

NEW BOUND ON THE ERROR PROBABILITY OF MAXIMUM LIKELIHOOD SEQUENCE DETECTION OF SIGNALS SUBJECT TO INTERSYMBOL INTERFERENCE

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1. Introduction

The Forney upper bound on the error probability of maximum likelihood sequence detection is the cornerstone of the analysis of optimum demodulators for channels with intersymbol interference. In this paper, we show a simple derivation of a tighter upper bound by applying the method of error-sequence decomposition.

Due to the dynamical component introduced by the presence of intersymbol interference into the data demodulation problem, optimum signal detection cannot be achieved on the basis of the independent observation of the time interval of each transmitted symbol. Rather it is necessary to treat the problem as one of sequence detection, whereby observation of the whole received waveform is required to produce a sufficient statistic. Since in this case, the transmitted symbols are not independent a-posteriori, there is not a unique optimality criterion even though all sequences are assumed to be equiprobable. In practice the main optimality criterion is maximum likelihood detection; i.e., the detector selects the sequence of symbols corresponding to the minimum energy noise realization. The pre-eminence of maximum likelihood sequence detection is due to two main reasons; first, unless the background is not, it achieves near-optimum error probability; and second, it can be implemented via the Viterbi algorithm in time-complexity per binary decision which is independent of the number of transmitted symbols and exponential in the number of interfering signals at any given time. The applicability of the Viterbi algorithm to the optimum demodulation of PAM sequences transmitted through white Gaussian dispersive linear channels was noticed independently by Forney [1], Kobayashi [2], and Omura [3]. Ungerboeck [4] streamlined the maximum likelihood sequence detector showing a particularly simple implementation. Forney obtained upper [1] and lower bounds [5] on the error probability, which are tight in the low-noise region. The lower bound analysis [5] is based on the error probability of receivers with side information that perform one-shot optimum decisions. The upper bound (see (8) below) was obtained by the proof of the *three subevents*, amended later by Foschini [6] and reproduced in a number of subsequent works (e.g. [7, 4, 8]). This argument was simplified in various ways by Mazo [9], Acampora [10] and Viterbi and Omura [11, 12].

The Forney bound (8) is an infinite series whose convergence and computation are nontrivial. Foschini [6] showed local convergence (i.e., for sufficiently low noise levels) of the bound for any intersymbol interference problem.¹ Forney [1] showed how to apply Viterbi's symbolic flowgraph technique in order to compute the bounding series; from the computational complexity viewpoint this approach is limited by the fact that inherently inefficient symbolic transfer function techniques have to be applied on graphs with exponential dimensionality in the length of the interference.

The new upper bound presented in Section 2 admits a simple proof and shows that a substantial number of terms can be excluded from the Forney bound.² The new upper bound is more general in the sense that nonlinear modulation formats are allowed and the interference length is not restricted to be finite. Section 3 is devoted to the problem of the computation of the new bound. The symbolic flowgraph approach is not suitable for this task and we present a *branch-and-bound* based combinatorial approach, which is of independent interest, to compute efficiently the bounding series up to any prespecified degree of accuracy.

2. New upper bound

The starting point of the proof of the new upper bound will be a conceptually straightforward derivation of the Forney bound which will lead naturally to the new result. Suppose that the receiver observes an antipodally modulated sequence of equiprobable and independent bits imbedded in additive white Gaussian noise whose two-sided power spectral level is equal to σ^2 ; i.e.,

$$r_t = \sum_{i=-M}^M b(i)s_{t-iT} + n_t, \quad t \in I, \quad b(i) \in \{-1, +1\}. \quad (1)$$

The objective of the maximum likelihood sequence detector is to select the most likely sequence $\mathbf{b} = \{b(i) \in \{-1, +1\}, i = -M, \dots, M\}$ given $r_t, t \in I$. Since the noise is Gaussian and all sequences are a priori equiprobable, this is equivalent to the mean-square criterion whereby the detector outputs the sequence $\hat{\mathbf{b}}$ that maximizes³

$$\Omega(\mathbf{b}) = 2 < S(\mathbf{b}), r > - \| S(\mathbf{b}) \|^2, \quad (2)$$

where $S_t(\mathbf{b}) = \sum_{i=-M}^M b(i)s_{t-iT}, t \in I$. The performance of any detector that maximizes (2) is obviously independent of the implementation of the decision algorithm; whether this is the Viterbi algorithm, brute force or any other approach is immaterial in the sequel. We are interested not in the probability that the detector outputs an erroneous sequence but in the probability that there is an error in the output sequence k^{th} component; i.e., $P_e(k) = P\{\hat{b}^t(k) \neq b^t(k)\}$ if \mathbf{b}^t is the transmitted sequence. Note that for every M , $P_e(k)$ need not be independent of k : however as $M \rightarrow \infty$ it converges to the sought-after bit error rate.

