MULTIUSER DEMODULATION

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1. Introduction

In recent years, Code Division Multiple Access (CDMA) has received considerable attention as a multiplexing technology in radio networks. In CDMA, the transmitters send their infomation simultaneously, asynchronously and through the same channel. Each transmitter is assigned a fixed, distinct signature waveform which he uses to modulate his message in the same fashion as in single-user communication. The destination receives a noisy version of the superposition of all the transmitted waveforms and the information sent by each user can be demodulated by correlating the received signal with each of the signature waveforms. This demodulator, whose use is widespread in practice, is referred to as the conventional single-user detector (Figure 1). As is well-known, when the channel output is corrupted by additive white Gaussian noise, the conventional single-user detector minimizes the probability of error in a single-user channel, i.e., in the absence of interfering users. In the presence of interfering users, the performance of the conventional single-user detector is acceptable provided that the energies of the received signals are not too dissimilar and that the signature waveforms are designed so that their crosscorrelations are low enough (this depends on the desired maximum number of simultaneous users). In practice, low crosscorrelations are usually achieved employing Spread-Spectrum Pseudonoise sequences of long periodicity. If the received signal energies are indeed dissimilar, i.e., some users are very weak in comparison to others, then the conventional single-user detector is unable to recover the messages of the weak users reliably, even if the signature waveforms have very low crosscorrelations. This is known as the *near-far problem* and is the main shortcoming of CDMA systems.

In contrast to the conventional single-user demodulator, the approach in *multiuser demodulation* (also known as multiuser detection and interference suppression) is to take into account the multiuser interference. Increasing the sophistication of the signal processing at the receiver, it is possible to greatly relax the power control requirements of the mobile transmitters. This philosophy is particularly indicated in applications where the base (not mobile) stations are few relative to the number of transmitters, thereby favoring designs with very simple transmitters.

In this tutorial, we give a brief explanation of the main results obtained in multiuser demodulation. The chief reason why multiuser detection did not develop until relatively recently was the belief shared by many a worker in Spread-Spectrum that multiuser interference is accurately modeled as a white Gaussian random process, and thus the conventional detector is essentially optimum. It is not difficult to build an infinite population multiuser signal model which can be rigorously shown to be asymptotically Gaussian as the individual amplitudes go to zero with the appropriate speed. The knowledge at the receiver, number of transmitters, signature waveforms, and power levels encountered in many practical situations (e.g. in near-far environments) render the Gaussian approximation completely useless.

Section 2 presents the models of synchronous and asynchronous Code-Division Multiple-Access channels subject to additive white Gaussian noise used throughout the paper. Section 3 defines the main performance measures of interest in the analysis of multiuser detectors, namely, bit-error-rate, asymptotic efficiency and near-far resistance. Those performance measures are obtained first for the simplest detector, the conventional multiuser detector, in Section 4. The derivation and analysis of multiuser detectors is reviewed in Section 5. For the sake of clarity and in contrast to [16], the synchronous channel is treated separately, thereby highlighting that all the major issues in the analysis (but not design) of the optimum multiuser detector are present in the simpler setting of synchronous transmitters. One of the most attractive tradeoffs between complexity and performance in multiuser detection is the decorrelating detector [5, 6], which is studied in Section 6 with a treatment that emphasizes simplicity. Finally, other solutions proposed so far in multiuser detection are reviewed in Section 7.

2. CDMA Channel Model

In this section, we present the model of the Code-Division Multiple-Access channel shared by K users. User $k \in \{1,...K\}$ is assigned a signature waveform $s_k(t)$ which is equal to zero outside the interval $[0,T]$ where T is the inverse of the data rate, assumed to be equal for all users. Throughout this work we assume that the modulation of the signature waveforms is linear, i.e., the data stream of the kth user

$$\mathbf{b}_k^T \triangleq [b_k(-M),...b_k(M)]$$

is transmitted as

$$A_k \sum_{i=-M}^{M} b_k(i)\, s_k(t - iT). \tag{2.1}$$

Moreover, we assume without loss of generality that the signature waveforms have unit energy

$$\int_0^T s_k^2(t)\, dt = 1, \quad k = 1,...,K\ . \tag{2.2}$$

For convenience, we further assume that the modulation is antipodal, i.e., $b_k(i) \in \{-1,1\}$. A_k is the received amplitude of the kth user, and the energy-per-bit is equal to

$$w_k \triangleq A_k^2 \int_0^T b_k^2(i)\, s_k^2(t)\, dt = A_k^2 \tag{2.3}$$

The demodulator obtains the sum of the K transmitted waveforms embedded in additive white Gaussian noise. If the transmitters cooperate to maintain symbol-synchronism among them, then the received signal is

$$r(t) = S_t(\mathbf{b}) + n(t) \tag{2.4}$$

where $\{n(t)\}$ is white Gaussian noise with power spectral density equal to σ^2, and

$$S_t(\mathbf{b}) = \sum_{k=1}^{K} A_k \sum_{i=-M}^{M} b_k(i)\, s_k(t - iT) \tag{2.5}$$

with the $K \times (2M + 1)$ matrix **b** made up of the rows $\mathbf{b}_k$, $k = 1,...K$. For all purposes, it is enough to consider the one-shot version of the channel in (2.4) corresponding to any arbitrary bit, e.g., for $i = 0$,

$$r(t) = \sum_{k=1}^{K} A_k\, b_k(0)\, s_k(t) + n(t) \tag{2.6}$$

and therefore the value of M is irrelevant in the case of symbol-synchronous users. However, one of the nice features of Code-Division Multiple-Access is that symbol synchronization is not required among the transmitters. When the users are symbol-asynchronous, the offset between their signals is modeled by the delays $\tau_k \in [0,T]$ $k = 1,...K$ such that the received signal can be expressed as

$$r(t) = S_t(\mathbf{b}) + n(t) \tag{2.7a}$$

with

$$S_t(\mathbf{b}) = \sum_{k=1}^{K} \sum_{i=-M}^{M} A_k\, b_k(i)\, s_k(t - iT - \tau_k) \tag{2.7b}$$

