MINIMUM PROBABILITY OF ERROR FOR ASYNCHRONOUS MULTIPLE ACCESS COMMUNICATION SYSTEMS

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ABSTRACT

Consider an ideal white gaussian channel shared by K users that transmit independent data streams by modulating, antipodally a set of assigned signal waveforms, without carrier phase or bit-interval synchronism among them. The coherent receiver commonly employed in this multiple access communication system [22] consists of a set of matched filters synchronized to the signal of each user not followed by a zero-threshold (Fig. 1), i.e., assuming that the channel introduces additive white gaussian noise, the conventional receiver can be viewed as a bank of optimum detectors for single user communication. However, since in general the input to each channel has an additive component of multiple access interference (cross-correlation with the signals of the other users), the conventional receiver is optimum for a given user assuming that all other users are known exactly or approx. These controllers do not consider the cooperative nature of the interference and the inherent interference caused by the presence of other users. The goal of this study is to obtain the minimum probability of error achievable for the usual synchronous multiaccess digital communications model; we will derive and analyze an optimum E-user coherent detector (the signal unknown relative delay

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Fig. 1. Conventional receiver for asynchronous multiple-access communications

and carrier phase of each active user are assumed to be acquired) which bases every bit decision on the whole observation interval. Under these conditions it is straightforward to show that the whole sequence of outputs of the bank of matched filters for all users is a sufficient statistic for the detection of the transmitted information sequence. The bank of matched filters is followed by a decision device (Fig. 2) that depends on the optimality criterion employed, the maximum or minimum bit error rate and maximum a posteriori sequence detector. Although the performance measure of interest is the set of error probabilities for each user, the MAP sequence detection may be preferable since it gives the most probable transmitted information sequence given the received data or equivalently it selects among the possible noise realizations the one with minimum power. Also, while for high SNR both receivers achieve the same error probabilities, the minimum SNR detector is considerably more complex (106, 171) than the MAP sequence detector due to the inherently nonstationary (compound hypothesis) character of the minimum error probability problem.

As is shown in Section II, the decision device for MAP sequence detection can be implemented by a Viterbi algorithm with 2K states and a K-dimensional version of the branch metric employed in the maximum likelihood sequence detection of PAM communications through linear channels that introduce intersymbol interference of approximately finite length. The main result of the minimum error probability analysis of Section III is that when signal sets with good cross-correlation properties are employed the optimum probability of error of every user has the same asymptotic behavior (increasing SNR) as that achievable in single-user communication.
Fig 2. Optimum sequence detector for asynchronous multiple-access communications.

every source of interest and unless the system operates in the low SNR regime, appreciably superior to that exhibited by the conventional receiver, the optimum receiver offers important performance advantages. Furthermore, even if signal waveforms with post-correlation properties are used, the optimum sequence receiver provides multi-access capability and it is shown that even for trivial noise, and in contrast with the conventional receiver, any pre-specified probability of error is achievable with high enough SN. Finally, in Section IV the computation of the average and worst-case of the error probability bound found in Section III is illustrated numerically.

II. OPTIMAL SEQUENCE DETECTOR

In the usual asynchronous K-user data communications model (cf. [12]), the input to the receiver is

\[ s(t) = h(t) + u(t) \]

where \( u(t) \) is zero-mean white gaussian noise (double-sided spectral width of \( 2W \)). \( h(t) \) is the bit interval duration assumed to be the same for all users) and the received information-bearing signal (with binary antipodal modulation) is

\[ h(t) = B(t) \]

where \( B(t) = B(1)(t - T) + B(0)(t - T) \)

The MAP bit-sequence detection scheme for the model (2.1-2) is to solve for the bit-sequence that maximizes the MAP probability

\[ \text{MAP} = \text{argmax} \left( \frac{P(B(t) | s(t))}{P(s(t))} \right) \]

where \( P(s(t)) \) is the signal waveform and the delay model (in bit) is independent of the bit symbol rate. Consequently, the detection rule that gives the MAP sequence can be implemented by a front end of matched filters (as in the conventional receiver) followed by a recursive shortest-paths decision algorithm whose computational complexity is equivalent to the K decoding of a convolutional code of rate \( \frac{1}{k} \) and constraint length equal to 2. Note that assumes the use of a bank of conventional receivers for single user communications with observations perturbed by independent noise processes, and that Subbaia [14] arrived at a discrete-time model with additive white noise and considered the use of a Viterbi algorithm with \( 2^k \) states.

