is the z-transform of the sequence $wb = \{[\sqrt{w_1(i)} b_1(i), \cdots, \sqrt{w_K(i)} b_K(i)]\}$, the matrix $S(z)$ is

$$S(z) = R^T(1)z + R(0) + R(1)z^{-1} \tag{4.17}$$

and $Y(z)$, $B(z)$ and $N(z)$ are, respectively, the vector-valued z-transforms of the matched filter output sequence, the transmitted sequence, and the noise sequence at the output of the matched filters. $S(z)$ can be interpreted as the equivalent transfer function of the multiuser communication system between transmitter and decision algorithm, as illustrated in Fig. 5. In this setting the optimal receiver problem is to find the transfer function matrix $G(z)$ of a K-input K-output linear time-invariant filter, at the output of which

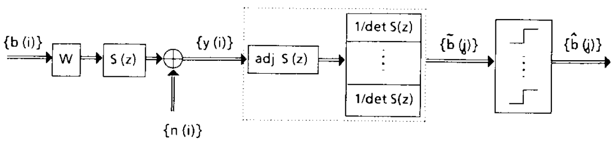


Fig. 6. Interpretation of the decorrelating detector.

a sign-decision yields estimates of the transmitted sequence which are optimal in a certain sense. In our case the optimality criterion is the near-far resistance, and we have demonstrated that the optimal filter is the decorrelating filter, which is the filter that eliminates the multiuser interference, i.e., is the K-input K-output time invariant linear filter which recovers the transmitted data in the absence of noise. Its transfer function is therefore the inverse of the equivalent transfer function $S(z)$:

$$G(z) = [S(z)]^{-1}. \tag{4.18}$$

◇

The effect of the inverse filter $[S(z)]^{-1}$ can be interpreted as illustrated in Fig. 6. The decorrelating filter can be viewed as the cascade of a finite impulse response filter with transfer function *adjoint* $S(z)$, which decorrelates the users, but introduces intersymbol interference among the previously noninterfering symbols of the same user, and of a second filter, consisting of a bank of K identical filters with transfer function $[\det S(z)]^{-1}$, which removes this intersymbol interference. Whereas the region of convergence of the z-transform can always be chosen so as to make $S(z)$ invertible, attention has to be paid to the issue of stability.

Proposition 3: There is a stable, noncausal realization of the decorrelating detector, if and only if the signal cross-correlations are such that

$$\det S(e^{j\omega}) = \det [R^T(1)e^{j\omega} + R(0) + R(1)e^{-j\omega}] \neq 0, \ \forall\, \omega \in [0, 2\pi]. \tag{4.19}$$

●

Proof: As long as $\det S(z)$ has no zeros on the unit circle, a nonempty convergence region of $S^{-1}(z)$ can be chosen which includes the unit circle. Thus, stability can be achieved. But, since $R(0)$ is symmetric,

$$\det S(z) = \det S^T(z) = \det S(z^{-1}).$$

Hence, the stable version of the decorrelating detector will be noncausal. (As a side remark, the matrix $S(e^{j\omega})$ is nonnegative define for all ω, cf. [15]). ◇

Condition (4.19) is equivalent to the limit of the LIA as $N \to \infty$. Both are necessary and sufficient conditions for system invertibility. The LIA requires that the output of a system (the system between the user bit-streams and the matched filter outputs) not be identically zero if the input is nonzero. Hence different inputs generate different outputs, i.e., the system is invertible. For a linear system the requirement that nonzero input produce nonzero output is equivalent to requiring that the transfer matrix be nonsingular on the unit circle. Assume the transfer matrix is singular at the angular frequency ω_0. Necessity follows since otherwise the input sequence consisting of a complex exponential at ω_0 times a vector in the nullspace of the transfer matrix evaluated at ω_0 yields zero output, since the transfer function on the unit circle gives the magnitude and phase of the system response to complex exponentials. On the other hand, sufficiency can be established by using Parseval's relation extended to multivariable systems:

$$\sum_{n=-\infty}^{\infty} \|y_n\|^2 = \frac{1}{2\pi}\int_0^{2\pi} \|Y(e^{j\omega})\|^2\, d\omega$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \|H(e^{j\omega})X(e^{j\omega})\|^2\, d\omega.$$

Hence, for a zero output sequence y_n the vector $H(e^{j\omega})X(e^{j\omega})$ has to vanish for all ω, which implies that $H(e^{j\omega})$ is singular whenever $X(e^{j\omega})$ is nonzero. This establishes the claimed equivalence.

Proposition 4: Condition (4.19) of Proposition 3 is equivalent to

$$\min_{\substack{x^*x=1\\ x\in\mathbb{C}}} \left(x^*R(0)x - \sqrt{(x^*R_+x)^2 + (x^*R_-x)^2}\right) > 0 \tag{4.20}$$

where $R_+ = R^T(1) + R(1)$ and $R_- = j(R^T(1) - R(1))$. The * denotes the complex conjugate. ●

Note that both R_+ and R_- are Hermitian. The proof of Proposition 4 is given in [15], together with the following two results.