Let us proceed to our derivation of the Forney upper bound on $P_e(k)$. If $\hat{b}^t(k) \neq b^t(k)$ then there exists an error sequence ϵ , i.e., a vector whose $2M+1$ components are drawn from $\{-1, 0, 1\}$, such that $\epsilon(k) \neq 0$, $\Omega(\mathbf{b}^t - 2\epsilon) = \max \Omega(\mathbf{b})$ and such that if $\epsilon(i) \neq 0$ then $b^t(i) = \epsilon(i)$, i.e., $\mathbf{b}^t - 2\epsilon$ is a sequence of ± 1 . For every sequence in $E_k = \{\epsilon \in \{-1, 0, 1\}^{2M+1}, \epsilon(k) \neq 0\}$, the probability of the latter event is equal to $2^{-w(\epsilon)}$, where $w(\epsilon)$ is

² For example, if the interference length is equal to two, the Forney bound is composed of all the finite sequences drawn from $\{-1, +1\}$, while the new bound only allows sequences of alternate $+1$ and -1 .

³ $\langle f, g \rangle = \int f_t g_t dt$ and $\|f\|^2 = \langle f, f \rangle$.

¹ Global convergence (i.e., for any signal-to-noise ratio) occurs only in special cases (e.g., when only two symbols interfere at a time - see Section 3).

the number of nonzero components of ϵ (recall that \mathbf{b}^t is drawn equiprobably from the ensemble $\{-1, +1\}^{2M+1}$). Unfortunately, unless we deal with the one-shot case $M=0$, the probability that $\mathbf{b}^t - 2\epsilon$ is the most likely sequence does not admit an explicit expression. However, this event is upper bounded by $\{\Omega(\mathbf{b}^t - 2\epsilon) \geq \Omega(\mathbf{b}^t)\}$, whose probability is easily seen to equal

$$P \{ | \langle \mathbf{b}^t, S(\epsilon) \rangle / \| S(\epsilon) \| > - \| S(\epsilon) \| \} = Q(\| S(\epsilon) \| / \sigma). \quad (3)$$

Hence, we have obtained an upper bound on $P_e(k)$, namely,

$$P_e(k) \leq \sum_{\epsilon \in E_k} 2^{-w(\epsilon)} Q(\| S(\epsilon) \| / \sigma) \quad (4)$$

This bound, however is of little or no use because it diverges as $M \rightarrow \infty$ for any σ . We are clearly summing over too many sequences.

Forney's approach was to assume that the intersymbol interference length K is finite (K is the smallest positive integer that satisfies $\int s_i s_{t+iT} = 0$, if $i \geq K$), and to exclude all the sequences in E_k containing at least $K-1$ zeros amid nonzero components. The remaining sequences $B_k \subset E_k$ will henceforth be referred to as *simple*. In order to justify that the resulting series is still an upper bound to the error probability, i.e.,

$$P_e(k) \leq \sum_{\epsilon \in B_k} 2^{-w(\epsilon)} Q(\| S(\epsilon) \| / \sigma) \quad (5)$$

we will show that if $\epsilon \in E_k$ is not simple and $\Omega(\mathbf{b}^t - 2\epsilon) = \max_b \Omega(\mathbf{b})$, then we can find a simple sequence $\epsilon^1 \in B_k$, such that $\Omega(\mathbf{b}^t - 2\epsilon^1) \geq \Omega(\mathbf{b}^t)$. To that end, note that we can always write ϵ as the sum of simple sequences. The desired sequence $\epsilon^1 \in B_k$ is the unique simple subsequence of ϵ whose k^{th} component is not equal to zero. If $\mathbf{b}^t - 2\epsilon$ is the sequence selected by the maximum likelihood sequence detector then necessarily we have

$$\Omega(\mathbf{b}^t - 2\epsilon) \geq \Omega(\mathbf{b}^t - 2(\epsilon - \epsilon^1)). \quad (6)$$

Moreover, it is easy to show for any arbitrary pair ϵ^a, ϵ^b of error sequences that

$$\Omega(\mathbf{b}^t - 2(\epsilon^a + \epsilon^b)) - \Omega(\mathbf{b}^t - 2\epsilon^a) + \Omega(\mathbf{b}^t) - \Omega(\mathbf{b}^t - 2\epsilon^a) = | | S(\epsilon^a + \epsilon^b) | |^2 + | | S(\epsilon^a) | |^2 + | | S(\epsilon^b) | |^2. \quad (7)$$

Since $S(\epsilon^1)$ and $S(\epsilon - \epsilon^1)$ are orthogonal (recall that ϵ^1 is a subsequence of ϵ flanked by at least $K-1$ zeros at each side), when we particularize (7) to the case $\epsilon^a = \epsilon^1, \epsilon^b = \epsilon - \epsilon^1$, its right-hand side is equal to zero; so together with (6), it implies that $\Omega(\mathbf{b}^t - 2\epsilon^1) \geq \Omega(\mathbf{b}^t)$ as we wanted to show.