\[ \xi(1) \]

\[ \xi(2) \]

\[ \xi(K) \]

\[ S(t) \]

\[ h(t) \]

\[ B(t) \]

\[ B(1)(t) \]

\[ B(0)(t) \]

\[ P(B(t) | s(t)) \]

\[ P(s(t)) \]

\[ \text{MAP} \]

\[ \text{argmax} \]

\[ \frac{P(B(t) | s(t))}{P(s(t))} \]

\[ P(s(t)) \]

\[ s(t) \]

\[ \text{MAP} \]

\[ \text{argmax} \]

\[ \frac{P(B(t) | s(t))}{P(s(t))} \]

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\[ P(s(t)) \]

\[ s(t) \]
Although the decision delay is unavoidable because the optimum decision cannot be made until all states since a common subpath, a well-known advantage of the Viterbi algorithm is that little degradation of performance occurs when the algorithm uses an adequacy chosen fixed decision lag. On the other hand, for large number of states, the dimensionality of the state space becomes infeasible and hence the designer must resort to suboptimal implementations based either on the Viterbi algorithm (e.g., [22,41]) or on other schemes such as sequential decoding (e.g., [19, 199]).

III. ERROR PROBABILITY ANALYSIS

III.1 Upper and lower bounds

This section is devoted to the analysis of the minimum bit error probability for the asynchronous multiple-access model (3.2-3) as a function of the received set of delays and carrier phases. Specifically, we analyze the probability $P_e$ that the optimum a posteriori decision corresponding to the $k$th bit of the $i$th user is erroneous; i.e., $P_e = P[b_i \neq \hat{b}_i | y_i(0) \neq y_i(0)]$, where

$$
\hat{b}_i(1) = \arg \max_k \left[ P[b_i = k | y_i(0)] \right] = \text{cf}(y_i(0)), \quad \text{cf}(y_i(0)) \equiv \begin{cases} 1 & y_i(0) - y_i(0) > \pi/2, \\ 0 & \text{otherwise} \end{cases}
$$

Define the set of sequence error sequences $E = \{y_i(0) = y_i(0) + 0.1 \pi, \ldots, y_i(0) + \pi \}$ for some $y_i(0)$. An error sequence is called decomposable if there exists a pair of sequence $x_i(0) \in E$, $x_i(0) \in E$, such that $x_i(0) = (x_i(0), x_i(0) + \pi, \ldots) = \hat{b}_i(x_i(0), x_i(0) + \pi, \ldots)$, and $b_i = \hat{b}_i(x_i(0), x_i(0) + \pi, \ldots)$, and $P_e = P[b_i \neq \hat{b}_i(x_i(0), x_i(0) + \pi, \ldots)]$ is the set of error sequences that affect the $k$th bit of the $i$th user and that are admissible given the channel is transmitted. Define the functions

$$
\psi(y_i(0)) = \sum_{k=1}^{n} x_i(0) \mid y_i(0) = y_i(0) + 0.1 \pi
$$

and

$$
\tilde{P}_e(x_i(0)) = \text{Pr}[\psi(y_i(0)) = \psi(x_i(0))] = \sum_{k=1}^{n} \beta_k(x_i(0)) \beta_k(y_i(0) + 0.1 \pi)
$$

and

$$
\hat{b}_i(1) = \arg \max_k \left[ P[b_i = k | y_i(0)] \right] = \text{cf}(y_i(0)), \quad \text{cf}(y_i(0)) \equiv \begin{cases} 1 & y_i(0) - y_i(0) > \pi/2, \\ 0 & \text{otherwise} \end{cases}
$$

The minimum error probability of any sequence such that $y_i(0) \neq 0$ will be denoted by $\epsilon_i(0) \text{ (i.e.,}

$$
\epsilon_i(0) = \min_k P[b_i = k | y_i(0) = y_i(0)]
$$

which corresponds to one half of the minimum BER of the difference between the signals of any pair of transmitted sequences that differ in the $k$th bit of the $i$th user.

3 We denote $n | x_i(1) | n$ for all $x_i(1)$ and $x_i(0)$.

**Proposition 1**

The minimum error probability of the $k$th user is upper bounded by

$$
P_e \leq P[b_i \neq \hat{b}_i | y_i(0) \neq y_i(0)] = \text{cf}(y_i(0)) \leq \epsilon_i(0)
$$

and

$$
P_e \leq P[b_i \neq \hat{b}_i | y_i(0) \neq y_i(0)](1 + \epsilon_i(0))
$$

where the expectation is over the ensemble of equally likely transmitted sequences $y_i(0)$.