—A necessary condition for (4.20) is that the matrices $R(0) + R(1) + R^T(1)$ and $R(0) - R(1) - R^T(1)$ be nonsingular.

—A sufficient condition for (4.20) is that

$$\lambda^2_{\min}(R(0)) > \max\{\lambda^2_{\max}(R_+), \lambda^2_{\min}(R_+)\} + \lambda^2_{\max}(R_-).$$

The following results quantify the asymptotic efficiency achieved by the limiting decorrelating detector.

Proposition 5: Let

$$[S(z)]^{-1} = \sum_{m=-\infty}^{\infty} D(m)z^{-m}. \tag{4.21}$$

Then the asymptotic efficiency of the limiting decorrelating detector for the kth user is given by

$$\eta_k^d = \frac{1}{D_{kk}(0)} \tag{4.22}$$
$$= \left[\frac{1}{2\pi}\int_0^{2\pi} [R^T(1)e^{j\omega} + R(0) + R(1)e^{-j\omega}]^{-1}_{kk}\, d\omega\right]^{-1}. \tag{4.23}$$

●

Proof: From Proposition 2 the z-transform of the decision statistic at the output of the limiting decorrelating detector is given by

$$G(z)Y(z) = [WB](z) + [S(z)]^{-1}N(z) = [WB](z) + N'(z)$$

where $N'(z)$ is the z-transform of the (stationary) filtered Gaussian background noise vector sequence. The z-transform of its covariance matrix sequence $E[n'(\cdot)n'^T(\cdot + i)]$ is equal to $\sigma^2[S(z)]^{-1}$, hence with (4.21) n_k' is a zero-mean Gaussian random variable with variance $\sigma^2 D_{kk}(0)$. Therefore, the probability of error for the kth user equals

$$P_k = P(n_k' > \sqrt{w_k}) = Q\left(\frac{\sqrt{w_k}}{\sigma\sqrt{D_{kk}(0)}}\right). \tag{4.24}$$

From here, using the definition of asymptotic efficiency, the first equality follows. For the second, note that applying the inverse z-transform and definition (4.21), we obtain

$$D_{kk}(0) = \frac{1}{2\pi}\int_0^{2\pi} [S(e^{j\omega})]^{-1}_{kk}\, d\omega,$$

and the result follows using (4.17). ◇

Proposition 6: The asymptotic efficiency of the limiting decorrelating detector for the kth user is strictly positive, and lower bounded by

$$\eta_k^d \geq \left[\max_{\omega\in[0,2\pi]} \left|[R^T(1)e^{j\omega} + R(0) + R(1)e^{-j\omega}]^{-1}_{kk}\right|\right]^{-1} > 0. \tag{4.25}$$

●

Proof: From (4.22), (4.23)

$$D_{kk}(0) \le \max_{\omega \in [0, 2\pi]} \left| [R^T(1)e^{j\omega} + R(0) + R(1)e^{-j\omega}]^{-1}_{kk} \right|. \quad (4.26)$$

Hence,

$$\eta_k^d = \frac{1}{D_{kk}(0)} \ge \left[\max_{\omega \in [0, 2\pi]} \left| [R^T(1)e^{j\omega} + R(0) + R(1)e^{-j\omega}]^{-1}_{kk} \right| \right]^{-1}$$

$$\ge \frac{\min_{\omega} |\det [R^T(1)e^{j\omega} + R(0) + R(1)e^{-j\omega}]|}{\max_{\omega} |\mathrm{adj}_k\, [R^T(1)e^{j\omega} + R(0) + R(1)e^{-j\omega}]|}, \quad (4.27)$$

which is positive by Proposition 3. ◇

Proposition 7: In the two-user case let $R_{12}(0) = \rho_{12}$ and $R_{12}(1) = \rho_{21}$. Then the asymptotic efficiency of the decorrelating detector for infinite sequence length is given by

$$\eta_1^d = \eta_2^d = \sqrt{(1 - \rho_{12}^2 - \rho_{21}^2)^2 - 4\rho_{12}^2\rho_{21}^2}$$
$$= \sqrt{[1 - (\rho_{12} + \rho_{21})^2][1 - (\rho_{12} - \rho_{21})^2]}. \quad (4.28)$$

●

Proof: This formula can be obtained by particularizing Proposition 5 or by minimizing the asymptotic efficiency of optimal multiuser detection in the two-user case with respect to energies. Alternatively, we will prove (4.28) by taking the limit as $N \to \infty$ of the asymptotic efficiency of the decorrelating filter for the central bits in a length N sequence. We will then have proved that in the two-user case the limit of the asymptotic efficiency of the finite-length decorrelating detector as $N \to \infty$ is indeed the asymptotic efficiency of the limiting decorrelating detector.