In Forney's work [1] the upper bound on the error probability of the maximum likelihood sequence detector is not given as in (5); rather, it is expressed in the form,

$$P_e \leq \sum_{d \in D} Q(d/2\sigma) \sum_{\xi \in E_d} w(\xi) 2^{-w(\xi)} \quad (8)$$

in the case of binary modulation. This corresponds to a rearrangement of the terms in the summation of (5) when $M = \infty$. In (8), E_d is the set of error events with Euclidean weight equal to d^2 , and D is the set of square roots of Euclidean weights attained by the error events. An error event is a finite string of elements drawn from $\{-1, 0, 1\}$ such that the first and the last elements are nonzero and such that there are no more than $K-2$ consecutive zeros between any pair of nonzero elements. Clearly, if $M = \infty$ there exists a one-to-one correspondence between the error events and a partition of the simple error sequences according to the equivalence relation of being a shifted version of one another. The Euclidean weight of the error event ξ is equal to $d^2(\xi) = 4 \| S(\epsilon) \|^2$, where ϵ is any simple sequence belonging to the equivalence class associated to ξ . Since $w(\xi)$ is equal to the number of elements in the intersection of B_k and the equivalence class associated to ξ it follows that the right-hand sides of (5)

and (8) coincide. Note further that in order to be able to apply the symbolic flowgraph technique to the computation of (8), the function $Q(x)$ is substituted in [1] by its upper bound $1/2 \exp(-x^2/2)$, resulting in the bound

$$P_e \leq 1/2 \sum_{d \in D} \exp(-d^2/8\sigma^2) \sum_{\xi \in E_d} w(\xi) 2^{-w(\xi)}. \quad (9)$$

Let us return to the derivation of (5) and see how to tighten the upper bound by the elimination of additional sequences. The main step of the above proof is the identification of a set of sequences, B_k , such that if $\epsilon \in E_k - B_k$, and $\mathbf{b}^t - 2\epsilon$ is most likely a posteriori, then there exists $\epsilon^1 \in B_k$ such that $\Omega(\mathbf{b}^t - 2\epsilon^1) \geq \Omega(\mathbf{b}^t)$. Now we will find a proper subset $F_k \subset B_k$ that still satisfies this property. Notice that we only used the fact that ϵ was not simple in order to decompose it into $\epsilon^1 \in B_k$ and $(\epsilon - \epsilon^1)$ such that $S(\epsilon^1)$ and $S(\epsilon - \epsilon^1)$ are mutually orthogonal. However, had $\langle S(\epsilon^1), S(\epsilon - \epsilon^1) \rangle \geq 0$ the proof would have held verbatim since the nonpositivity of the right-hand side of (7) is enough, along with (6), to conclude $\Omega(\mathbf{b}^t) \leq \Omega(\mathbf{b}^t - 2\epsilon^1)$. Therefore we can exclude from the bound all the sequences ϵ that already have a subsequence ϵ^1 in the series satisfying $\langle S(\epsilon^1), S(\epsilon - \epsilon^1) \rangle \geq 0$. Summarizing, we have shown the following result.

Proposition 1:

We say that an error sequence $\epsilon \in E_k$ is *decomposable* into nonzero error sequences ϵ^1 and ϵ^2 if $\epsilon = \epsilon^1 + \epsilon^2$, $\epsilon^1(i) = 0$ implies that $\epsilon^2(i) = 0$ (and $\epsilon^2(i) = 0$), and $\langle S(\epsilon^1), S(\epsilon^2) \rangle \geq 0$.

Let $F_k = \bigcup_{n=1}^{2M+1} F_k(n)$, where

$$F_k(n) = \{ \epsilon \in E_k : w(\epsilon) = n, \text{ and if } \epsilon \text{ is decomposable}$$

$$\text{into } \epsilon^1 \text{ and } \epsilon^2, \text{ then } \epsilon^1, \epsilon^2 \notin \bigcup_{m=1}^{n-1} F_k(m) \}.$$

Then,

$$P_e(k) \leq \sum_{\epsilon \in F_k} 2^{-w(\epsilon)} Q(\| S(\epsilon) \| / \sigma) \quad (10)$$