In addition to the signal-to-noise ratios, the parameters that determine the performance of a Code-Division Multiple-Access system are the crosscorrelations between every pair of signature waveforms

$$\rho_{ij}(\tau) = \int_0^T s_i(t)\, s_j(t - \tau)\, dt \qquad i \le j \tag{2.8a}$$

$$\rho_{ji}(\tau) = \int_0^T s_i(t)\, s_j(t + T - \tau)\, dt \qquad i > j \tag{2.8b}$$

The value of τ of interest in (2.8) is equal to the difference between the offsets $\tau_j - \tau_i$, and the users can be labeled so that their offsets satisfy $\tau_1 \le \tau_2 \cdots \le \tau_K$, in which case the argument of the crosscorrelation functions in (2.8) takes values on $[0,T]$. For notational convenience, the dependence of the crosscorrelations on the relative offsets will be dropped in the sequel, as the delays remain fixed. Note finally that if the channel is symbol-synchronous, the crosscorrelations defined in (2.8b) are equal to zero and only one crosscorrelation (2.8a) need be defined for each pair of users.

3. Performance Measures

The main performance measure of interest in multiuser detection is the bit error rate. In general, the probability that the demodulator makes the wrong decision on the ith bit of the kth user, $P[b_k(i) \ne \hat{b}_k(i)]$ depends on M, $i \in \{-M,..M\}$ and $k \in \{1,...K\}$. However, as the frame length is usually a large integer, it makes sense to consider the bit error rate defined as

$$P_k(\sigma) = \lim_{M \to \infty} P[b_k(i) \ne \hat{b}_k(i)] \tag{3.1}$$

In the single-user case, the minimum probability of error is attained by a demodulator which passes the received waveform through a matched filter for the signature waveform of the active user and compares the output sample to a zero threshold. Thus the matched filter output is a conditionally Gaussian random variable with variance equal to σ^2 and mean equal to A_k. Therefore, the single-user bit error rate is equal to

$$P_k(\sigma) = Q(\frac{A_k}{\sigma}) \qquad (3.2)$$

where

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \qquad (3.3)$$

The presence of other users in the channel can only increase $P_k(\sigma)$, and it is of interest to quantify the multiuser error probability relative to (3.2). In order to do that, we define the *effective energy* of user k, $e_k(\sigma)$, as the energy that that user would require to achieve bit-error-rate equal to $P_k(\sigma)$ in the same Gaussian channel, but without interfering users, i.e., (cf. (3.2))

$$P_k(\sigma) = Q(\frac{\sqrt{e_k(\sigma)}}{\sigma}) \qquad (3.4)$$

Thus, the *efficiency* or ratio between the effective and actual energies $e_k(\sigma)/w_k$ is an alternative way to characterize the multiuser bit-error-rate. As $e_k(\sigma) \le w_k$, the efficiency belongs to the interval $[0,1]$ and quantifies the performance loss (in dB) due to the existence of other users in the channel. The *asymptotic efficiency* is defined as

$$\eta_k = \lim_{\sigma \to 0} e_k(\sigma)/w_k \qquad (3.5)$$

and measures the slope with which $P_k(\sigma)$ goes to 0 in the high signal-to-noise ratio region, i.e.,

$$\eta_k = \sup\{0 \le r \le 1;\ \lim_{\sigma \to 0} P_k(\sigma)/Q(\frac{\sqrt{r}\,A_k}{\sigma}) < +\infty\} \qquad (3.6)$$

It turns out that the asymptotic efficiency is very close to the efficiency for the signal-to-noise ratios encountered in most CDMA applications. Moreover, in some cases the asymptotic efficiency is easier to compute than the bit-error-rate and it allows intuitive and easy-to-grasp depictions of the effects of unequal received energies.

The near-far problem is the main impairment suffered by conventional CDMA systems. Therefore, it is important to quantify the degree of robustness against the near-far problem achieved by each of the demodulators we will review in the sequel. To that end, we define the *near-far resistance* as the minimum asymptotic efficiency over the relative energies of all the other users,

$$\bar{\eta}_k = \inf_{\substack{w_j > 0 \\ j \ne k}} \eta_k \qquad (3.7)$$

Asymptotic efficiency (3.5) was first proposed as a performance measure of multiuser demodulators in [14-16] It is unrelated to similarly termed performance measures encountered in statistical inference. The near-far resistance measure in (3.7) was proposed in [3, 5, 18] in the context of synchronous channels and in [4, 6] in order to study the near-far problem in asynchronous channels.