Recall that the asymptotic efficiency of the decorrelating detector is given by the reciprocal of the corresponding diagonal element of $\mathcal{R}^{-1}$. We need to find explicit expressions for the central diagonal elements of the inverse of the matrix $\mathcal{R}$ as a function of N. We have

$$\mathcal{R} = \begin{pmatrix} 1 & \rho_{12} & 0 & 0 & \\ \rho_{12} & 1 & \rho_{21} & 0 & \\ 0 & \rho_{21} & 1 & \rho_{12} & \ddots \\ 0 & 0 & \rho_{12} & 1 & \ddots \\ & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \quad (4.29)$$

Denote by Δ_n the determinant of the above $n*n$ matrix. It is easy to see from the structure of $\mathcal{R}$ that Δ_n satisfies the recursion

$$\Delta_n = \Delta_{n-1} - \begin{cases} \rho_{12}^2\,\Delta_{n-2}, & n \text{ even} \\ \rho_{21}^2\,\Delta_{n-2}, & n \text{ odd.} \end{cases} \quad (4.30)$$

Hence, we can write

$$\begin{bmatrix} \Delta_{2n} \\ \Delta_{2n-1} \end{bmatrix} = \begin{bmatrix} 1 - \rho_{12}^2 & -\rho_{21}^2 \\ 1 & -\rho_{21}^2 \end{bmatrix} \begin{bmatrix} \Delta_{2n-2} \\ \Delta_{2n-3} \end{bmatrix}. \quad (4.31)$$

If we consider the sequence of $4n*4n$ matrices for simplicity, the central diagonal element of the inverse of $\mathcal{R}$ is $\Delta_{4n}/(\Delta_{2n-1}\,\Delta_{2n})$. Hence, after introducing the state vector

$$x_n = \begin{bmatrix} \Delta_{2n} \\ \Delta_{2n-1} \end{bmatrix}, \quad (4.32)$$

we see that finding Δ_{2n}, Δ_{2n-1} requires finding the trajectory of the unforced linear dynamic system

$$x_n = \begin{bmatrix} 1 - \rho_{12}^2 & -\rho_{21}^2 \\ 1 & -\rho_{21}^2 \end{bmatrix} x_{n-1}, \quad x_1 = \begin{bmatrix} 1 - \rho_{12}^2 \\ 1 \end{bmatrix},$$

i.e.,

$$x = \begin{bmatrix} 1 - \rho_{12}^2 & -\rho_{21}^2 \\ 1 & -\rho_{21}^2 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (4.33)$$

The eigenvalues of this system are found to be

$$\lambda_{1,2} = \frac{1 - \rho_{12}^2 - \rho_{21}^2 \pm \sqrt{(1 - \rho_{12}^2 - \rho_{21}^2)^2 - 4\rho_{12}^2\rho_{21}^2}}{2}.$$

We see $0 < \lambda_1 < \lambda_2 < 1$. After finding the corresponding eigenvectors it follows that:

$$x_n = \begin{bmatrix} \lambda_1 + \rho_{12}^2 & \lambda_2 + \rho_{21}^2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} \cdot \begin{bmatrix} 1 & -(\lambda_2 + \rho_{21}^2) \\ -1 & \lambda_1 + \rho_{21}^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{\lambda_1 - \lambda_2}$$
$$= \begin{bmatrix} \lambda_1 + \rho_{12}^2 & \lambda_2 + \rho_{21}^2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^n \\ -\lambda_2^n \end{bmatrix} \frac{1}{\lambda_1 - \lambda_2}. \quad (4.34)$$

Hence the central diagonal element of the inverse of $\mathcal{R}$ is

$$\frac{\Delta_{4n}}{\Delta_{2n-1}\,\Delta_{2n}} = \frac{[1 \;\; 0]x_{2n}}{[0 \;\; 1]x_n[1 \;\; 0]x_n}$$
$$= \frac{(\lambda_1/\lambda_2)^{2n}(\lambda_1 + \rho_{21}^2) - (\lambda_2 + \rho_{21}^2)}{[(\lambda_1/\lambda_2)^n - 1][(\lambda_1/\lambda_2)^n(\lambda_1 + \rho_{21}^2) - (\lambda_2 + \rho_{21}^2)]} \cdot (\lambda_1 - \lambda_2). \quad (4.35)$$

So finally

$$\eta^d = \lim_{n \to \infty} \frac{\Delta_{4n}}{\Delta_{2n-1}\,\Delta_{2n}} = \lambda_2 - \lambda_1 = \sqrt{(1 - \rho_{12}^2 - \rho_{21}^2)^2 - 4\rho_{12}^2\rho_{21}^2}.$$

◇

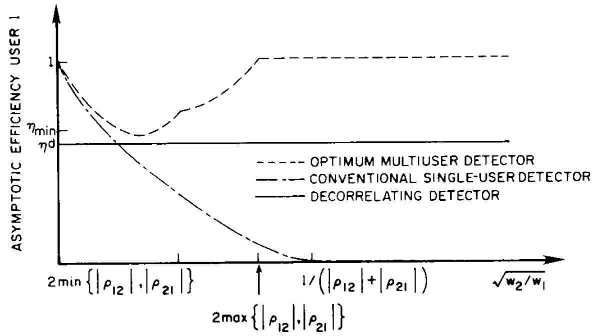


Fig. 7. Asymptotic efficiencies in the two-user case for infinite transmitted sequence length, when the user energies are constant over time (here we chose $|\rho_{12}|$, $|\rho_{21}| = 0.3, 0.5$ which yields $\eta_{\min} = 0.68$, $\eta^d = 0.59$).