It was recognized in Forney [1], that the greatest theoretical deficiency of the error probability analysis of the maximum-likelihood sequence detector was the assumption of finite intersymbol interference length. Placing further restrictions on the channel impulse response, Wyner [13] showed that as $\sigma \rightarrow 0$ the asymptotic behavior (dominated by the minimum Euclidean distance) of the Forney bound is retained even in the case of unbounded intersymbol interference length. More significantly, we can see that K does not play any role in the derivation of (10), and hence the new bound holds for any noise level regardless of whether the interference length is finite. Also, it is interesting to notice that the foregoing result holds even if the modulation format is not restricted to be linear; i.e., if the received process is

$$r_t = \sum_{i=-M}^M s_{t-iT}(b(i)) + n_t, \quad (11)$$

where $s(+1), s(-1)$ are arbitrary L_2 waveforms. The only required change is the definition of $S(\epsilon)$, which now has to be generalized to

$$S_t(\epsilon) = 1/2 \sum_{i=-M}^M [s_{t-iT}(\epsilon(i)) \cdot s_{t-iT}(-\epsilon(i))].$$

How does the new bound compare to the Forney bound? It is easy to see that all the sequences in F_k are simple: if $\epsilon \in E_k - B_k$ then one can always decompose⁴ ϵ into $\epsilon^1 \in F_k$ and $\epsilon - \epsilon^1$. Moreover at least half of the sequences of the Forney bound are excluded from the new bound. To see this, denote the sequence whose only nonzero component is $\epsilon(k) = 1$ by $e_k \in F_k$. Obviously, if $\epsilon \in B_k - \{e_k\}$, then $2e_k - \epsilon \in B_k$, however ϵ and $2e_k - \epsilon$ cannot both belong to F_k for either ϵ is decomposable into e_k and $(\epsilon - e_k)$ or $2e_k - \epsilon$ is decomposable into e_k and $(e_k - \epsilon)$. Usually, however, a much greater proportion of error sequences are excluded. For example, in the case $K=2$, and $\int s_i s_{i-T} > 0$, the simple sequences are those that do not contain any zero amid nonzero components, while F_k consists exclusively of the simple sequences whose nonzero components have alternate signs.

3. Computation of the bounding series

In this section we study the problem of evaluating the new bound on the bit error rate of the maximum likelihood sequence detector. This problem is nontrivial because when $M=\infty$, the right-hand side of (10) contains infinite terms whose summation does not admit a closed-form expression due to the nature of the Q-function and the set F_0 (we restrict attention to $k=0$ since (10) is independent of k in the infinite-horizon case). The results of this section show a procedure that computes in finite time an approximation to the bounding series up to any predetermined degree of accuracy. To this end we first show a combinatorial algorithm that approximates the sum of the values of an infinite tree provided that a convergent bound for the sum of the values of all successors is available for each node. Then, taking advantage of the fact that sums of $2^{-u(k)} \exp(-\|S(\epsilon)\|^2/2\sigma^2)$ over simple sequences can be computed exactly, we show how to compute the bounding series when $K < \infty$ by using the aforementioned algorithm.

Suppose that it is desired to compute (an approximation to) the sum of the values of an infinite-depth tree, and suppose that for each node we can compute an upper bound on the sum of the values of all its successors. Then, successive approximations to the value of the series can be obtained by using a procedure similar to the *branch-and-bound* algorithm (e.g., [14]). This is a combinatorial optimization procedure that searches for the node with largest payoff in a tree. It is necessary that for each node an upper bound on the payoff of its successors is available. The basic idea of the branch-and-bound algorithm is that when the upper bound on the payoff of the successors of a node n is found to be less than or equal to the payoff of the visited nodes, it removes all the successors of n from further consideration. Hence, an exhaustive search of all the nodes in the tree is avoided. A similar idea can be used to compute bounds on the sum of the values of all nodes in the tree. The lower bound is the sum of the values of the visited nodes and the upper bound is equal to the lower bound plus the sum of the offspring bounds of the leaves of the visited subtree. A new pair of bounds can be obtained by branching a node, subtracting its offspring bound from the upper bound, adding the value of its children to the lower and the upper bounds, and adding their offspring bounds to the upper bound. Under certain conditions on the offspring bounds (Proposition 2 below), it can be shown that the bounds converge to the value of the series. The formal algorithm appears in Figure 1 (cf. [14, Figure 18-5]), and its correctness and conditions for convergence are stated in the following result.

⁴ If $\epsilon^1 \in B_k$ found in the proof of (5) does not belong to F_k , then it is decomposable into $\epsilon^2 \in F_k$ and $\epsilon^1 - \epsilon^2$. But since $S(\epsilon^2)$ and $S(\epsilon - \epsilon^1)$ are orthogonal, ϵ is decomposable into ϵ^2 and $(\epsilon - \epsilon^2)$.

