Multi-Antenna Capacity in Interference-Limited Low-Power Conditions

Angel Lozano
Bell Labs (Lucent Technologies)
Holmdel, NJ 07733, USA
e-mail: aloz@lucent.com

Antonia M. Tulino
Universita Degli Studi di Napoli
81125 Napoli, Italy
e-mail: atulino@ee.princeton.edu

Sergio Verdú
Princeton University
Princeton, NJ 08544, USA
e-mail: verdu@ee.princeton.edu

Abstract — **This paper provides an analytical characterization of the multi-antenna capacity for scenarios where the noise is dominated by out-of-cell interference, spatially colored and subject to fading. The analysis is carried out in the region of energy per bit close to its minimum value.**

I. Introduction

In most multi-antenna capacity analyses, the noise is restricted to be spatially white. This adequately models the thermal noise encountered by wireless systems in early phases of deployment, but not the interference from adjacent cells—spatially colored and subject to fading—that dominates the noise in mature stages. As the radio spectrum is aggressively reused across cells, not only does interference dominate the noise but, since a majority of locations lie in the cell periphery, users very often operate at low signal-to-noise ratio (SNR).

As shown in [1], the key performance measures at low SNR are $\frac{E_b}{N_0}_{\min}$ (minimum energy per information bit required to convey data reliably) and S_0, the capacity slope therein in bits/s/Hz/(3 dB), such that the capacity can be expressed as

$$\mathsf{C}(\tfrac{E_b}{N_0}) = \frac{S_0}{3_{\text{dB}}}\left(\frac{E_b}{N_0}|_{\text{dB}} - \frac{E_b}{N_0}_{\min}|_{\text{dB}}\right) + \epsilon$$

with ϵ a second-order term. In contrast with C, whose evaluation may be feasible only in simple canonical cases, both $\frac{E_b}{N_0}_{\min}$ and S_0 can be expressed in closed-form very generally.

Consider n_T transmit and n_R receive antennas with the channel represented by a $(n_R \times n_T)$ random matrix $\sqrt{\mathsf{g}}\mathbf{H}$ whose entries are zero-mean i.i.d. Gaussian.[1] The scalar gain g is such that the entries of $\mathbf{H}$ have unit variance. Comprising both a thermal component and interference, the noise—whose spectral density is N_0—adopts the form

$$\mathbf{n} = \sum_{\ell=1}^{L} \sqrt{\mathsf{g}_\ell}\,\mathbf{H}_\ell \mathbf{x}_\ell + \mathbf{w}$$

with L the number of interferers, $\mathbf{x}_\ell$ the i.i.d Gaussian input and m_ℓ the number of antennas at the ℓ-th interferer, $\sqrt{\mathsf{g}_\ell}\,\mathbf{H}_\ell$ the $(n_R \times m_\ell)$ channel from such interferer, and $\mathbf{w}$ Gaussian thermal noise with spectral density γ. The short-term covariance of $\mathbf{n}$ depends on $\mathbf{H}_\ell$, $\ell=1\ldots L$, whose entries are zero-mean i.i.d. Gaussian. Assembling them into $\overline{\mathbf{H}}=[\mathbf{H}_1|\ldots|\mathbf{H}_L]$, the normalized noise covariance conditioned on $\overline{\mathbf{H}}$, is

$$\mathbf{\Phi}_{\mathbf{n}}(\overline{\mathbf{H}}) = \frac{E[\mathbf{n}\mathbf{n}^\dagger|\overline{\mathbf{H}}]}{N_0} = \frac{1}{N_0}\left(\sum_{\ell=1}^{L} \mathcal{I}_\ell \frac{\mathbf{H}_\ell \mathbf{H}_\ell^\dagger}{m_\ell} + \gamma \mathbf{I}\right)$$

with $\mathcal{I}_\ell \stackrel{\text{def}}{=} \mathsf{g}_\ell\, E[\|\mathbf{x}_\ell\|^2]$ and hence with $N_0 = \gamma + \sum_{\ell=1}^{L} \mathcal{I}_\ell$.