Fig. 7 shows the asymptotic efficiency of the decorrelating detector for infinite transmitted sequence length in the two user case. Note its invariance with respect to energies. The discrepancy between η^d and $\eta_{\min}$, defined in (3.4), is due to the fact that $\eta_{\min}$ is higher than the near-far resistance of optimum multiuser detection, since for $\eta_{\min}$ the energies are constrained to be constant over time.

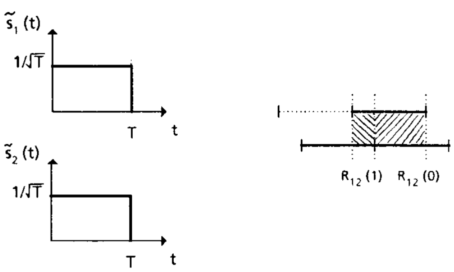


Fig. 8. Signals and crosscorrelations of example (4.42).

The fact that the stable version of the decorrelating filter turns out to be noncausal is not surprising. Due to the lack of synchronism among the users any decision based on less than the entire received waveform is suboptimal. In practice, since the filter is stable, the more remote symbols will count less heavily, and truncation of the noncausal part will be performed after a suitable delay without affecting performance appreciably. For illustration consider the two-user case where we let $R_{12}(0) = \rho_{12}$ and $R_{12}(1) = \rho_{21}$. Then

$$S(z) = \begin{pmatrix} 1 & \rho_{12} + \rho_{21} z^{-1} \\ \rho_{12} + \rho_{21} z & 1 \end{pmatrix}$$

and the transfer function of the decorrelating detector as given by (4.27) is

$$S^{-1}(z) = \frac{1}{1 - \rho_{12}^2 - \rho_{21}^2 - \rho_{12}\rho_{21} z - \rho_{12}\rho_{21} z^{-1}} \cdot \begin{pmatrix} 1 & -(\rho_{12} + \rho_{21} z^{-1}) \\ -(\rho_{12} + \rho_{21} z) & 1 \end{pmatrix}. \quad (4.36)$$

We are interested in the impulse response $f(n)$ of the IIR part of the above filter. Taking the inverse z-transform it is found to be

$$f(n) = Z^{-1}\left[\frac{1}{1 - \rho_{12}^2 - \rho_{21}^2 - \rho_{12}\rho_{21} z - \rho_{12}\rho_{21} z^{-1}}\right] = \frac{\xi^{|n|}}{\eta} \quad (4.37)$$

where $\xi = (1 - \rho_{12}^2 - \rho_{21}^2 - \eta)/(2\rho_{12}\rho_{21})$ and η is the asymptotic efficiency which is given by Proposition 7. It can be checked that $|\xi| \le 1$, with equality if $|\rho_{12}| + |\rho_{21}| = 1$, which can be shown to coincide with the condition imposed by Proposition 3 for the two-user case. In the latter case the asymptotic efficiency is zero, which follows from Proposition 7. Otherwise, since $|\xi| < 1$ the limiting filter is stable, with symmetric coefficients which decay with rate ξ. In practical applications the filter will be approximated up to any desired precision by truncation of the noncausal part to a finite number of filter coefficients. For illustration the decay rate ξ of the filter coefficients and the achievable asymptotic efficiency η are plotted in Fig. 9 as functions of ρ_{12} and ρ_{21}.

Poor cross-correlation properties among the signature waveforms could imply that the limiting filter $G(z)$ does not exist, although the decorrelating detector exists for finite-length transmitted sequences. We give an example to illustrate this fact. For $K = 2$ it is straightforward to show that the condition of Proposition 3 is satisfied for all signal constellations for which $|R_{12}(0)| = |R_{12}(1)| \neq 1$. This is the case unless the normalized waveforms coincide modulo circular shifts and sign changes.

Consider the trivial signal case where both users are assigned the same rectangular waveform, as shown in Fig. 8. Abbreviate $R_{12}(0)$, which is the crosscorrelation between bits in the same signaling interval, by $r = \tau/T \in [0, 1)$, then in this case $R_{12}(1)$, which is the crosscorrelation between bits in adjacent intervals, is $1 - r$. Then,

$$S(z) = \begin{pmatrix} 1 & r + (1-r)z^{-1} \\ r + (1-r)z & 1 \end{pmatrix} \quad (4.38)$$

becomes singular for $z = 1$, hence there is no stable limiting inverse filter. And if it existed its asymptotic efficiency, as given by (4.28), would be zero. This is not surprising, for an infinite sequence of transmitted bits where both users use the same waveform. However, for finite length sequences advantage can be taken of the marginal effects of having bits which are not affected by either past or future bits. For finite N the decorrelating detector exists unless $r = 0$, i.e., when the transmissions are not synchronous. This is in accord with the multiarrival condition given in Appendix 2, and with the results obtained in the synchronous case [7].