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input:  tree with node set = V
        nonnegative function "value"  $v[n]$ ,  $n \in V$ 
        nonnegative function "offspring"  $\theta[n]$ ,  $n \in V$ 
        note: if  $\Psi[n]$  is the set of successors of  $n$  then

$$\sum_{n \in \Psi[m]} v[n] \leq \theta[m], m \in V.$$

         $A \subset R^2$ ; set of "acceptable" approximations.

output: (lobound, upbound)  $\in A$ , s.t.
        lobound  $\leq \sum_{n \in V} v[n] \leq$  upbound

begin
  lobound := v[root];
  upbound := v[root] +  $\theta$ [root];
  activeset := {root};
  visitset := {root};
  while (lobound, upbound)  $\notin A$  and
    activeset is nonempty do
    begin
      choose branching node  $n \in$  activeset;
      activeset := activeset - { $n$ };
      upbound := upbound -  $\theta$ [ $n$ ];
      for all children  $m$  of node  $n$  do
        begin
          visitset := visitset  $\cup$  { $m$ };
          lobound := lobound +  $v$ [ $m$ ];
          upbound := upbound +  $v$ [ $m$ ];
          if  $m$  has children then do
            begin
              activeset := activeset  $\cup$  { $m$ };
              upbound := upbound +  $\theta$ [ $m$ ];
            end
          end
        end
      end
    end
end

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Figure 1. Branch and bound series approximation algorithm.

Proposition 2:

Suppose $\bar{v}[n]$, $n \in V$ is a nonnegative function defined on the set V of nodes of the tree such that

$$i) \theta[m] \leq \sum_{n \in \Psi[m]} \bar{v}[n], \text{ for all } m \in V, \text{ and} \quad (12)$$

$$ii) \sum_{n \in V} \bar{v}[n] < \infty. \quad (13)$$

Let u_i, l_i, V_i , be the values of *upbound*, *lobound* and *visitset*, respectively, at the beginning of the i^{th} iteration of the algorithm of Figure 1 where $A = \{(x, y), x \in R\}$. Then,

$$l_i \leq \sum_{n \in V} v[n] \leq u_i,$$

and for every $\delta > 0$ there exists i_0 and V_{i_0} such that $u_i - l_i < \delta$ for $i \geq i_0$.

Proof: Let A_i be the activeset at the beginning of the i^{th} iteration. Using (12) we have

$$l_i = \sum_{n \in V_i} v[n] \leq \sum_{n \in V} v[n] \leq u_i$$

$$\sum_{n \in V_i} v[n] + \sum_{n \in A_i} \theta[n] \leq \sum_{n \in V_i} v[n] + \sum_{n \in V-V_i} \bar{v}[n]$$

Because of (13) we have on the one hand that $\sum_{n \in V} v[n]$ converges, and on the other hand, for every $\delta > 0$ we find a valid visitset W (a subtree that contains the root and the node n only if all the children of the father of n are also contained) such that $\sum_{n \in V-W} \bar{v}[n] < \delta$, since the cardinality of the children of each node is assumed to be bounded. Hence if $V_{i_0} = W$ we have

$$u_i - l_i \leq \sum_{n \in V-V_i} \bar{v}[n] \leq \sum_{n \in V-W} \bar{v}[n] < \delta, \text{ for } i > i_0.$$

We show now how to pose the problem of computing $\sum_{\epsilon \in F_0} 2^{-w(\epsilon)} Q(\|S(\epsilon)\|/\sigma)$ as a set of tree summation problems. Define for each vector $z \in \{-1,0,1\}^{K-1}$,

$$H(z) = \{\epsilon: \epsilon_i = z_i, i=0, \dots, K-2\}, z \in \{-1,0,1\}^K,$$

and

$$\Gamma(z) = \sum_{\epsilon \in H(z)} 2^{-w(\epsilon)} Q(\|S(\epsilon)\|/\sigma).$$

Note that the sought-after bound can be written

$$\sum_{\epsilon \in F_0} 2^{-w(\epsilon)} Q(\|S(\epsilon)\|/\sigma) = \sum_{\substack{z \in \{-1,0,1\}^{K-1} \\ z_0 \neq 0}} \Gamma(z).$$

In order to compute $\Gamma(z)$, we construct a tree $V(z)$ with value function, $v[n]$, and an offspring-bound function $\theta[n]$, such that for each $z \in \{-1,0,1\}^{K-1}$

- a) $\Gamma(z) = \sum_{n \in V(z)} v[n]$, and
- b) $V(z)$, v , θ satisfy i) and ii) in Proposition 2.