**Proposition 2**

The minimum error probability of the $k$th user is lower bounded by

$$
P_e \geq P[b_i \neq \hat{b}_i | y_i(0) \neq y_i(0)] = \epsilon_i(0)
$$

and

$$
P_e \geq P[b_i \neq \hat{b}_i | y_i(0) \neq y_i(0)]
$$

The bounds (3.2) and (3.4) are, respectively, the $k$th user error probability achieved by the conventional receiver (Fig. 1) and the $k$th user optimum error probability in no other user was active. It is easy to see that they are tight in the low SNR regime. On the other hand, the bounds (3.3) -- upper bound to the $k$th user error probability of the optimum sequence detector of Section II and (3.4) are tight in the high SNR regime as is demonstrated in the following subsection.

III.2 Asymptotic Probability of Error

Because of the asymptotic behavior of the $Q$-function $Q(z)$, when $Q(z)/Q(z) \to 0$, if $z \to \infty$, then the only sequences that contribute appreciably to the upper bound (3.2) when $y_i(0) > 0$, are those with minimum energy. Therefore (3.2) and (3.4) result in

$$
\frac{Q(d_k \epsilon_{k,m} \epsilon \epsilon_i(0))}{\epsilon \epsilon_i(0)} \leq P_e \leq \frac{Q(d_k \epsilon_{k,m} \epsilon \epsilon_i(0))}{\epsilon \epsilon_i(0)}
$$

where

$$
\epsilon_i(0) = \frac{1}{2} \sum_{k=1}^{n} \beta_k(x_i(0)) = \frac{1}{2} \sum_{k=1}^{n} \beta_k(y_i(0) + 0.1 \pi)
$$

and

$$
\epsilon_i(0) = \frac{1}{2} \sum_{k=1}^{n} \beta_k(x_i(0)) = \frac{1}{2} \sum_{k=1}^{n} \beta_k(y_i(0) + 0.1 \pi)
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$$

Since the upper and lower asymptotic bounds of (3.3) differ by a multiplicative constant independent of $y_i(0)$, the $k$th user error probability for high signal to noise ratio is equivalent to that of an antipodal single-user system with bit-energy equal to $\epsilon_i(0)$. This suggests a measure of the degradation in

**Proofs are omitted in the Conference Record because of space limitations.**
terms that this can occur only if the cross-
correlations are at least comparable to the energies of the rest of the signals. A simple sufficient con-
dition for $h_k = 1$ less (2.9) that formalizes this reasoning is that for all $a$ and $a' \neq 0$ in $S_{11}$.

$$\sum_{a \neq 0} |a^H e(a')| \geq |a^H v|$$

The following result provides a more sufficient condition for $h_k = 1$, which is stronger than the condition for $h_k = 1$. (Concrete examples can be found in which $h_k = 0$ and $h_k = 1$).

**Proposition 4**

*Propose that*

1. $h_k = 1$.
2. $h_k = 0$ or $a \neq 0$. $P$

The expression is to find out under what conditions on the signal cross-correlations there is no degradation in the effective SIR of a particular user for the maximum of other active users. It can be seen from (3.3) that for the conventional receiver this is only possible when the cross-correlations with the rest of the users are zero, i.e., with zero probability. Fortunately, the behavior of the optimum error probability is at. In the case that $h_k = 1$, there must exist an error sequence with $h_k = 0$ and at least another component different from zero, whose energy $|a^H e|_F$ is smaller than $h_k = 0$. From this inspection of (3.4) or (3.9) one can see that in case

$$\text{Hence the discrepancy between } h_k \text{ and the previously proposed measure of SIR degradation based on the Gaussian white modeling of the multiple access interference} [20]$$

**IV. NUMERICAL EXAMPLES**

Two pairs of lower and upper bounds to the $h_k$ user minimum error probability have been presented in Section III and they have been shown to be tight asymptotically. Nonetheless, it remains to determine the SIR level for which such asymptotic approximation is sufficiently accurate. In the sequel, this question is illustrated by several samples of the computation of average and extreme cases of the above bounds. Moreover, the explicit expressions of the asymptotic efficiency factor found in the case of two users will be employed to investigate the relative effects of the near-far problem on the conventional and optimum detectors.