V. Error Probabilities: Numerical Examples

In the sequel, we compare the performances of the conventional and of the decorrelating detector. Without loss of generality we consider the error probability of user 1 in a channel shared by several active users. The conventional detector decides for the sign of the kth component of the matched filter output vector, given by (2.14). Therefore its average error probability over the bit sequences of the interfering users equals

$$\frac{1}{2^2(K-1)} \sum_{b_j(0),\, b_j(-1),\, j \neq 1} Q\left(\frac{\sqrt{w_1} - \sum_{j=2}^{K} [R_{1j}(0) b_j(0) + R_{1j}(1) b_j(-1)] \sqrt{w_j}}{\sigma}\right), \quad (5.1)$$

whereas its worst case error probability over the interfering bit sequences equals

$$Q\left(\frac{\sqrt{w_1} - \sum_{j=2}^{K} [|R_{1j}(0)| + |R_{1j}(1)|] \sqrt{w_j}}{\sigma}\right). \quad (5.2)$$

The probability of error of the decorrelating detector equals, from (4.24),

$$Q\left(\frac{\sqrt{w_1}}{\sigma\sqrt{D_{11}(0)}}\right),$$

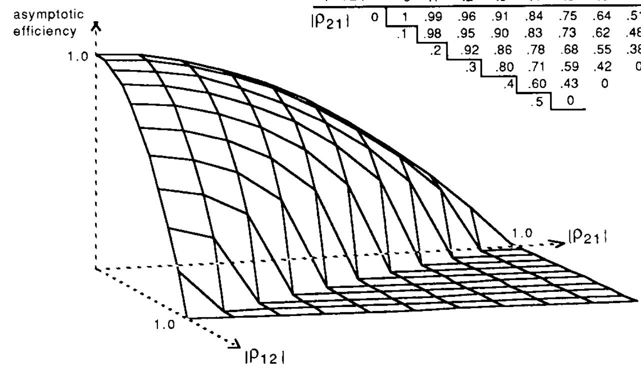
with (the equivalence with (4.23) is easy to show, cf. [15])

$$D_{11}(0) = \frac{1}{\pi}\int_0^{\pi} [R(1)^T e^{j\omega} + R(0) + R(1) e^{-j\omega}]_{11}^{-1}\, d\omega. \quad (5.3)$$

The delays enter the above formulas implicitly via the crosscorrelation matrices, which are functions thereof and of the chosen signature sequences. In the following examples, we have chosen a set of spread-spectrum m-sequences of length 31.

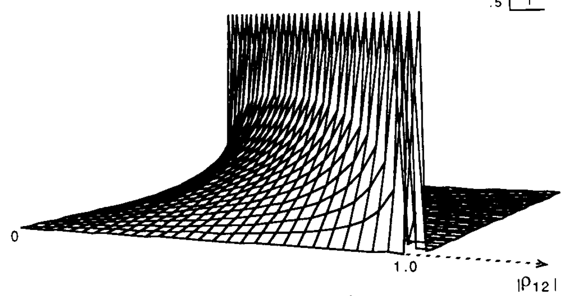
In Fig. 10 we use, for comparison purposes to previous works ([14], [1]), the set of 3 sequences reported in [12, Table V] to be optimal with respect to a signal-to-multiple-access interference parameter when the conventional detector is used. We consider a baseband environment with $K - 1$ active equal energy interferers, whose delay relative to each other is fixed. Fig. 10, for $K = 3$, shows the 1st user error probability of the conventional receiver versus SNR_1, the signal-to-background-noise ratio of user 1, for different values

$\lvert\rho_{21}\rvert$ \ $\lvert\rho_{12}\rvert$	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
0	1	.99	.96	.91	.84	.75	.64	.51	.36	.19	0
.1		.98	.95	.90	.83	.73	.62	.48	.31	0	
.2			.92	.86	.78	.68	.55	.38	0		
.3				.80	.71	.59	.42	0			
.4					.60	.43	0				
.5						0					



(a)

$\lvert\rho_{21}\rvert$ \ $\lvert\rho_{12}\rvert$	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
0	0	0	0	0	0	0	0	0	0	0	0
.1		.01	.02	.03	.05	.07	.1	.14	.24	1	
.2			.04	.07	.1	.14	.21	.33	1		
.3				.11	.16	.24	.37	1			
.4					.25	.39	1				
.5						1					



(b)

Fig. 9. (a) Asymptotic efficiency of the decorrelating detector for two users as a function of the partial crosscorrelations of their signature waveforms. (b) Decay rates of the coefficients of the IIR part of the decorrelating detector for two users, symmetric in ρ_{12} and ρ_{21}.