In fact, only $(3^{K-1}-1)/2$ of these trees are needed because $\Gamma(z) = \Gamma(-z)$. The definition of the tree $V(z)$ is as follows:

Definition of $V(z)$: Besides the aforementioned value and offspring functions, we associate to each node a sequence $u[n] \in E_0$, a left-state $x_L[n] \in \{-1,0,1\}^{K-1}$ and a right-state $x_R[n] \in \{-1,0,1\}^{K-1}$. The root of $V(z)$ has the following labels:

$$u_i[\text{root}] = \begin{cases} 0 & i < 0 \text{ or } i \geq K-1 \\ z^i & 0 \leq i < K-1 \end{cases}$$

$$x_L^i[\text{root}] = z_{i-1}; \quad x_R^i[\text{root}] = z_{K-1-i}; \quad i=0, \dots, K-2.$$

$$v[\text{root}] = \begin{cases} 2^{-w(u[\text{root}])} Q(\|S(u[\text{root}])\|/\sigma) & \text{if } u[\text{root}] \in F_0 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\bar{v}[\text{root}] = 2^{-w(u[\text{root}])} \exp(-\|S(u[\text{root}])\|/2\sigma^2).$$

If $x_L[n] = x_R[n] = 0$ then node n has no children. Otherwise, the children of node n are $n_{j_L j_R}$ where $j_L \in \{-1,0,1\}$ if $x_L[n] \neq 0$ and $j_L = 0$ if $x_L[n] = 0$ (analogously j_R). If the depth of $n_{j_L j_R}$ is equal to l , then

$$u_i[n_{j_L j_R}] = \begin{cases} j_L & i = l \\ j_R & i = K-2+l \\ u_i[n] & \text{otherwise,} \end{cases}$$

$$x_R[n_{j_L j_R}] = f(x_R[n], j_R), \text{ and } x_L[n_{j_L j_R}] = f(x_L[n], j_L),$$

where f is a $K-1$ dimensional shift-register system, $f(x, u) = [x^2, \dots, x^{K-1}u]^T$.

The value and offspring functions of the children of node n are defined as follows:

$$v[n_{j_L j_R}] = \begin{cases} 0 & \text{if } j_L = j_R = 0 \text{ or } u[n_{j_L j_R}] \notin F_0 \\ 2^{-w(u[n_{j_L j_R}])} Q(\|S(u[n_{j_L j_R}])\|/\sigma) & \text{otherwise} \end{cases}$$

and

$$\bar{v}[n_{j_L j_R}] = \begin{cases} 0 & \text{if } j_L = j_R = 0 \\ 2^{-w(u[n_{j_L j_R}])} \exp(-\|S(u[n_{j_L j_R}])\|^2/2\sigma^2) & \text{otherwise} \end{cases}$$

It is straightforward to show that since j_L and j_R are more than $K-1$ symbols apart, we have

$$\|S(u[n_{j_L j_R}])\|^2 = \|S(u[n])\|^2 + \rho(x_R[n], j_R) + \rho(x_L[n], j_L),$$

where

$$\rho(x, u) = |u|w + 2u \sum_{i=1}^{K-1} x_i h_{K-i},$$

$$w = \int s_i^2 dt, \text{ and } h_i = \int s_i s_{i-T} dt.$$

The upper bound on the sum of the values of the successors of node n (whose depth is equal to l) is given by

$$\theta[n] = \phi[n] [T(x_R[n]) T(x_L[n]) - 1]$$

where

$$\phi[n] = \begin{cases} 0 & \text{if } \Psi[n] = \phi \\ 2^{-w(u[n])} \exp(-\|S(u[n])\|^2/2\sigma^2) & \text{otherwise,} \end{cases}$$

and for all $x \in \{-1,0,1\}^{K-1}$,

$$T(x) = \sum_{u \text{ simple}} \prod_{i=0}^{\infty} 2^{-1 \cdot u_i} \exp(-\rho(x_i, u_i)/2\sigma^2) \quad (14)$$

where $x_{i+1} = f(x_i, u_i)$, $x_0 = x$.

The next result shows that the values of $\Gamma(z)$ can be obtained by applying the branch-and-bound series approximation algorithm (Figure 1) to the tree $V(z)$ defined above.

Proposition 3: The following three equalities are valid:

- i) $\sum_{n \in V(z)} \bar{v}[n] = \sum_{\epsilon \in B(z)} 2^{-w(\epsilon)-1} \exp(-\|S(\epsilon)\|^2/2\sigma^2)$,
- ii) $\sum_{n \in V(z)} v[n] = \sum_{\epsilon \in H(z)} 2^{-w(\epsilon)} Q(\|S(\epsilon)\|/\sigma)$, and
- iii) $\sum_{m \in \Psi[n]} v[m] \leq \theta[n] = \sum_{m \in \Psi[n]} \bar{v}[m]$, $n \in V(z)$;

where $B(z) = \{\epsilon: \epsilon \text{ is simple and } \epsilon_i = z_i, i=0, \dots, K-2\}$.