4. Conventional Single-user Detector

For the sake of clarity, we will first discuss the case when the users are synchronous. The output of the k^{th} matched filter is equal to

$$y_k = A_k \int_0^T r(t)\, s_k(t)\, dt$$

$$= A_k\, b_k(0) + \sum_{j \neq k} A_k A_j\, b_j(0)\, \rho_{jk} + n_k \qquad (4.1)$$

where n_k is Gaussian with zero mean and variance equal to $A_k^2\,\sigma^2$. Acceptable performance of this system will require that the second and third terms are comparatively small relative to the first one. This is quantified by the error probability, which is obtained recalling that the $b_j(0)$, $j = 1,...K$ are independent, equally likely to be -1 or 1.

$$P_k^c(\sigma) = P\,[b_k(0) \neq \hat{b}_k(0)] = \frac{1}{2^{K-1}} \sum_{e_1 \in \{-1,1\}} \cdots \sum_{\substack{e_j \in \{-1,1\} \\ j \neq k}} \cdots \sum_{e_K \in \{-1,1\}} Q\left[\frac{A_k + \sum_{j \neq k} e_j\, A_j\, \rho_{jk}}{\sigma}\right] \qquad (4.2)$$

which shows that any user which is not orthogonal to the k^{th} user can drive the error probability to intolerable levels with sufficiently high power. In the two-user case (4.2) reduces to

$$P_1^c(\sigma) = \frac{1}{2}\, Q\left[\frac{A_1 + A_2\, \rho_{12}}{\sigma}\right] + \frac{1}{2}\, Q\left[\frac{A_1 - A_2\, \rho_{12}}{\sigma}\right] \qquad (4.3)$$

If the k^{th} user signature waveform is orthogonal to each of the other waveforms, i.e., $\rho_{jk} = 0$, $j \neq k$, then

$$P_k^c(\sigma) = Q\left[\frac{A_k}{\sigma}\right] \qquad (4.4)$$

and the efficiency and near-far resistance are equal to unity. If, on the other hand, the signature waveforms are not orthogonal, e.g., due to bandwidth or complexity constraints, then the bit error rate depends on the strength of the interferers. According to (3.6) and (4.2), the asymptotic efficiency is equal to

$$\eta_k = \max{}^2\{0,\, 1 - \sum_{j \neq k} \frac{A_j}{A_k}\, |\rho_{jk}|\}. \qquad (4.5)$$

From (4.5) it follows that the near-far resistance is equal to

$$\bar{\eta}_k = \min_{A_j > 0} \max{}^2\{0,\, 1 - \sum_{j \neq k} \frac{A_j}{A_k}\, |\rho_{jk}|\}$$

$$= \begin{cases} 1 & \text{if } \rho_{jk} = 0 \text{ for all } j \neq k \\ 0 & \text{otherwise} \end{cases} \qquad (4.6)$$

In the asynchronous case, the analysis of bit-error-rate yields

$$P_k^c(\sigma) = \frac{1}{2^{K-2}} \sum_{\substack{e_1 \in \{-1,1\}}} \cdots \sum_{\substack{e_K \in \{-1,1\} \\ j \neq k}} \sum_{d_1 \in \{-1,1\}} \cdots \sum_{\substack{d_K \in \{-1,1\} \\ j \neq k}} Q\left[\frac{A_k + \sum_{j \neq k} e_j A_j \rho_{jk} + \sum_{j \neq k} d_j A_j \rho_{kj}}{\sigma}\right] \tag{4.7}$$

with efficiency and near-far resistance given by the corresponding expressions in the synchronous case replacing $|\rho_{jk}|$ therein by $|\rho_{jk}| + |\rho_{kj}|$, i.e.,

$$\eta_k = \max{}^2\{0,\ 1 - \sum_{j \neq k} \frac{A_j}{A_k} (|\rho_{jk}| + |\rho_{kj}|)\} \tag{4.8}$$

and

$$\bar{\eta}_k = \begin{cases} 1 & \text{if } \rho_{jk} = \rho_{kj} = 0 \text{ for all } j \neq k \\ 0 & \text{otherwise} \end{cases} \tag{4.9}$$

When the asymptotic efficiency becomes zero, one of the arguments in the Q-function appearing in (4.7) is negative, and therefore

$$\lim_{\sigma \to 0} P_k^c(\sigma) > 0 \tag{4.10}$$

i.e., even in the absence of noise the conventional detector makes errors due to the large amount of multiaccess interference. The precise level at which this phenomenon (akin to a closed eye diagram in classical communication) occurs is obtained from (4.8):

$$A_k \leq \sum_{j \neq k} A_j\ (|\rho_{jk}| + |\rho_{kj}|) \tag{4.11}$$

Since it is not possible to design two signature waveforms j and k such that $\rho_{jk} = \rho_{kj} = 0$ for all relative delays, the conventional demodulator for a CDMA system is always vulnerable to a high enough multiaccess interference level that renders the system performance useless.

5. Optimum Detectors

We now address the design and analysis of optimum multiuser detectors. In this section, we will assume that the receiver knows the signature waveform employed by each user and has acquired the timing epochs and amplitudes of the transmitted signals. We assume that the transmitted bits are equiprobable and independent. Because of the interdependence of the a posteriori probabilities, there exist several optimality criteria that make sense. We can select the most likely sequence of bits

$$\mathbf{b} = \begin{bmatrix} b_1(-M) & & b_1(M) \\ \vdots & \cdots & \vdots \\ b_K(-M) & & b_K(M) \end{bmatrix} \tag{5.1}$$

i.e.,

$$\arg\max_{\mathbf{b}} P[\mathbf{b} \mid \{r(t),\, t \in (-MT,\, MT+2T)\}] \tag{5.2}$$

or we can select the decisions that minimize the probability of error, i.e., for each k and i we select

$$\arg\max_{b\in\{-1,1\}} P[b_k(i) = b \mid \{r(t),\, t \in (-MT,\, MT+2T)\}] \tag{5.3}$$