[1] For the extension to correlated and Ricean channels, see [2].

We consider $\mathbf{H}$ and $\mathbf{\Phi}_{\mathbf{n}}$—but not $\overline{\mathbf{H}}$—known to the receiver. Such information, though, is not available to the transmitter.

II. Interference-Limited Low-SNR Capacity

Given a $(n \times n)$ random matrix $\mathbf{A}$, define

$$\zeta(\mathbf{A}) \stackrel{\text{def}}{=} n\frac{E[\text{Tr}\{\mathbf{A}^2\}]}{E^2[\text{Tr}\{\mathbf{A}\}]} \qquad \varphi(\mathbf{A}) \stackrel{\text{def}}{=} \frac{E[\text{Tr}^2\{\mathbf{A}\}]}{E^2[\text{Tr}\{\mathbf{A}\}]}.$$

Proposition 1. *If $\mathbf{\Phi}_{\mathbf{n}}(\overline{\mathbf{H}})$ is the normalized conditional covariance of the noise,*

$$\frac{E_b}{N_0}_{\min} = \frac{\log_e 2}{\mathsf{g}} \frac{1}{E\left[\text{Tr}\left\{\mathbf{\Phi}_{\mathbf{n}}^{-1}(\overline{\mathbf{H}})\right\}\right]} \tag{1}$$

$$S_0 = \frac{2\, n_T n_R}{n_T\, \zeta\left(\mathbf{\Phi}_{\mathbf{n}}^{-1}(\overline{\mathbf{H}})\right) + n_R\, \varphi\left(\mathbf{\Phi}_{\mathbf{n}}^{-1}(\overline{\mathbf{H}})\right)}$$

with expectation over the fading of the interferers, $\overline{\mathbf{H}}$.

Proof: See [2].

Let us now consider pure interference-limited conditions, i.e. $\sum_{\ell=1}^{L} \mathcal{I}_\ell/\gamma \to \infty$.

Proposition 2. *In the presence of a single interferer with m_1 antennas,*

$$\lim_{(\mathcal{I}_1/\gamma)\to\infty} \frac{E_b}{N_0}_{\min} = \frac{\log_e 2}{\mathsf{g}}\left[\frac{1}{n_R} - \frac{1}{m_1}\right]^+$$

with $[x]^+ = x$ if $x \geq 0$ and $[x]^+ = 0$ otherwise. Correspondingly,

$$\lim_{(\mathcal{I}_1/\gamma)\to\infty} S_0 = \frac{2\, n_T n_R}{n_T \Psi + n_R \Upsilon}$$

where, for $m_1 \leq n_R$,

$$\Psi = \frac{n_R}{n_R - m_1} \qquad \Upsilon = 1$$

while, for $m_1 > n_R + 1$,

$$\Psi = \frac{m_1(m_1 - n_R)}{(m_1 - n_R)^2 - 1} \qquad \Upsilon = \frac{\Psi}{n_R} + \frac{(n_R - 1)(m_1 - n_R)}{n_R(m_1 - n_R + 1)}$$

Proof: See [2].

The value of the above result is reinforced by the fact that the capacity with L equal-energy interferers equals the capacity with a single *equivalent* interferer equipped with the same total number of antennas and aggregate energies as those L individual interferers.

References

[1] S. Verdú, "Spectral Efficiency in the Wideband Regime," *IEEE Trans. Inform. Theory*, vol. 48, pp. 1319–1343, Jun. 2002.

[2] A. M. Tulino, A. Lozano and S. Verdú, "Power-Bandwidth Tradeoff of Multi-Antenna Systems in the Low-Power Regime," *Multiantenna channels: Capacity, Coding and Signal Processing (Editors: G. Foschini and S. Verdú), American Mathematical Society Press*, 2003.

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