Proof: First, we show that there is a one-to-one correspondence between $B(z)$ and the subset of $V(z)$ with $\bar{v}[n] \neq 0$. For each node in the tree, $u[n] \in B(z)$ because if $x_L[n] = x_R[n] = 0$ then node n has no children. On the other hand, for each sequence $u \in B(z)$, there exist several $n \in V(z)$ such that $u = u[n]$; however, only one of those (the one with minimum depth) is such that $\bar{v}[n] \neq 0$. Furthermore, if $\bar{v}[n] \neq 0$ then $\bar{v}[n] = 2^{-w(u[n])-1} \exp(-\|S(u[n])\|^2/2\sigma^2)$, and i) follows. The image of the set $\{n \in V(z); v[n] \neq 0\}$ according to the above correspondence is $H(z) \subset B(z)$, and if $v[n] \neq 0$ then $v[n] = 2^{-w(u[n])} Q(\|S(u[n])\|/\sigma)$, and ii) follows. Equation iii) is a consequence of $Q(x) \leq 1/2 \exp(-x^2/2)$ and the fact that if $\Psi[n] \neq \emptyset$ (otherwise $\theta[n] = 0$), then

$$\begin{aligned} \sum_{m \in \Psi[n]} \bar{v}[m] &= 2^{-w(u[n])-1} \exp(-\|S(u[n])\|^2/2\sigma^2) \\ &\quad [T(x_R[n]) T(x_L[n]) - 1] \\ &= \phi[n] [T(x_R[n]) T(x_L[n]) - 1] = \theta[n]. \end{aligned}$$

The question that remains to be elucidated is how to compute the "tail function" $T(x)$ defined in (14). Note that the convergence of the right-hand side of (14) is equivalent to the convergence of the Forney bound which is guaranteed in a neighborhood of $\sigma=0$ ([6]). From (14) it follows that $T(x) = T(-x)$, $T(0) = 1$, and if $x \neq 0$ then

$$\begin{aligned} T(x) &= T(f(x, 0)) \\ &+ 1/2 \exp\left[-\frac{w}{2\sigma^2}\right] \left\{ \exp\left(\sum_{i=1}^{K-1} z_i h_{K-i}/\sigma^2\right) T(f(x, -1)) \right. \\ &\quad \left. + \exp\left(-\sum_{i=1}^{K-1} z_i h_{K-i}/\sigma^2\right) T(f(x, 1)) \right\}. \end{aligned}$$

These relationships can be written in matrix form using the following notation. Let $x(1), \dots, x(N)$ be a listing of the elements modulo sign of $\{-1, 0, 1\}^{K-1} - \{0\}$, denote $\xi(i) = T(x(i))$ and define the $N \times N$ matrix:

$$\Phi_{i,j} = \begin{cases} 2^{-|a|} \exp\left[-\left(\|s\|w + 2a \sum_{k=1}^{K-1} z_k(i) h_{K-k}\right)/2\sigma^2\right] \\ \quad \text{if } x(j) = \pm f(x(i), a); a \in \{-1, 0, 1\} \text{ and} \\ \quad \quad x(i) \neq \pm[1, 0, \dots, 0]^T \\ \exp(-w/2\sigma^2) \cosh(h_{K-1}/\sigma^2) \\ \quad \text{if } x(i) = \pm[1, 0, \dots, 0]^T \text{ and } x(j) = \pm[0, \dots, 0, 1]^T \\ 0 \\ \quad \text{otherwise} \end{cases}$$

Then we can write

$$\xi = \Phi \xi + e \tag{15}$$

where the only nonzero component of the N -dimensional vector e is equal to 1 and corresponds to the equivalence class $\pm[1, 0, \dots, 0]$. Now, the computation of the tail function reduces to the solution of the linear system (15). Note that this equation may have a solution even if the series in (14) diverges; a necessary and sufficient condition for this series to converge is that the eigenvalues of Φ lie inside the unit circle. This fact implies global convergence of (14), and hence of the Forney bound (9), in the case $K=2$. (In this case, $N=1, |h_1| < w/2$ and $\Phi = \exp(-w/2\sigma^2) \cosh(h_1/\sigma^2) < 1$.)