1. Synchronous Channel

Since the noise is white and Gaussian we have

$$P[\{r(t),\, t \in [0,T]\} \mid \mathbf{b}] = C \exp\left(-\frac{1}{2\sigma^2}\int_0^T [y(t) - \sum_{k=1}^{K} b_k(0)\, A_k\, s_k(t)]^2\, dt\right) \tag{5.4}$$

where $\mathbf{b} = [b_1(0),\ldots,b_k(0)]^T$. Expanding the integral in (5.4) it can be verified that the vector $\mathbf{b}^*$ that maximizes (5.4) satisfies:

$$\mathbf{b}^* = \arg\max_{\mathbf{b}\in\{-1,1\}^K} \Omega(\mathbf{b}) \tag{5.5}$$

where

$$\Omega(\mathbf{b}) = 2\int S_t(\mathbf{b})\, r(t)\, dt - \int S_t^2(\mathbf{b})\, dt = 2\mathbf{b}^T\mathbf{y} - \mathbf{b}^T\, \mathbf{H}\, \mathbf{b}. \tag{5.6}$$

In Equation (5.6) the coordinates of the vector $\mathbf{y}$ are the matched filter outputs in (4.1),

$$\mathbf{H} = \mathbf{A}\,\mathbf{R}\,\mathbf{A} \tag{5.7}$$

where $\mathbf{A} = \mathrm{diag}\{A_1, \ldots, A_K\}$ and $\mathbf{R}$ is the normalized crosscorrelation matrix whose entries are $\{\rho_{ij}\}$.

The decision that minimizes the probability of error of the k^{th} user is the element $b^* \in \{-1,1\}$ that maximizes

$$\sum_{\substack{\mathbf{b} \\ b_k = b}} \exp(\Omega(\mathbf{b})/2\sigma^2) \tag{5.8}$$

This criterion leads to smooth decision regions in the K-dimensional decision space of the observables in (5.7), whereas the decision regions corresponding to the globally optimum criterion in (5.5) are polytopes. In the majority of applications, (5.5) is preferred over (5.8) due to its lower complexity and the fact that the bit-error-rates obtained under both criteria are very close for signal-to-noise ratios of interest in practice.

The maximization in (5.5) is a combinatorial optimization problem which has been shown to be NP-hard in K [15, 20]. This means that for arbitrary crosscorrelation matrices, no algorithm whose complexity is polynomial in the number of users is feasible unless such an algorithm exists for long-standing combinatorial problems such as the *traveling salesman* and *integer linear programming* problems.

Let us now direct our attention to the analysis of the minimum probability of error achievable in the synchronous channel. We will obtain lower and upper bounds on the error rate for each user which give a close approximation over the whole SNR range. Upper bounds can be obtained by analyzing detectors that are suboptimum in terms of error probability. For example, the probability of error of the conventional detector (4.2) gives such an upper bound. As we will see this upper bound is very loose unless the SNR is low, in which case the background noise is the main factor limiting performance, and thus, the conventional detector which neglects the interference from the other users is near optimum in that situation. A more useful upper bound is obtained by upper bounding the error probability of the globally optimum multiuser detector in (5.5). To that end we will introduce the following notation.

The normalized difference between any pair of distinct transmitted vectors is referred to as an *error vector*. The set of error vectors that affects the k^{th} user is

$$E_k = \{\boldsymbol{\varepsilon} \in \{-1,0,1\}^K,\ \varepsilon_k \neq 0\}$$

and the set of (nonzero) error vectors is denoted by

$$E = \bigcup_{k=1}^{K} E_k$$

The set of error vectors that are admissible conditioned on $\mathbf{b} \in \{-1,1\}$ being transmitted, that is, those that correspond to the difference between $\mathbf{b}$ and the vector selected by the detector is denoted by

$$A(\mathbf{b}) = \{\boldsymbol{\varepsilon} \in E,\ 2\boldsymbol{\varepsilon} - \mathbf{b} \in \{-1,1\}^K\} \quad (5.9)$$

The admissible error vectors that affect the k^{th} user are

$$A_k(\mathbf{b}) = A(\mathbf{b}) \cap E_k \quad (5.10)$$

The number of nonzero components of an error vector and the energy of a hypothetical multiuser signal modulated by $\boldsymbol{\varepsilon}$ are denoted, respectively, by

$$w(\boldsymbol{\varepsilon}) = \sum_{k=1}^{K} |\varepsilon_k| \quad (5.11)$$

and

$$\|S(\boldsymbol{\varepsilon})\|^2 = \int_{-\infty}^{\infty} \left(\sum_{k=1}^{K} \varepsilon_k A_k s_k(t)\right)^2 dt \quad (5.12)$$

An error vector $\boldsymbol{\varepsilon} \in E$ is *decomposable* into $\boldsymbol{\varepsilon}' \in E$ and $\boldsymbol{\varepsilon}'' \in E$ if

1) $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}' + \boldsymbol{\varepsilon}''$
2) $\varepsilon_k = 0 \Rightarrow \varepsilon_k' = \varepsilon_k'' = 0$
3) $\langle S(\boldsymbol{\varepsilon}'), S(\boldsymbol{\varepsilon}'')\rangle = \boldsymbol{\varepsilon}'^T \mathbf{H} \boldsymbol{\varepsilon}'' \geq 0$

The set of *indecomposable* vectors that affects the k^{th} user (i.e., $\varepsilon_k \neq 0$) is denoted by F_k. The upper bound on the probability of error of the k^{th} user is

$$P_k \leq \sum_{\boldsymbol{\varepsilon} \in F_k} 2^{-w(\boldsymbol{\varepsilon})}\, Q\left(\frac{\|S(\boldsymbol{\varepsilon})\|}{\sigma}\right) \quad (5.13)$$

This bound is found by bounding the error probability of an 2^K-ary hypothesis test in terms of the error probabilities of binary tests.