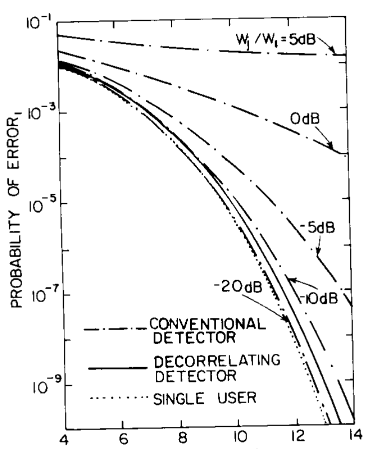


Fig. 10. Error probability of user 1 with 2 active equal energy interferers, each of energy w_j, averaged over the interfering bit sequences and over the delay of user 1, for the decorrelating and conventional receiver versus the SNR of user 1, for m-sequences of length 31 and different interference levels.

of the energy ratio SNR_j/SNR_1, averaged over the bit sequences of the two interferers and over the delay of user 1. Also shown are the 1st user error probability of the decorrelating detector and the error probability of the single user channel. From Fig. 10 we see the strong dependence of the performance of the conventional receiver on the relative energies of the active users. While the error probability of the decorrelating detector is invariant to the energy of interfering users, the performance of the conventional receiver deteriorates rapidly for increasing interference, till for an energy ratio above 5 dB the conventional receiver becomes practically multiple-access limited. (For a sufficiently high level of nonorthogonal interference the error probability of the conventional receiver can be seen to become irreducible. E.g., in the two-user synchronous case, for $\sqrt{w_2}/\sqrt{w_1} = (1+\Delta)/\rho$ where ρ is the normalized crosscorrelation coefficient between the two signature signals and $\Delta \geq 0$, the error probability of the conventional receiver tends to 1/4 if $\Delta = 0$ and to 1/2 if $\Delta > 0$ for increasing SNR of user 1). Note that if the energies of all the users are equal the decorrelating detector is around two orders of magnitude better than the conventional receiver at 10 dB. Only if the multiple-access interference level plays a subordinate role compared to the background noise does the conventional detector outperform the decorrelating detector, which pays a penalty for combatting the interference instead of ignoring it. Similar results were obtained regardless of the actual value of the relative delay between the two interfering users.

Fig. 11 shows the same setting as above, in the case $K = 6$. We have used the set of autooptimal m-sequences of length 31 found in

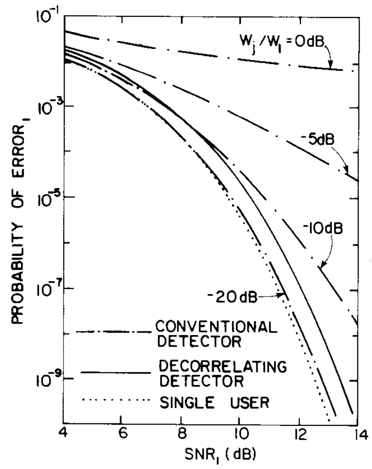


Fig. 11. Same as Fig. 10, with 5 active equal energy interferers.

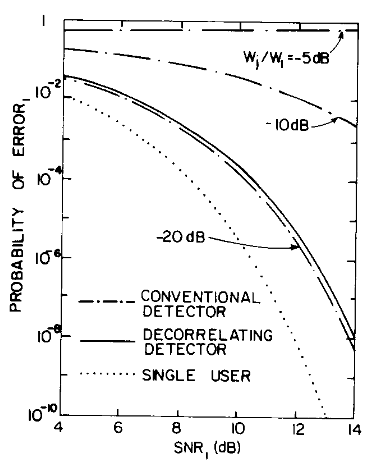


Fig. 12. Worst case probability of user 1 w.r.t. the bit sequences of the interfering users, with 9 equal energy interferers.

[13, Fig. A.1] to be optimal with respect to certain peak and mean-square correlation parameters which play an important role in the error probability analysis of the conventional detector. Comparing Fig. 11 to Fig. 10 we see the same qualitative error probability relation between the two detectors, and again the strong near-far limitation of the conventional receiver. Since there are more active interferers the performance advantage of the decorrelating detector in a near-far environment is even more pronounced: if the energies of all the users are equal the decorrelating detector is almost three orders of magnitude better than the conventional receiver at 10 dB.

Finally, Fig. 12 shows the worst case probability of the conventional detector over the sequences of interfering users, as given by (5.2), for $K = 10$. The signature sequence set used for $K = 6$ has been expanded—without trying to optimize, as before, with respect to the performance of the conventional detector. The shown error probabilities are typical, varying very little if different sets of delays are used because of the good crosscorrelation properties of m-sequences.

Overall the generated error probability curves show the pronounced superiority of the decorrelating receiver in a near-far environment, and whenever sufficiently many users are active even if their energies are well below the energy of the desired user. Note, finally, that the selected signature sequences were optimal with respect to the performance of the conventional receiver. It would be interesting to investigate the possible performance gain of using the decorrelating detector in conjunction with a set of signature sequences optimized for its use.