4. Concluding Remarks

The method of error sequence decomposition, originally developed to analyze optimum multi-user detectors [15], has been successfully applied to the bit error rate analysis of maximum likelihood sequence detection of signals imbedded in white Gaussian noise and subject to intersymbol interference. This method reduces the analysis of the sequence detector (an m -ary hypothesis testing problem) to the analysis of a collection of binary hypothesis testing problems. The derivation of the error probability of the maximum likelihood sequence detector, which is also optimum in a mean-square sense, can be carried over to the non-Gaussian case simply by modifying the right-hand side of (3). Also, the generality of the method has been illustrated in

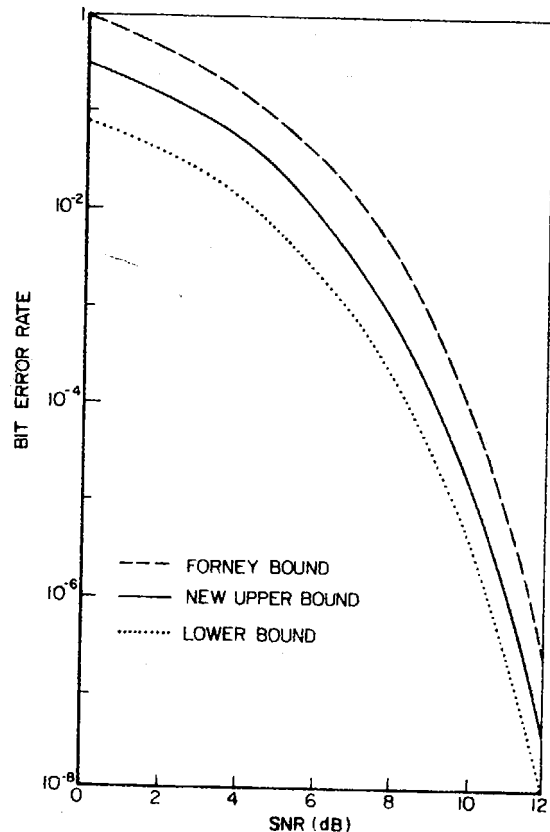


Figure 2. Bit error rate for duobinary signaling with rectangular pulse.

the case of Poisson point-process observations [16], where a different definition of decomposability (which takes into account the non-Euclidean structure of the signal space) allows for the derivation of a bound similar to (10).

Numerical examples comparing the Forney bound (9) computed through the transfer function of the state diagram, the new upper bound (10) computed via the branch-and-bound technique of Section 3, and the one-shot error probability $Q(w^{1/2}/\sigma)$ appear in Figures 2 and 3. Figure 2 corresponds to the simplest intersymbol interference problem, namely, duobinary signaling with a rectangular pulse ($K = 2, h_1 = w/2$). Figure 3 corresponds to an exponential pulse truncated to $5T$ and whose time-constant is equal to $3T/2$ ($K = 5, h_1 = 0.512w, h_2 = 0.260w, h_3 = 0.126w, h_4 = 0.051w$). In the high SNR region (in which the Forney bound is tight), the upper bounds are dominated by the minimum Euclidean distance (or error energy $\|S(\epsilon)\|^2$) terms of the series; in this region the difference between both upper bounds is due to the substitution of $Q(x)$ by $1/2 \exp(-x^2/2)$ in (9). As the SNR decreases the effect of the error sequences eliminated from the Forney bound becomes noticeable, enlarging appreciably the region on which the upper and lower bounds provide a tight approximation to the uncoded bit error rate.

The new bound is not only tighter but more general than the Forney bound, in that it does not require the intersymbol interference length to be bounded. However, it remains to obtain a computational method and a proof of local convergence that do not hinge on this restriction.

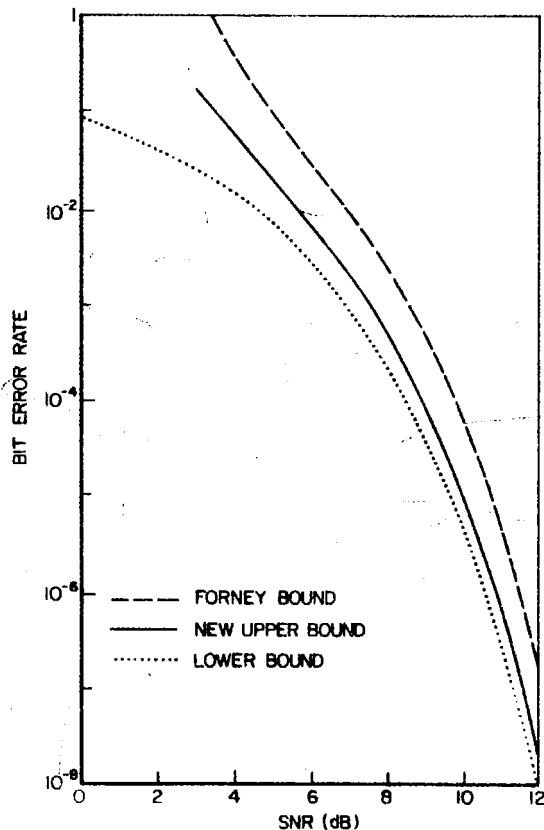


Figure 3. Bit error rate for truncated exponential pulse ($K = 5$).

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