Using a hypothetical optimum detector that has some side information, it is shown in [15, 16] that the probability of error is lower bounded by

$$P_k \geq P[d_k(\mathbf{b}) = d_{k,min}]\, Q(d_{k,min}/\sigma) \tag{5.14}$$

where

$$d_k(\mathbf{b}) = \min_{\epsilon \in A_k(\mathbf{b})} \|S(\epsilon)\| \tag{5.15}$$

and

$$d_{k,min} = \min_{\mathbf{b} \in \{-1,1\}^K} d_k(\mathbf{b}) = \min_{\substack{\epsilon \in \{-1,0,1\}^K \\ \epsilon_k = 1}} \|S(\epsilon)\| \tag{5.16}$$

i.e., $d_{k,min}^2$ is equal to one-half of the minimum energy of the difference between two multiuser signals that differ in the k^{th} bit.

Asymptotically as $\sigma \to \infty$, the bound in (5.13) is dominated by the terms corresponding to the error vectors that achieve $d_{k,min}$. Thus,

$$\lim_{\sigma \to 0} \sum_{\epsilon \in F_k} 2^{-w(\epsilon)}\, Q(\|S(\epsilon)\|/\sigma)/Q(d_{k,min}/\sigma) = \sum_{\substack{\epsilon \in F_k\, s.t. \\ \|S(\epsilon)\| = d_{k,min}}} 2^{-w(\epsilon)} \tag{5.17}$$

and the upper and lower bounds differ asymptotically by at most a constant. If the minimizer of the energy is unique (modulo sign) then the ratio of both bounds is equal to unity asymptotically. In any case, (3.6), (5.14), and (5.17) imply that the asymptotic efficiency is given by

$$\eta_k = d_{k,min}^2 / w_k, \tag{5.18}$$

which in the two-user case, reduces to

$$\eta_1 = \min\{1,\, 1 + \frac{A_2^2}{A_1^2} - 2|\rho_{12}|\, \frac{A_2}{A_1}\} \tag{5.19}$$

It is useful to compare this expression to the asymptotic efficiency of the conventional receiver, $\eta_1 = \max^2\{0, 1 - \frac{A_2}{A_1}|\rho_{12}|\}$. In both cases, η_1 is close to unity if $A_2 \ll A_1$. However, (5.19) is not monotonic in A_2/A_1. Actually, if

$$A_2/A_1 \geq 2\,|\rho_{12}| \tag{5.20}$$

then $\eta_1 = 1$. Therefore, as long as the energy of user 2 exceeds the threshold given by (5.20) the bit-error-rate of user 1 is equivalent to the single-user case where use 2 is not active. The explanation of this behavior of the optimum receiver is that if the interfering user is sufficiently powerful, then the primary source of errors committed in the optimum demodulation of user 1 is the background Gaussian noise, rather than the randomness of the information carried by the interfering signal. Note that according to the threshold in (5.20), an interferer who is 3 dB weaker than the user of interest has no appreciable effect on the bit-error-rate as long as the maximum crosscorrelation is below 0.35 (a mild condition on the signal design). Interestingly, the same is true if the relative energy of the interferer is *higher* than -3 dB.

The near-far resistance is obtained from (3.7) and (5.16) as

$$\bar{\eta}_k = \min_{\substack{w_i \geq 0 \\ i \neq k}} \min_{\substack{\varepsilon \in \{-1,0,1\}^K \\ \varepsilon_k = 1}} \frac{1}{w_k} \varepsilon^T \mathbf{H} \varepsilon = \min_{\substack{x \in R^K \\ x_k = 1}} \mathbf{x}^T \mathbf{R} \mathbf{x} = 1/\mathbf{R}^+_{kk} \tag{5.21}$$

where the last equality is shown in [5] and $\mathbf{R}^+$ denotes the (Moore-Penrose generalized) inverse of $\mathbf{R}$. In the two-user case (5.21) results in

$$\bar{\eta}_k = 1 - \rho_{12}^2 \tag{5.22}$$

which implies that in the present synchronous case the optimum multiuser detector (whether in the globally optimum sense, or minimum bit-error-rate sense) does not suffer from the near-far problem as long as the set of signature waveforms is linearly independent. This should be contrasted with the behavior of the conventional detector which is not near-far resistant unless the signature waveforms are orthogonal.

2. Asynchronous Channel

Many of the results in the previous subsection can be generalized in a fairly straightforward fashion to the more important asynchronous problem. In essence, this is bectransmitted by a different fictitious user over the interval $[-MT, MT + 2T]$. This results in an equivalent synchronous channel with $(2M + 1)K$ fictitious users. This approach leads to a straightforward generalization of the performance analysis reviewed in the previous subsection. It also shows a way to obtain the optimum decisions; however this has no practical value as its complexity is exponential in the product of the number of users and the frame length. In order to obtain a useful detector one needs to exploit the fact that each symbol overlaps with at most $2K - 2$ symbols. This makes it possible to decompose $\Omega(\mathbf{b})$ in a sequential fashion that lends itself to efficient optimization. Let $z_{k+iK}(t) = A_k\, s_k(t - iT - \tau_k)$ and denote the matched filter output

$$y_j = \int_{-\infty}^{\infty} z_j(t) r(t) dt. \tag{5.23}$$

Then we can write (cf. 5.6)

$$\Omega(\mathbf{b}) = 2 \sum_{j=1-MK}^{MK+K} b_j y_j - \sum_{j=1-MK}^{MK+K} \sum_{l=1-MK}^{MK+K} b_j b_l h(j,l) \tag{5.24}$$