VI. Conclusions

In this paper, we have obtained a linear multiuser detector, the decorrelating detector, for demodulation of asynchronous code-division multiplexed signals in white Gaussian channels. The bit-error-rate of this detector is independent of the energy of the interfering users and exhibits the same degree of near-far resistance as the optimum multiuser detector obtained in [1]. Since the decorrelating detector does not require knowledge of the received energies and its complexity is only linear in the number of users, it emerges as the solution of choice in near-far environments with a large number of users.

In applications where each receiver is interested in demodulating the information transmitted by only one user, it is easy to decentralize the K-user decorrelating receiver since it can be implemented as K separate (continuous-time) single-input (discrete-time) single-output filters. Each of those filters can be viewed as a modification of the conventional single-user matched filter where instead of correlating the channel output with the signature waveform of the user of interest, we use its projection on the subspace orthogonal to the space spanned by the interfering signals.

Note, finally, that if the filter is actually an approximation to the decorrelating receiver, due to, for example, finite accuracy in the computation of the crosscorrelations or truncation of the impulse response (Section IV-B), it will no longer be orthogonal to the subspace of the interfering signals and therefore it will not be near-far resistant in the worst case sense adopted in this paper. However, the effect on the bit-error-rate will be arbitrarily small with a good enough approximation to the decorrelating receiver, and therefore the bit-error-rate will be very insensitive to the energy level of the interferers. Hence, the resistance to the near-far problem can be preserved within any desired energy range.

Appendix 1

Saddle-Point Property in (3.19)

Though the penalty function of (3.18) looks similar to the signal-to-noise ratio functional encountered in the robust matched filtering problem [5], $|\langle h, s\rangle|^2/\langle h, \sum h\rangle$, the problem is different here because the numerator can be negative. Thus we have to establish the result "from scratch." In order to show that $M(v, y)$ has a saddle point, i.e., satisfies (3.19), we show that it satisfies the requirements of the following theorem.

Theorem [4, Thm. 2.1]: Suppose Q is a convex set and $M(v, \cdot)$ is convex on Q for every $v \in H$. Then if (v_L, y_L) is a *regular pair*[3] for (H, Q, M), the following are equivalent:

a) $y_L \in \arg\min_{y \in Q} \sup_{v \in H} M(v, y)$,

b) (v_L, y_L) is a saddle point solution for (H, Q, M).

This theorem establishes that if we exhibit a regular pair whose second argument satisfies a), the game (H, Q, M) has a saddle point, which means that the sequence of max and min in (3.18) can be interchanged. In the following, we find a suitable regular pair, thereby proving (3.19).

Clearly the convexity conditions are satisfied (the set of detectors is not required to be topologized). We need to find a candidate regular pair. Note that the value of inf term in (3.18) is $-\infty$ (which gives a near-far resistance of zero) unless v is picked such that $\mathcal{R}v = u^{k,i}$ (η is invariant with respect to scaling of v). $u^{k,i}$ is the (k, i)th unit vector in the Hilbert space L, defined as $u_j^{k,i}(l) = \delta_{kj}\delta_{li}$. This gives us a candidate for an optimal detector v_L: d, with $\mathcal{R}d = u^{k,i}$. Existence of such a vector is shown in (4.6) to follow from the LIA of (2.3).

[3] $(v_L, y_L) \in H \times Q$ is a regular pair for (H, Q, M) if, for every $y \in Q$ such that $y_\alpha = (1-\alpha)y_L + \alpha y \in Q$ for $\alpha \in [0, 1]$, we have

$$\sup_{v \in H} M(v, y_\alpha) - M(v_L, y_\alpha) = o(\alpha).$$

(If this detector is indeed optimal, which follows if the candidate pair is regular and satisfies a), and coincides with v^*).

Next we find a y_L which meets the requirement of point a) of the theorem. Using the Cauchy–Schwarz inequality, we find that

$$\sup_{v \in H} M(v, y) = \sup_{\substack{v \in L \\ v^T \mathcal{R} v \neq 0}} \frac{y^T \mathcal{R} v}{\sqrt{v^T \mathcal{R} v}} = \sqrt{y^T \mathcal{R} y} \tag{A.1}$$

where the inner product is maximized for $v = ky + \{x \in L\colon \mathcal{R}x = 0\}$.