The computation of the upper bound to the error probability of the optimum sequence detector is carried out by generating a subset of $P_k$ that achieves a sufficient degree of approximation to the series of the right-hand side of (5.2). This subset of $P_k$ is chosen via a binary tree in which the root represents the error sequence whose only nonzero element is $h_k = 0$ and in which the score of each node is its indecomposable augmented sequence (from the left and right with no element of the set $\{-1,0,1\}$ whose absolute contribution to the partial sum is greater than the convergence parameter. For a given accuracy, the speed
of convergence of this procedure depends on the max-
imum value of $R_0$ for which the bound is computed and
on the magnitude of the signal cross-correlations.

Denoting by $\hat{B}_{\text{err}}(\omega)$ the explicit dependence of
the error variance on the set of delays, it is easy to
see that if the signal waveforms are constant on the
intervals $[\tau_k, (\tau_k + 1)]$ where $\tau_0, \ldots, \tau_{k-1}$ and $\tau_k, \ldots, \tau_N$, then $\hat{B}_{\text{err}}(\omega)$ is affine in $\omega$ in every code
interval $[\tau_{k-1}, \tau_k + 1]$. This property and the fact that the function $\hat{B}_{\text{err}}(\omega)$ is convex imply that the worst-case of the optimum sequence lower
bound occurs for $\omega = \omega_{\tau_0}, \ldots, \omega_{\tau_{k-1}}$ and that an upper
bound to the average $\tilde{P}_e$ is test probability of error is
given by

$$\tilde{P}_e \leq \frac{1}{N_0} \sum_{\tau_k < \omega} \frac{1}{2N_0} \left| \hat{B}_{\text{err}}(\omega) \right|^2 / \sigma_0^2.$$  
(4.1)

Our first example is a baseband asynchronous sys-
tem with two equal-energy users that employ a single
set of signal waveforms (Fig.3). In this figure the
upper bounds to the best and worst cases of the
optimum detector are indistinguishable from each
other, and for SNR higher than about 6 dB, from the
single user lower bound (which is also the minimum
energy lower bound since $\eta = 0$). Note also that the
maximum interference coefficient is $\sqrt{2}/3$ for all
delays and the performance of the conventional
receiver varies very slightly with the relative delay.

**Fig. 3.** Best and worst-cases of the error probability
of user 1 achieved by conventional and optimum
detectors.

![Fig. 3. Best and worst-cases of the error probability of user 1 achieved by conventional and optimum detectors.](image)

**Fig. 4.** Worst-case and average error probabilities
achieved by conventional and optimum detectors
with 2 active users employing max-SNR =
sequences of length 31.

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sequences of length 31.
The next example investigates the near-far problem (i.e., the effects of unequal received energies) for two users that employ a subset of the previously used maximal-length signature sequences. The conventional and optimum asymptotic efficiency of user 2 obtained via (3.10) and (3.11) is shown in Fig. 5 as a function of the relative energy of the interfering user. As should be expected, the conventional detector approaches the optimum efficiency as the interfering user becomes weaker, and decreases monotonically with the energy ratio until it reaches zero (multiple-access limitation) for relative energies in the intervals \((-4.3, 30)\) dB -users 2 and 1- and \((-9, 90)\) dB -users 2 and 3-.

By means of (3.13) it is easy to show that in the case of two users the optimum asymptotic efficiency is equal to one if and only if the relative energy of the interfering user is greater than 40 dB, where

\[
\rho = \left( \frac{\beta^2}{1 + \rho_{12}^2} \right)^{1/2}.
\]

Further-more if the relative energy of the interfering user is less than 40 dB then the lowest possible efficiency is

\[
1 - \max_{r} \rho^2 (r^2) / 2
\]

(ef. Fig. 5). The intuitive reason for this behavior is that noise, not the randomness of the information of the interfering users, is the primary source of the errors committed in the optimum detection of the user of interest, if the interfering users are sufficiently powerful. When users 1 and 2 are active, the worst-case optimum asymptotic efficiency of Fig. 5 corresponds to \(\rho^2 (1/2, \beta^2 (3/2))\) and shows clearly that the minimum energy is achieved by three different error sequences depending on the relative energy of the interfering user. Over the range of error probabilities corresponding to this example (worst-case relative delay between users 1 and 2) are illustrated in Fig. 6. For three relative energies, namely, SMR(30 dB) -10 dB, -5 dB, 0 dB, wherein the optimum asymptotic efficiencies are 0.88, 0.93 and 1.0 respectively (Fig. 5). It is interesting to observe in the graphics corresponding

![Fig 5. Asymptotic efficiency as a function of the relative energy of the interfering user.](image)

![Fig 6. Bounds on the minimum error probability of user 1. Worst-case delays and 1 active users.](image)