where

$$h(j,l) = \int_{-\infty}^{\infty} z_j(t) z_l(t) dt \tag{5.25}$$

It follows immediately from the definition that these coefficients satisfy

(1) $h(k + iK, k + iK) = A_k^2 = w_k$

(2) $h(k + iK, n + iK) = h(k,n)$ for all i

(3) $h(j,l) = A_j A_l \rho_{jl}$ if $1 \leq j < l \leq K$

(4) $h(j + K, l) = A_j A_l \rho_{lj}$ if $1 \leq j < l \leq K$

(5) $h(j,l) = 0$ unless $|j - l| < K$

Using those properties, and letting $\kappa(j) \in \{1,...,K\}$ be the modulo-K remainder of j (i.e., for some i, $j = \kappa(j) + iK$), we can decompose the quadratic form

$$\sum_{j=1-MK}^{MK} \sum_{l=1-MK}^{MK} d_j d_l h(j,l) = \sum_{j=1}^{MK} d_j \left[w_{\kappa(j)} + 2 \sum_{l=j-K=1}^{j-1} d_l h(j,l) \right] = \sum_{j=1-MK}^{MK} d_j \left[w_{\kappa(j)} + 2 \sum_{n=1}^{K-1} d_{j-n} \mathbf{g}_{\kappa(j)}(K-n) \right] \quad (5.26)$$

where $\mathbf{g}_\kappa(m) = h(\kappa + K, \kappa + m)$. Plugging (5.26) into (5.24), we can express $\Omega(\mathbf{b})$ as a sum of $(2M+1)K$ terms, each of which depends on K components of $\mathbf{b}$ and such that consecutive terms depend on the same components but one. Specifically, we can write

$$\Omega(\mathbf{b}) = \sum_{j=1}^{MK} \lambda_j(x_j, b_j) \quad (5.27)$$

where

$$\lambda_j(\mathbf{x}, u) = u\,[2y_j + u w_{\kappa(j)} - 2\mathbf{x}^T \mathbf{g}_{\kappa(j)}] \quad (5.28)$$

and $\mathbf{x}_j$ is the state of a shift-register $K-1$ dimensional system

$$\mathbf{x}_{j+1}^T = [x_{j+1}(1), \cdots, x_{j+1}(K-1)] = [x_j(2), \cdots, x_j(K-1), b_j]; \;\; x_0 = 0. \quad (5.29)$$

It is now apparent that the maximization of (5.27) can be carried out by a Viterbi algorithm. Thus, the structure of the optimum multiuser detector for the asynchronous channel is a bank of matched filters, which are sampled in a round-robin fashion, and those observables are processed by a Viterbi algorithm (Figure 2).

The error probability analysis of the optimum synchronous detector carries over to the asynchronous case with few modifications. In the two-user case, the expression found in the synchronous case (5.19) generalizes to [17]

$$\eta_1 = 1 - \frac{A_2}{A_1} \left[\left\{ 2|\rho_{12}| - \frac{A_2}{A_1} \right\}^+ + \left\{ 2|\rho_{21}| - \frac{A_2}{A_1} \right\}^+ \right]. \quad (5.30)$$

and the near-far resistance becomes

$$\bar{\eta}_k = [1 - (\rho_{12} + \rho_{21})^2]^{1/2} [1 - (\rho_{12} - \rho_{21})^2]^{1/2}. \quad (5.31)$$

The analysis and derivation of optimum multiuser detectors was first carried out in the asynchronous case [15]. An earlier attempt by Schneider [10] to obtain an optimum multiuser detector for the synchronous channel turned out to be erroneous (cf. [15, 20]). The first optimum multiuser detector was published in [14]. Its complexity is higher than that of the demodulator we saw in this section, which appeared in [13, 15, 16]. The analysis presented here appeared in [16]. The same method has been applied [19] to the single-user intersymbol interference problem to yield a bound which is tigher than the Forney bound [2]. The optimum near-far resistance was obtained in [5, 18] in the synchronous case and in [4, 6] in the asynchronous case.

6. Decorrelating Detector

The optimum detector reviewed in the last section, affords important performance gains over the conventional single-user detector, including the solution of the near-far problem. The price is an increase in implementation costs, in particular, the exponential complexity of the decision algorithm in the number of users and the need to acquire the actual values of the received energies. It is therefore of interest to explore whether there are multiuser demodulators which are more economical to implement than the optimum detector while retaining its superior performance.

Let us examine first the situation when the demodulator is constrained to ignore the received amplitudes of the active transmitters. Unless a prior distribution is known for each of those amplitudes, we are led to consider joint maximum likelihood estimation of amplitudes and transmitted bits.

1. Synchronous channel

The most likely bits and amplitudes are those that best explain the received waveform in a mean-square sense, i.e.,

$$\min_{b\in\{-1,1\}^K} \min_{\substack{A_k\in[0,+\infty)\\ k=1,\ldots,K}} \int_0^T [r(t) - \sum_{k=1}^{K} A_k b_k s_k(t)]^2 \, dt \qquad (6.1)$$

Denoting the normalized matched filter outputs by

$$\tilde{y}_k = \int_0^T r(t) s_k(t) \, dt \qquad (6.2)$$

we see that the minimization in (6.1) is equivalent to

$$\max_{c\in R^K} 2\, \mathbf{c}\tilde{\mathbf{y}}^T - \mathbf{c}^T \mathbf{R} \mathbf{c} \qquad (6.3)$$

where $c_k = A_k b_k$. If the crosscorrelation matrix $\mathbf{R}$ is invertible, the solution to (6.3) is

$$\mathbf{c}^* = \mathbf{R}^{-1}\tilde{\mathbf{y}} \qquad (6.4)$$

and the most likely bits are given by

$$\hat{\mathbf{b}} = \mathrm{sgn}\,(\mathbf{c}^*) = \mathrm{sgn}\,(\mathbf{R}^{-1}\tilde{\mathbf{y}}) \qquad (6.5)$$

whereas the absolute values of the components of $\mathbf{c}^*$ give estimates of the received amplitudes which may or may not be of interest. The demodulator in (6.5) subjects the matched filter outputs to a linear transformation prior to threshold comparison. This receiver can be implemented using the same structure as the conventional receiver where demodulation of each user is decoupled. Note that this is particularly advantageous if the receiver is not interested in the demodulation of each active user in the channel. Instead of matched filtering the incoming signal with the signature waveform of the k^{th} user $s_k(t)$, we use a matched filter for

$$\sum_{j=1}^{K} R_{kj}^{+} \, s_j(t) \qquad (6.6)$$

where R_{kj}^{+} is shorthand for $(\mathbf{R}^{-1})_{kj}$.