We now need to solve the Hilbert space optimization problem

$$\inf y^T \mathcal{R} y \tag{A.2}$$

$$\text{subject to } y_k(i) = 1.$$

Using (2.21) and the definition of d we can rewrite the minimization problem under consideration as

$$\inf \|y\|_R \tag{A.3}$$

$$\text{subject to } \langle d, y \rangle_R = 1.$$

$\|\cdot\|_R$ is a norm since $\mathcal{R}$ is positive definite. We have obtained a minimum-norm optimization problem in Hilbert space. To prove existence of a solution we need to show that constraint set to be closed, which holds since the Hilbert space is finite dimensional. (Even for $N \to \infty$, when we have an infinite dimensional optimization problem, we could use the fact that the codimension is finite. The problem there is that the signals are no longer square integrable.) The constraint, $y_k(i) = 1$, is equivalent to $y = u^{k,i} + \{x\colon \langle x, d\rangle_R = 0\}$. $\mathcal{A} = [d]$, the subspace generated by d, is a closed subspace of dimension 1. Hence the constraint set $\{x\colon \langle x, d\rangle_R = 0\} = \mathcal{A}^\perp$ is closed. We now have a minimum-norm optimization problem in Hilbert space over a closed subspace. Hence, the Projection Theorem, [6], guarantees existence [so we can replace the inf by a min, as required in a)] and uniqueness of a minimizing equivalence class y^*, with

$$y^* \in \{\mathcal{A}^\perp + u^{k,i}\} \cap \mathcal{A}^{\perp\perp} = \{\mathcal{A}^\perp + u^{k,i}\} \cap \mathcal{A} \tag{A.4}$$

where equality holds since A is closed. Hence $y_k^*(i) = 1$ and $y^* = kd$, which implies

$$y^* = \frac{1}{d_k(i)} d. \tag{A.5}$$

We now have a candidate regular pair which satisfies a): $(v_L, y_L) = (d, (d_k(i))^{-1} d)$. From (A.1) and the definition of regularity we have to check the dependence on α of

$$\begin{aligned}
\sqrt{y_\alpha^T \mathcal{R} y_\alpha} - \frac{y_\alpha^T \mathcal{R} v_L}{\sqrt{v_L^T \mathcal{R} v_L}}
&= \sqrt{d^T \mathcal{R} d + 2\alpha (y - d)^T \mathcal{R} d + \alpha^2 (y - d)^T \mathcal{R} (y - d)} \\
&\quad - \sqrt{\frac{1}{d_k(i)}} \\
&= \sqrt{\frac{1}{d_k(i)} + \alpha^2 (y - d)^T \mathcal{R} (y - d)} - \sqrt{\frac{1}{d_k(i)}}.
\end{aligned} \tag{A.6}$$

We have repeatedly used the decorrelating property of d. Since $\sqrt{1 + x} \le 1 + 1/2x$, the above quantity lies in the interval $[0, (y - d)^T \mathcal{R} (y - d) \sqrt{d_k(i)}/2\alpha^2]$, hence divided by α goes to 0 when $\alpha \downarrow 0$. Thus $(d, (d_k(i))^{-1} d)$ is a regular pair which satisfies point a) of the theorem. Hence it follows from the theorem that the penalty function $y^T \mathcal{R} v / \sqrt{v^T \mathcal{R} v}$ has a saddle point, i.e.,

$$\sup_{\substack{v \in L \\ v^T \mathcal{R} v \neq 1}} \inf_{\substack{y \in L \\ y_k(i) = 1}} \frac{y^T \mathcal{R} v}{\sqrt{v^T \mathcal{R} v}} = \inf_{\substack{y \in L \\ y_k(i) = 1}} \inf_{\substack{y \in L \\ v^T \mathcal{R} v \neq 1}} \frac{y^T \mathcal{R} v}{\sqrt{v^T \mathcal{R} v}}. \tag{A.7}$$

◇

Appendix 2

Sufficient Conditions for Linear Independence

Suppose that, for a fixed signal set,

i) $\{\tau_1, \cdots, \tau_K\}$ are continuous random variables,

ii) $\{\tau_1, \cdots, \tau_K\}$ are independent random variables,

iii) $w_k(i) \neq 0$.

Then almost surely there is no $v \in L, v_k(i) \neq 0$ such that $\tilde{S}(t, v) = 0$.

Proof: Define the times of effective arrival and departure of the ith signal of the kth user [1], as

$$\lambda_{i,k}^a = \tau_k + iT + \sup\left\{\tau \in [0, T),\ \int_0^\tau s_k^2(t)\, dt = 0\right\} \tag{A.8}$$

and

$$\lambda_{i,k}^d = \tau_k + iT + \inf\left\{\tau \in (0, T],\ \int_\tau^T s_k^2(t)\, dt = 0\right\}, \tag{A.9}$$

respectively.

Since $v_k(i) \neq 0$ there is a first and a last symbol that differs from zero. It is readily apparent that in order to have $\tilde{S}(t, v) = 0$, the effective arrival of the first (and the effective departure of the last) symbol that differs from zero must be a point of effective multiarrival (respectively multideparture). Note that this property does not depend on the particular v chosen, but only on the set of delays. From (A.8), (A.9), the effective times of arrival and departure inherit from the delays the properties of being continuously valued and mutually independent. Therefore, the result follows, since the set of delays $\{\tau_1, \cdots, \tau_K\}$ for which multiarrival points result has measure zero. ◇

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