For example, in the two-user case the receiver correlates with respect to

$$s_1(t) - \rho\, s_2(t) \tag{6.7}$$

where $\rho = \int_0^T s_1(t)s_2(t)dt$.

We can write the matched filter outputs in (6.2) in vector notation:

$$\tilde{\mathbf{y}} = \mathbf{RAb} + \tilde{\mathbf{n}} \tag{6.8}$$

where $\tilde{\mathbf{n}}$ is a Gaussian K-vector with zero mean and covariance matrix equal to $\sigma^2\mathbf{R}$. Note that in the hypothetical case when there is no background Gaussian noise ($\sigma^2 = 0$), the multiuser detector in (6.5) achieves perfect detection, regardless of the values of the received energies. A nice property, which augurs well for the performance of this receiver (at least in the high SNR region). Indeed, it is the lack of this feature that makes the conventional receiver so sensible to the near-far problem. Moreover, the output of the filter matched to (6.6) contains no trace of the signals modulated by the interfering transmitters, as for any $\{a_j \in R\}$,

$$\int_0^T [\sum_{i \neq k} a_i s_i(t)]\, [\sum_{j=1}^{K} R_{kj}^{+} s_j(t)]\, dt = \sum_{i \neq k} a_i (\mathbf{R}^{-1}\mathbf{R})_{ik} = 0 \tag{6.9}$$

In other words, the detector correlates with the projection of $s_k(t)$ on the subspace orthogonal to the subspace spanned by the other signature waveforms $\{s_j(t),\, j \neq k\}$. Thus, it effectively tunes out the multiuser interference, and hence its name: *decorrelating detector*. Note that this strategy is not feasible if $s_k(t)$ belongs to the linear span of the interfering signature waveforms, and thus $\mathbf{R}$ is not invertible. On the other hand, it should be noted that the decorrelating detector for the k^{th} user exists as long as $s_k(t)$ is not spanned by the other signals, even if $\mathbf{R}$ is singular. This requires substituting the inverse of $\mathbf{R}$ in (6.4) by a generalized inverse [5].

Among all multiuser detectors, the decorrelating detector admits the simplest performance analysis. Due to property (6.9), the output of the matched filter in (6.6) has only two components: $A_k b_k$ due to the signal of user k, and $\tilde{n}_k$ due to the background noise which is Gaussian with zero mean and variance equal to $\sigma^2 R_{kk}^{+}$. Therefore its probability of error is

$$P_k^d(\sigma) = Q(A_k / \sigma \sqrt{R_{kk}^{+}}) \tag{6.10}$$

which means that its efficiency is equal to

$$\eta_k^d = 1/R_{kk}^{+}. \tag{6.11}$$

In this case, the efficiency is independent of the background noise level or the energies of the interferers. Therefore, the efficiency of the decorrelating detector is equal to its asymptotic efficiency and to its near-far resistance.

Comparing (6.11) to (5.21), we see that the decorrelating detector achieves optimum near-far resistance. This means that knowledge of the received energies is not required to combat the near-far problem, and a receiver whose complexity is similar to that of the conventional detector achieves the same degree of robustness against imbalances in the received energies as the optimum detector, which has exponential complexity in the number of users.

Figure 3 depicts the asymptotic efficiency of user 1 as a function of the relative energy of the interferer in a 2-user case. The conventional single-user detector asymptotic efficiency is monotonically decreasing with the relative energy of the interferer and it is equal to 0 for (cf. (4.5))

$$A_2 \geq A_1 / |\rho_{12}|$$

For very low interference, the conventional detector achieves nearly optimum asymptotic efficiency. The optimum asymptotic efficiency is a parabola in A_2/A_1 in the interval $[0, 2|\rho_{12}|]$ and it is equal to unity for larger interference levels (cf (5.19)-(5.20)). Its minimum value (near-far resistance) coincides with the asymptotic efficiency of the decorrelating detector, namely, $1 - \rho_{12}^2$ which is independent of A_2/A_1 (cf. (5.22)).

2. Asynchronous channel

The analysis and derivation of the decorrelating detector for the synchronous channel can be extended without much conceptual difficulty to the asynchronous case. This can be accomplished by using the equivalence to a synchronous problem introduced in Section 5.2, whereby each of the $(2M + 1)K$ symbols can be viewed as being transmitted by a different fictitious user. However, the resulting detector would be impossible to implement as it would involve the inversion of a crosscorrelation matrix of dimension $(2M + 1)K$. Fortunately, as $M \to \infty$ such a detector is equivalent to a much more manageable structure.

In the asynchronous case, the matched filter output corresponding to the i^{th} bit of the k^{th} user becomes

$$\tilde{y}_k(i) = \int_{iT-\tau_k}^{iT-\tau_k+T} s_k(t - iT - \tau_k)\, r(t)\, dt \tag{6.12}$$

(i.e., the observables used in the derivare $y_{k+iK} = A_k \tilde{y}_k(i)$). Analogously to (6.8) we can write the matched filter outputs as a function of the transmitted bits:

$$\tilde{\mathbf{y}}(i) = \mathbf{R}(1)\mathbf{A}\mathbf{b}(i-1) + \mathbf{R}(0)\mathbf{A}\mathbf{b}(i) + \mathbf{R}^T(1)\mathbf{A}\mathbf{b}(i+1) + \tilde{\mathbf{n}}(i) \tag{6.13}$$

where $\tilde{\mathbf{y}}(i) = [\tilde{y}_1(i), \ldots, \tilde{y}_K(i)]^T$, $\mathbf{b}(i) = [b_1(i), \ldots, b_K(i)]^T$ and $\tilde{\mathbf{n}}(i)$ is a K-vector Gaussian sequence with zero-mean and correlation:

$$E[\tilde{\mathbf{n}}(i)\tilde{\mathbf{n}}(j)^T] = \sigma^2 \mathbf{R}(i-j) \tag{6.14}$$

where $\mathbf{R}^T(i) = \mathbf{R}(-i)$ and $\mathbf{R}(i)$ is defined in (5.41) for $i \geq 0$.

If the amplitudes of the transmitters are not constrained to be constant, then the decorrelating detector (for finite M) entails multiplying the vector $[\tilde{\mathbf{y}}^T(-M), \ldots, \tilde{\mathbf{y}}^T(M)]^T$ by the inverse of the matrix

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}(0) & \mathbf{R}(-1) & 0 & \cdots & 0 \\ \mathbf{R}(1) & \mathbf{R}(0) & \mathbf{R}(-1)' & & \vdots \\ 0 & \mathbf{R}(1) & \mathbf{R}(0) & \ddots & 0 \\ \vdots & & \ddots & \ddots & \mathbf{R}(-1) \\ 0 & \cdots & 0 & \mathbf{R}(1) & \mathbf{R}(0) \end{bmatrix}, \tag{6.15}$$

As $M \to \infty$, this is equivalent [6] to passing the sequence $\{y(i)\}$ through a K-input K-output linear-time invariant filter with transfer function

$$[\mathbf{R}^T(1)\, z + \mathbf{R}(0) + \mathbf{R}(1)\, z^{-1}]^{-1}$$

To fix ideas, consider the two-user case in which the transfer function becomes

$$[\mathbf{R}^T(1)\, z + \mathbf{R}(0) + \mathbf{R}(1)\, z^{-1}]^{-1} = \frac{1}{1 - \rho_{12}^2 - \rho_{21}^2 - \rho_{12}\rho_{21}z - \rho_{12}\rho_{21}z^{-1}} \begin{bmatrix} 1 & -\rho_{12} - \rho_{21}z^{-1} \\ -\rho_{12} - \rho_{21}z & 1 \end{bmatrix}$$

Let us now turn our attention to the performance of the decorrelating detector in the asynchronous channel. The analysis is very simple and similar to the synchronous case. Since the decorrelating filter is the inverse of the transfer function from $\{\mathbf{b}(i)\}$ to $\{\mathbf{y}(i)\}$ in (6.13), the i^{th} output of the k^{th} component of the decorrelating filter only has one component due to the transmitted signals, namely $A_k b_k(i)$. The output of the decorrelating filter due to the noise sequence $\{\tilde{\mathbf{n}}(i)\}$ is a stationary Gaussian K-vector process with power spectral density matrix equal to

$$[\mathbf{R}^T(1)\, e^{j\omega} + \mathbf{R}(0) + \mathbf{R}(1)\, e^{-j\omega}]^{-1}$$

Thus, the i^{th} output of the k^{th} component of the decorrelating filter due to the noise is a Gaussian random variable with variance equal to

$$\frac{1}{2\pi} \int_0^{2\pi} [\mathbf{R}^T(1)\, e^{j\omega} + \mathbf{R}(0) + \mathbf{R}(1)\, e^{-j\omega}]_{kk}^{-1}\, d\omega$$

which implies that the efficiency and near-far resistance of the decorrelating detector is equal to

$$\bar{\eta}_k^d = \left[\frac{1}{2\pi} \int_0^{2\pi} [\mathbf{R}^T(1)\, e^{j\omega} + \mathbf{R}(0) + \mathbf{R}(1)\, e^{-j\omega}]_{kk}^{-1}\, d\omega \right]^{-1}$$

which, in turn, shows (cf. (5.40)) that the optimal near-far resistance property of the decorrelating detector carries over to the asynchronous setting.

7. Other Multiuser Demodulators

We discussed three advantages of the decorrelating detector over the optimum detector from the standpoint of implementation: its lower computational complexity, its suitability for decentralized implementation, and the fact that it does not require knowledge of the received energies. Reference [5] examine sthe degree of improvement in performance which is attainable by knowing the relative energies of the interferers while retaining the linear structure of the receiver. Another way to trade off suppression of multiuser interference with noise reduction when the energies are known by the receiver is to use, in lieu of the decorrelating detector (6.5), the linear transformation

$$\hat{b} = \mathrm{sgn}\left[(\mathbf{H} + \sigma^2\mathbf{I})^{-1}\mathbf{y}\right] = \mathrm{sgn}\left[(\mathbf{R} + \sigma^2\mathbf{A}^{-2})^{-1}\tilde{\mathbf{y}}\right]$$

which is justified in [21] as the solution to a minimum mean-square error problem where one minimizes

$$E[(\mathbf{b} - \mathbf{M}\mathbf{y})^T (\mathbf{b} - \mathbf{M}\mathbf{y})]$$

over all nonnegative definite $K \times K$ matrices $\mathbf{M}$. Since as $\sigma \to 0$ this detector coincides with the